J. Inst. Math. Jussieu (2026), **25**(1), 477–513
doi:10.1017/S1474748025101357

COHOMOLOGICAL INTEGRALITY FOR WEAKLY SYMMETRIC REPRESENTATIONS OF REDUCTIVE GROUPS

LUCIEN HENNECART

LAMFA, CNRS UMR 7352, CNRS Université de Picardie Jules Verne, Amiens, France
(lucien.hennecart@u-picardie.fr)

(Received 14 November 2024; revised 12 September 2025; accepted 15 September 2025; first published online 3 December 2025)

Abstract In this paper, we prove the integrality conjecture for quotient stacks arising from weakly symmetric representations of reductive groups. Our main result is a decomposition of the cohomology of the stack into finite-dimensional components indexed by some equivalence classes of cocharacters of a maximal torus. This decomposition enables the definition of new enumerative invariants associated with the stack, which we begin to explore.

Contents

1 Introduction 478

2 Parabolic induction for representations of reductive groups 482

3 Action of tautological classes on the cohomology of the stack 490

4 Cohomological integrality for weakly symmetric representations of reductive groups 492

5 Vanishing bounds for the BPS spaces 503

6 Examples 504

Competing interests 511

References 511

Keywords: reductive groups; cohomological integrality; intersection cohomology

2020 Mathematics Subject Classification: Primary 14L30; 14L24; 13A50

© The Author(s), 2025. Published by Cambridge University Press.

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

1. Introduction

1.1. Invariant theory

Given a connected reductive group G and a finite-dimensional representation V of G, the quotient stack $\mathfrak{X} = V/G$ and the Geometric Invariant Theory (GIT) quotient $\mathcal{X} = V/\!/G$ are fundamental objects in invariant theory [MFK94; DC70; PV94], with significant applications to geometry and representation theory, e.g. [Nak94; Rei08; MO19] for connections with Lie theory.

Recent research has continued to explore the geometry of the quotient V/G. For instance, in [ŠV17; ŠV21], the authors study semiorthogonal decompositions of the coherent derived category $\mathcal{D}(V/G)$ to obtain noncommutative resolutions of $V/\!/G$ [Van04; Lun10; KL15], using the framework of window categories developed by Halpern-Leistner [Hal15]. Their approach focuses on *quasi-symmetric* representations of reductive groups, which are the representations for which the sum of weights on any line through the origin in the character lattice is zero.

In this paper, we consider the case where V is a *weakly symmetric* representation of G (Definition 2.3). This encompasses the case of self-dual representations. We prove a decomposition of the (infinite-dimensional) cohomology vector space $\mathrm{H}^*(V/G)$ into a finite collection of finite-dimensional, cohomologically graded vector spaces $\mathcal{P}_\lambda$ indexed by cocharacters $\lambda \in X_*(T)$ of a maximal torus T of G (Theorem 1.1). This decomposition, which employs parabolic induction (§2), is known as *cohomological integrality* and can be seen as a cohomological version of a generalised Springer decomposition [Lus84] in this context. We then define the refined Donaldson–Thomas invariants of V/G as the Betti numbers of these vector spaces $\mathcal{P}_\lambda$ (Definition 1.4) following [MR19; Mei15; DM20a] and initiate their study (Proposition 1.5). These invariants are expected to encode rich representation-theoretic information about the action of G on V and topological information about the GIT quotient $V/\!/G$. In particular, we expect that the vector space $\mathcal{P}_\lambda$ computes the intersection cohomology of some GIT quotient, thereby providing an efficient algorithmic and inductive way to determine the intersection Betti numbers of a large class of GIT quotients. We study this question in [Hen24] and in forthcoming work.

