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JV-algebras

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Introduction. Let J be a complex Banach space and a complex Jordan algebra equipped with an algebra involution $*$. Then J is a Jordan C^*-algebra if the following conditions are satisfied:

(1) $\|ab\| \leqslant \|a\| \, \|b\|$ for all $a, b \in J$,

(2) $\|U_a(a^*)\| = \|a\|^3$ for all $a \in J$.

(U_a is defined on page 3).

We define a JV-algebra as a complex unital complete normed Jordan algebra J such that

$$J = H(J) + iH(J)$$

in which $$H(J) = \{a \in J : a'(a) \in \mathbb{R};\ \forall\, a' \in J', a'(I) = \|a'\| = 1\},$$

where J' is the conjugate space of J and I the unit of J.

The most important consequence of the main result of this paper is that, if J is a JV-algebra, $H(J)$ is a real subalgebra. This, together with the results of (6) and (7), permits the identification of the class of the JV-algebras with that of the Jordan C^*-algebras. Based on my Theorem 2·3, in (5) the identification is also achieved independently.

1. *The spectrum and the numerical range of an element.*

A unital normed algebra will be a non-associative real or complex algebra A with unit I on which a norm ($\| \ \|$) is defined such that

$$\|ab\| \leqslant \|a\| \, \|b\| \quad \text{for all} \quad a, b \in A \quad \text{and} \quad \|I\| = 1.$$

If a is an element of such an algebra A, we define the numerical range of a by

$$V(A, a) = \{a'(a) : a' \in A',\ \|a'\| = a'(I) = 1\}$$

If A is a complex unital normed algebra we denote by $H(A)$ the set of all hermitian elements of A ($a \in H(A)$ if and only if $V(A, a) \subset \mathbb{R}$). Clearly $H(A)$ is a closed real linear subspace of A.

Let J be a Jordan algebra with identity element I. Then an element $a \in J$ is called invertible with b as an inverse if the equations

$$ab = I, \quad a^2 b = a$$

hold in J.

If J is a complex Jordan algebra with unit, the spectrum of an element a of J is the set $\mathrm{Sp}\,(J, a)$ of complex numbers defined as follows

$$\mathrm{Sp}\,(J, a) = \{z \in \mathbb{C} : zI - a \text{ is not invertible in } J\}$$

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Finally, if A is any algebra with product ab, the mapping

$$x \to xa (\text{resp. } x \to ax) \text{ in } A \text{ will be denoted as } R_a (\text{resp. } L_a)$$

THEOREM 1.1. *Let J be a complex Jordan algebra with unit, let $a \in J$ and p be a non-constant polynomial. Then*

$$\mathrm{Sp}\,(J, p(a)) = \{p(z)/z \in \mathrm{Sp}\,(J, a)\}$$

Proof. Let B be the subalgebra of J generated by I and a. It is well known that B is a strongly associative subalgebra of J. Since for $z_0 \in \mathbb{C}$ there exists $x_0, x_1, \ldots, x_n \in \mathbb{C}$ such that

$$z_0 I - p(a) = x_0(x_1 I - a) \ldots (x_n I - a),$$

then by ((2), section 8, chapter I; lemma 8):

$$U_{z_0 I - p(a)} = U_{x_0(x_1 I - a)} \ldots U_{x_n I - a},$$

where $U_a = 2R_a^2 - R_{a^2}$.

Hence by ((2), section 11, chapter I; theorem 13) $z_0 I - p(a)$ is not invertible if and only if $x_i I - a$ is also not invertible for some i.

THEOREM 1·2. *Let J be a complex complete normed Jordan algebra with unit. Then for each element a of J*

$$\mathrm{Sp}\,(BL(J), R_a) \subseteq \tfrac{1}{2}(\mathrm{Sp}\,(J, a) + \mathrm{Sp}(J, a)).$$

Proof. $\{R_a, R_{a^2}\}$ is a commutative subset of $BL(J)$. By Zorn's lemma there exists a commutative full and norm-closed subalgebra B of $BL(J)$ such that $\{R_a, R_{a^2}\} \subseteq B \subseteq BL(J)$ and therefore

$$\mathrm{Sp}\,(BL(J), b) = \mathrm{Sp}\,(B, b) \quad (b \in B).$$

Also for every $z \in \mathbb{C}$ it follows that $U_{a-zI} \in B$. Let $z \in \mathrm{Sp}\,(BL(J), R_a)$. Then $z \in \mathrm{Sp}\,(B, R_a)$ and so by the Gel'fand theory for commutative Banach algebras there exists a non-zero multiplicative linear functional g on B such that $g(R_a) = z$. If z_1 and z_2 are the roots of the equation

$$g(U_{a-zI}) = g(U_a) - 2zg(R_a) + z^2 = 0$$

then

$$g(R_a) = \tfrac{1}{2}(z_1 + z_2).$$

Clearly $U_{a-z_i I}$ $(i = 1, 2)$ is not invertible and therefore

$$a - z_i I \quad (i = 1, 2)$$

is not invertible either, i.e. $z_1, z_2 \in \mathrm{Sp}\,(J, a)$.