Stacks of the form V/G and their moduli spaces $V/\!/G$ serve as local models for smooth stacks $\mathfrak{X}$ with good moduli spaces $\mathcal{X}$ by the Luna étale slice theorem [AHR20]. Therefore, our results provide a foundation for exploring the geometry and topology of morphisms $\mathfrak{X} \to \mathcal{X}$ from a weakly symmetric smooth stack to its good moduli space [Alp13]. While this paper develops the core algebraic results, the applications to the geometry of a class of significant stacks in algebraic geometry and representation theory are investigated in [Hen24].

1.2. Cohomological integrality

Cohomological integrality results, originally developed by Kontsevich and Soibelman in the context of cohomological Hall algebras and wall-crossing for moduli spaces [KS08; KS11], involve factorisations of generating series into plethystic exponentials. For *symmetric* quivers without potential, Efimov [Efi12] proved a cohomological integrality isomorphism, answering a conjecture by Kontsevich and Soibelman [KS11], which

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

identifies the cohomological Hall algebra (CoHA) to a supercommutative algebra over a graded vector space with finite-dimensional graded components. Efimov's isomorphism can be seen as a categorification of integrality formulas, in the sense that an equality between two generating series is turned into an isomorphism between two graded vector spaces whose graded characters recover the series.

Efimov's integrality theorem forms the basis of all subsequent cohomological integrality results for CoHAs [MR19; DM20a; KV22; DHM22; DHM23]. These results provide deep structural insights and link the topology of the stack to the good moduli space. In this context, cohomological integrality is often expressed through Poincaré-Birkhoff-Witt (PBW) theorems, analogous to those in the theory of Lie algebras, and leads to refined cohomological invariants of interest in algebraic geometry, forming the basis of *cohomological Donaldson–Thomas theory* [Sze14; DM20a; Dav23].

In contrast to the traditional approach involving cohomological Hall algebras, our work deals directly with parabolic induction morphisms (§2) for more general smooth weakly symmetric stacks arising from weakly symmetric representations of reductive groups. We define the cohomological integrality morphism explicitly in Theorem 1.1. It does not involve the symmetric power of a vector space as in the quiver case, but some combinatorial replacement of it for any Weyl group, defined from the lattice of cocharacters of a maximal torus.

This paper can be viewed as a generalization of Efimov's integrality isomorphism [Efi12] to all weakly symmetric stacks of the form V/G, not just those arising from quivers. It lays the groundwork for extending cohomological Donaldson–Thomas theory to more general stacks and moduli spaces beyond those parameterising objects in an Abelian category. The cohomological integrality we present here is a purely algebraic result, with a purely algebraic proof. We explain in [Hen24] the geometric implications of this result.

There are also motivic versions of integrality results, such as those studied in [Bu24] following [KS08] and in the context of motivic Donaldson–Thomas theory [KS10; Mei17; BJM19], which are different from ours.

1.3. Main results

In this paper, we work over an algebraically closed field of characteristic zero. Moreover, cohomology is considered with $\mathbf{Q}$-coefficients.

1.3.1. Parabolic induction. We refer to §2.1 for the precise definitions. Let G be a connected reductive group and V a representation of G. We let $T \subseteq G$ be a maximal torus. For any cocharacter $\lambda\colon \mathbf{G}_{\mathrm{m}} \to T$, we let $G^{\lambda} \subseteq G$ be the corresponding Levi subgroup of G [DM20b, Proposition 1.22] and V^{λ} the representation of G^{λ} given by the λ-fixed subspace of V. If $\lambda = 0$ is the trivial cocharacter, we write $V = V^0$ and $G = G^0$.

There is an induction map

$$\mathrm{Ind}_{\lambda}\colon \mathrm{H}^{*+d_{\lambda}}(V^{\lambda}/G^{\lambda}) \to \mathrm{H}^{*+d_0}(V/G), \tag{1.1}$$

where $d_{\lambda} = \dim V^{\lambda} - \dim G^{\lambda}$ (§2.2). If V is symmetric (i.e. V and V^* have the same sets of weights, Definition 2.3), then the induction map preserves shifted cohomological

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

degrees (Lemma 2.7), that is $\mathrm{Ind}_\lambda(\mathrm{H}^{i+d_\lambda}(V^\lambda/G^\lambda)) \subseteq \mathrm{H}^{i+d_0}(V/G)$ for any $i \in \mathbf{Z}$ (this is the reason for the notation H^{*+d_λ} in (1.1), which emphasises that the natural grading is the cohomological grading shifted by d_λ).

We define an order on the set $X_*(T)$ of cocharacters of T as follows:

$$\lambda \preceq \mu \overset{\text{def.}}{\iff} \begin{cases} V^\lambda \subseteq V^\mu \\ \mathfrak{g}^\lambda \subseteq \mathfrak{g}^\mu. \end{cases} \tag{1.2}$$

We let $\mathscr{P}_V := X_*(T)/\sim$, where $\sim$ is the equivalence relation given by $\lambda \sim \mu \iff (\lambda \preceq \mu$ and $\mu \preceq \lambda)$. In other words, two cocharacters are equivalent if their fixed-point sets inside V and $\mathfrak{g}$ (or equivalently inside V and G) both coincide. The set $\mathscr{P}_V$ is finite. If V is symmetric, the induction map Ind_λ only depends on the class $\overline{\lambda} \in \mathscr{P}_V$ up to a non-zero scalar (Lemma 2.16).

We let $W = N_G(T)/T$ be the Weyl group of G. The natural action of W on $X_*(T)$ descends to a W-action on $\mathscr{P}_V$ (§2.2). The image of the induction map Ind_λ only depends on the class $\widetilde{\lambda} \in \mathscr{P}_V/W$ of λ (Lemma 2.17).

1.3.2. Cohomological integrality. Let G be a reductive group with maximal torus T and V be a finite-dimensional *weakly symmetric* representation of G (Definition 2.3). For any $\overline{\lambda} \in \mathscr{P}_V$, we let $\lambda \in X_*(T)$ be an arbitrary lift. For $\widetilde{\lambda} \in \mathscr{P}_V/W$, we let $\lambda \in X_*(T)$ be an arbitrary lift.

For $\lambda \in X_*(T)$, we let $W_\lambda = \{w \in W \mid V^{w\cdot\lambda} = V^\lambda \text{ and } \mathfrak{g}^{w\cdot\lambda} = \mathfrak{g}^\lambda\}$. This is a subgroup of W (Lemma 4.2).

We let $G_\lambda \subseteq G$ be the intersection of the kernel of the action map $G^\lambda \to \mathrm{GL}(V^\lambda)$ with the center of G^λ and $\mathfrak{g}_\lambda := \mathrm{Lie}(G_\lambda)$. Note that the group G_λ is not necessarily connected, but this can essentially be ignored (Lemma 4.1) because we consider cohomology with $\mathbf{Q}$-coefficients.

For $\alpha \in X^*(T)$, we let $V_\alpha := \{v \in V \mid \forall t \in T, t\cdot v = \alpha(t)v\}$ be the α-weight space of V. We define $\mathfrak{g}_\alpha$ similarly.

We let

$$k_\lambda := \frac{\prod_{\alpha \in X^*(T), \langle \lambda, \alpha\rangle < 0} \alpha^{\dim V_\alpha}}{\prod_{\alpha \in X^*(T), \langle \lambda, \alpha\rangle < 0} \alpha^{\dim \mathfrak{g}_\alpha}}.$$

We may call k_λ the *induction kernel*, see the shuffle-like formula in Proposition 2.11. We define $\varepsilon_{V,\lambda} \colon W_\lambda \to \mathbf{C}^*$ to be the character such that for any $w \in W_\lambda$, $w(k_\lambda) = \varepsilon_{V,\lambda}(w)^{-1}k_\lambda$ (Proposition 4.6).