If A is a unital normed algebra, the mapping $a \to R_a (a \to L_a)$ will be a linear isometry of A into $BL(A)$ preserving the unit and so we obtain the following proposition.

PROPOSITION 1·3. *Let A be a unital normed algebra and a an element of A. Then*

(a) $V(A, a) = V(BL(A), R_a) = V(BL(A), L_a)$.

(b) *If B is a subalgebra of A with $\{a, I\} \subseteq B$ then*

$$V(B, a) = V(A, a)$$

(c) $|z| \leqslant \|a\|$ $(z \in V(A, a))$.

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THEOREM 1·4. *Let J be a complex unital complete normed Jordan algebra. Then*

$$\mathrm{Sp}\,(J,a) \subset V(J,a).$$

Proof. By the associative theorem ((1), proposition 10·6), and the Proposition 1·3 we have

$$\mathrm{Sp}\,(J,a) \subseteq \mathrm{Sp}\,(\bar{B},a) \subset V(\bar{B},a) = V(J,a),$$

where B is the subalgebra of J generated by I and a.

2. *The main result.*

Notation. J will denote a complex unital complete normed Jordan algebra and $[.]$ the Lie product in $BL(J)$.

LEMMA 2·1. $[R_a,[R_b,R_c]]\in H(BL(J))$ *whenever* $a,b,c\in H(J)$.

Proof. By Proposition 1·3 and ((1), lemma 38·2)

$$i[R_a,i[R_b,R_c]] = -[R_a,[R_b,R_c]]\in H(BL(J)).$$

LEMMA 2·2. *Let* $a\in H(J)$ *be such that* $a^2 = b+ic$ *where* $b,\,c\in H(J)$. *Then* $[R_b,R_c]=0$.

Proof.

$$(a)\quad 0 = [R_a,R_{a^2}] = [R_a,R_b]+i[R_a,R_c].$$

By Proposition 1·3 and ((1), lemma 38.2)

$$i[R_a,R_b],\quad i[R_a,R_c]\in H(BL(J))$$

Since $0\in H(BL(J))$, we have $[R_a,R_b]\in H(BL(J))$. Hence

$$[R_a,R_b] = [R_a,R_c] = 0. \tag{1}$$

(b) We have, by ((2), section 7, chapter I; identity (54)); and since $[R_a,R_b]$ is a derivation in J ((2), section 7, chapter I),

$$[R_{a^2},[R_b,R_c]] = R_{[R_c,R_b]a^2} = R_{2a[R_c,R_b]a}.$$

It follows from (1) that

$$[R_c,R_b]a = 0.$$

Hence

$$0 = [R_{a^2},[R_b,R_c]] = [R_b,[R_b,R_c]]+i[R_c,[R_b,R_c]].$$

As in (a) we conclude now

$$[R_b,[R_b,R_c]] = [R_c,[R_b,R_c]] = 0.$$

The rest follows from Kleinecke's theorem (3) and Sinclair's theorem ((1), theorem 10·17).

THEOREM 2·3. *Let* $a\in H(J)$ *be such that* $a^2 = b+ic$ *where* $b,c\in H(J)$. *Then* $V(J,a^2)\subseteq\mathbb{R}^+$ *and so* $a^2\in H(J)$.

Proof.

$$R_{a^2} = R_b+iR_c$$

Hence by Lemma 2·2 and ((1), lemma 38.4):

$$V(BL(J),R_{a^2}) = c\circ\mathrm{Sp}\,(BL(J),R_{a^2}).$$

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Since a is hermitian, by Theorems 1·1, 12, 1·4:

$$\mathrm{Sp}\,(BL(J), R_{a^2}) \subseteq \tfrac{1}{2}(\mathrm{Sp}\,(J,a)^2 + \mathrm{Sp}\,(J,a)^2) \subseteq \mathbb{R}^+.$$

Hence by definition of $c \circ \mathrm{Sp}\,(BL(J), R_{a^2})$ and by Proposition 1·3

$$V(J,a^2) = V(BL(J), R_{a^2}) \subseteq \mathbb{R}^+.$$

DEFINITION 2·4. *A JV-algebra is a complex unital complete normed Jordan algebra J such that*

$$J = H(J) + iH(J).$$

THEOREM 2·5. *Let J be a JV-algebra. Then the mapping $a+ib \to a-ib$ $(a,b \in H(J))$ is an algebra involution on J which converts it into a Jordan C^*-algebra.*

Proof. The sum $H(J)+iH(J)$ is direct since

$$b+ic = 0 \Rightarrow R_b + iR_c = 0 (R_b, R_c \in H(BL(J))) \Rightarrow b = c = 0.$$

From Theorem 2·3, $H(J)$ is a real subalgebra of J. Therefore our theorem is proved as a consequence of (7), theorem 8.

The main result of this work is a part of the author's Ph.D. thesis (4). The author wishes to express his deepest gratitude to Dr Rodriguez Palacios for suggesting this topic and constant guidance in the preparation of the thesis.

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