Theorem 1.1 (Proposition 4.16 + Corollary 4.24). *There exist W_λ-stable cohomologically graded vector spaces $\mathcal{P}_\lambda \subseteq \mathrm{H}^{*+d_\lambda}(V^\lambda/G^\lambda)$, $\lambda \in X_*(T)$, such that the map*

$$\bigoplus_{\widetilde{\lambda} \in \mathscr{P}_V/W} (\mathcal{P}_\lambda \otimes \mathrm{H}^*(\mathrm{pt}/G_\lambda))^{\varepsilon_{V,\lambda}} \xrightarrow{\sum_{\widetilde{\lambda} \in \mathscr{P}_V/W} \mathrm{Ind}_\lambda} \mathrm{H}^*(V/G)$$

induced by the induction morphisms (1.1) is a graded isomorphism, where $\mathrm{H}^*(\mathrm{pt}/G_\lambda) \cong \mathrm{Sym}(\mathfrak{g}_\lambda^*)$, $\mathfrak{g}_\lambda^*$ sits in degree 2 and $(\mathcal{P}_\lambda \otimes \mathrm{H}^*(\mathrm{pt}/G_\lambda))^{\varepsilon_{V,\lambda}}$ denotes the $\varepsilon_{V,\lambda}$-isotypic

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

component for the natural W_λ-action. Moreover, the vector spaces $\mathcal{P}_\lambda$ are finite-dimensional and the inclusion $\mathcal{P}_\lambda \subseteq \mathrm{H}^{*+d_\lambda}(V^\lambda/G^\lambda)$ factors through the inclusion $\mathrm{H}^{*+d_\lambda}(V^\lambda/(G^\lambda/G_\lambda)) \to \mathrm{H}^{*+d_\lambda}(V^\lambda/G^\lambda)$. In addition, the vector spaces $\mathcal{P}_\mu$ for $\mu \preceq \lambda$ satisfy the cohomological integrality isomorphism for (V^λ, G^λ).

This type of theorems is sometimes referred to as *cohomological integrality*. We can reconstruct the infinite dimensional vector space $\mathrm{H}^*(V/G)$ using a finite number of finite-dimensional vector spaces $\mathcal{P}_\lambda$, $\widetilde{\lambda} \in \mathscr{P}_V/W$, allowing us to extract fundamental enumerative invariants associated to the stack V/G (Definition 1.4).

For a cocharacter $\lambda \in X_*(T)$, there is a natural W-equivariant map $\mathscr{P}_{V^\lambda} \to \mathscr{P}_V$, which can be defined as follows. Given a cocharacter $\mu \in X_*(T)$, there exists a cocharacter $\nu \in X_*(T)$ such that $V^\lambda \cap V^\mu = V^\nu$ and $G^\lambda \cap G^\mu = G^\nu$ (Lemma 2.8). We have $\nu \preceq \lambda$ and moreover, $\overline{\nu} = \overline{\mu}$ in $\mathscr{P}_{V^\lambda}$. The map sends $\overline{\mu} = \overline{\nu} \in \mathscr{P}_{V^\lambda}$ to $\overline{\nu} \in \mathscr{P}_V$. Then, the last sentence of Theorem 1.1 means that the vector spaces $\mathcal{P}_\nu$ satisfy the cohomological integrality isomorphism for (V^λ, G^λ).

Remark 1.2. Let $Q = (Q_0, Q_1)$ be a symmetric quiver with set of vertices Q_0 and set of arrows Q_1. Let $\mathbf{d} \in \mathbf{N}^{Q_0}$ be a dimension vector. If $V_\mathbf{d}$ is the representation space of $\mathbf{d}$-dimensional representations of Q which is acted on by $\mathrm{GL}_\mathbf{d} := \prod_{i \in Q_0} \mathrm{GL}_{\mathbf{d}_i}$, then the isomorphism given by Theorem 1.1 essentially recovers [Efi12, Theorem 1.1]. In *loc.cit*, it was convenient for the author to twist the cohomological Hall algebra multiplication to make it supercommutative, as in [KS11]. This twist does not change the images of the induction/multiplication morphisms which means that it is possible to have cohomological integrality isomorphisms even without this twist. This remark inspired the present work, in which the twist is replaced by the character $\varepsilon_{V,\lambda}$ of the group W_λ (Proposition 4.6). When V is *symmetric* (and not only *weakly symmetric*), the character $\varepsilon_{V,\lambda}$ takes values in $\{\pm 1\}$.

Remark 1.3. In this paper, we consider cohomology with $\mathbf{Q}$-coefficients, which is crucial for the cohomological integrality map (Theorem 1.1) to be an isomorphism. It is crucial to work with $\mathbf{Q}$-coefficients from §2.5 until the end of the paper. However, one can define the induction morphisms (§2) for any ring of coefficients. It then becomes an interesting question to study the analogous question of cohomological integrality in positive characteristic, especially when the cohomology of the classifying space $\mathrm{H}^*(\mathrm{pt}/G)$ has torsion.

Definition 1.4. We define the *cohomologically refined Donaldson–Thomas invariants* of V/G as the dimensions $p_{\lambda,i} := \dim \mathcal{P}_\lambda^i \subseteq \mathrm{H}^{i+d_\lambda}(V^\lambda/G^\lambda)$ of the cohomological degree i piece of the vector space $\mathcal{P}_\lambda$ for $\lambda \in X_*(T)$. We may call the Euler characteristics $p_\lambda := \sum_{i \in \mathbf{Z}} (-1)^i p_{\lambda,i}$ of $\mathcal{P}_\lambda$ the *Donaldson–Thomas invariants of V/G*.

1.3.3. Vanishing bound. The vector spaces $\mathcal{P}_\lambda$ given by Theorem 1.1 are finite dimensional (Proposition 4.16), and we can give a rough bound on the cohomological degrees for which they possibly do not vanish. We let $\mathcal{H}_\lambda := \mathrm{H}^{*+d_\lambda}(V^\lambda/G^\lambda)$, where $d_\lambda = \dim V^\lambda - \dim G^\lambda$. The natural grading on $\mathcal{H}_\lambda$ is given by $\mathcal{H}_\lambda^k = \mathrm{H}^{k+d_\lambda}(V^\lambda/G^\lambda)$. It induces a cohomological grading on $\mathcal{P}_\lambda$.

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

Proposition 1.5. *The cohomologically graded vector space $\mathcal{P}_\lambda$ given by Theorem 1.1 are concentrated in cohomological degrees $[\dim G^\lambda - \dim V^\lambda, \dim G^\lambda]$, that is $\mathcal{P}_\lambda^k = 0$ unless $k \in [\dim G^\lambda - \dim V^\lambda, \dim G^\lambda]$.*

1.4. Notations and conventions

- If V is a representation of a finite group W and χ an irreducible character of W, we let V^χ be the χ-isotypic component of V.
- If X is a complex algebraic variety acted upon by an algebraic group G, we let $X/G = [X/G]$ be the quotient stack; that is, we omit the brackets.
- The multiplicative group is denoted by $\mathbf{G}_\mathrm{m}$. The Abelian group of characters of an algebraic torus T is $\mathrm{Hom}_\mathbf{Z}(T, \mathbf{G}_\mathrm{m})$. The set of cocharacters of T is the Abelian group $\mathrm{Hom}_\mathbf{Z}(\mathbf{G}_\mathrm{m}, T)$. We have natural embedding $X_*(T) \subseteq \mathfrak{t} \coloneqq \mathrm{Lie}(T)$ and $X^*(T) \subseteq \mathfrak{t}^*$ respectively given by $\lambda \mapsto \frac{\mathrm{d}\lambda}{\mathrm{d}t}_{|t=1}$ and $\alpha \mapsto \mathrm{d}\alpha(1)$, the differential of a character α at the neutral element $1 \in T$. Moreover, these inclusions are compatible with the natural pairings $\langle -, - \rangle \colon X_*(T) \times X^*(T) \to \mathbf{Z}$ and $\mathfrak{t} \times \mathfrak{t}^* \to \mathbf{C}$.
- If A is an Abelian group and $S \subseteq A$ is a subset of A, we let $-S \coloneqq \{-s \colon s \in A\}$ be the opposite of S. We will only consider the case $A = X^*(T)$.
- Let $H \subseteq G$ be algebraic groups and X an H-variety. We let $X \times^H G \coloneqq (X \times G)/H$ where H acts on $X \times G$ by $h \cdot (x, g) = (h \cdot x, gh^{-1})$. The formula $g \cdot (x, g') \coloneqq (x, gg')$ gives a G-action on $X \times^H G$.
- If R is a domain (e.g. $R = \mathrm{H}_T^*(\mathrm{pt})$ for some algebraic torus T), we denote by $\mathrm{Frac}(R)$ its fraction field.
- If G is a possibly disconnected algebraic group, then we denote by G° its neutral component.
- Given a scheme X, we denote by $\mathcal{O}_X$ its sheaf of regular functions. If X is a complex scheme, then $\mathbf{C}[X] \coloneqq \mathcal{O}_X(X)$ set ring of regular functions on X.
- If S is a set (for example, a vector space) with the action of a group G, we denote by S^G the fixed subset.

2. Parabolic induction for representations of reductive groups

2.1. The induction diagram

Let G be a reductive group, $T \subseteq G$ a maximal torus of G and V a finite-dimensional representation of G. For a cocharacter $\lambda \colon \mathbf{G}_\mathrm{m} \to T$, we let $G^\lambda \coloneqq \{g \in G \colon \lambda(t) g \lambda(t)^{-1} = g\}$ be the centraliser of λ, a Levi subgroup of G [DM20b, Proposition 1.22] and $V^\lambda \coloneqq \{v \in V \mid \lambda(t) \cdot v = v\}$ be the fixed locus, a representation of G^λ. We also let $P_\lambda = G^{\lambda \geqslant 0} \coloneqq \{g \in G \mid \lim\limits_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists}\}$, a parabolic subgroup of G and $V^{\lambda \geqslant 0} \coloneqq \{v \in V \mid \lim\limits_{t \to 0} \lambda(t) \cdot v \text{ exists}\}$, a representation of P_λ.

A weight of V is a character $\alpha \colon T \to \mathbf{G}_\mathrm{m}$ such that there exists $v \in V \setminus \{0\}$ such that for any $t \in T$, $t \cdot v = \alpha(t) v$.

We let $\mathcal{W}'(V) \subseteq X^*(T)$ be the set of weights of the representation V. We have a direct sum decomposition

$$V \cong \bigoplus_{\alpha \in \mathcal{W}'(V)} V_\alpha$$

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

where V_α consists of 0 and nonzero vectors of weight α, where $V_\alpha \coloneqq \{v \in V \mid t \cdot v = \alpha(t)v \text{ for any } t \in T\}$.

We let $\mathcal{W}(V)$ be the collection of weights of V, counted with multiplicities: $\alpha \in \mathcal{W}'(V)$ appears $\dim V_\alpha$ times in $\mathcal{W}(V)$.

In particular, we have $\mathcal{W}'(\mathfrak{g})$ and $\mathcal{W}(\mathfrak{g})$ where $\mathfrak{g}$ is seen as the adjoint representation of G. We have the natural pairing

$$\langle -,- \rangle \colon X_*(T) \times X^*(T) \to \mathbf{Z}$$

between characters and cocharacters. If $(\lambda,\alpha) \in X_*(T) \times X^*(T)$, then $\alpha \circ \lambda \in \mathrm{End}(\mathbf{G}_\mathrm{m})$, and so it is given by raising to some power, which is $\langle \lambda,\alpha \rangle \in \mathbf{Z}$. We let $\mathcal{W}^{\lambda>0}(V) \coloneqq \{\alpha \in \mathcal{W}(V) \mid \langle \lambda,\alpha \rangle > 0\}$ and we define similarly $\mathcal{W}^{\lambda=0}(V)$, $\mathcal{W}^{\lambda\geqslant 0}(V)$, $\mathcal{W}^{\lambda<0}(V)$ and $\mathcal{W}^{\lambda\leqslant 0}(V)$ and their $'$ versions, by forgetting the multiplicities.

It is immediate that $V^{\lambda\geqslant 0} = \bigoplus_{\alpha \in \mathcal{W}'^{\lambda\geqslant 0}(V)} V_\alpha$.

We consider the (commutative) induction diagram

$$\begin{array}{ccccc} & & V^{\lambda\geqslant 0}/P_\lambda & & \\ & \overset{q_\lambda}{\swarrow} & & \overset{p_\lambda}{\searrow} & \\ V^\lambda/G^\lambda & & & & V/G \\ \downarrow{\scriptstyle \pi_\lambda} & & & & \downarrow{\scriptstyle \pi} \\ V^\lambda/\!/G^\lambda & & \xrightarrow{\qquad\qquad \imath_\lambda \qquad\qquad} & & V/\!/G \end{array} \tag{2.1}$$

where $V/\!/G \coloneqq \mathrm{Spec}(\mathbf{C}[V]^G)$ is the affine GIT quotient of V by G and the map $\imath_\lambda$ is induced by the equivariant closed immersion $(V^\lambda,G^\lambda) \to (V,G)$.

Lemma 2.1. *The map q_λ is a vector bundle stack of rank $r_\lambda \coloneqq \#\mathcal{W}^{\lambda>0}(V) - \#\mathcal{W}^{\lambda>0}(\mathfrak{g})$ and the map p_λ is a proper and representable morphism of stacks.*

Proof. The map p_λ can be presented as the map of quotient stacks $(V^{\lambda\geqslant 0} \times^{P_\lambda} G)/G \to V/G$ coming from the G-equivariant map

$$\widetilde{p}_\lambda \colon \widetilde{V}^{\lambda\geqslant 0} \coloneqq V^{\lambda\geqslant 0} \times^{P_\lambda} G \to V, \quad (v,g) \mapsto g \cdot v,$$

which can be factored as a closed immersion followed by a projective map

$$\begin{array}{ccccc} V^{\lambda\geqslant 0} \times^{P_\lambda} G & \to & V \times G/P_\lambda & \to & V \\ (v,g) & \mapsto & (g \cdot v, gP_\lambda) & & \\ & & (v,gP_\lambda) & \mapsto & v \end{array}$$

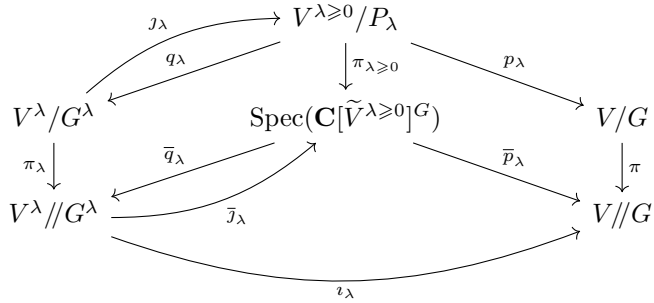
and is therefore projective. Therefore, p_λ is representable and projective.

The fact that q_λ is smooth, surjective of relative dimension $\#\mathcal{W}^{\lambda>0}(V) - \mathcal{W}^{\lambda>0}(\mathfrak{g})$ is easily seen. That it is in addition a vector bundle stack is a rather classical fact (which is not used in this paper). It comes from the facts that $V^{\lambda\geqslant 0} \to V^\lambda$ is a vector bundle (by the Białinicky-Birula decomposition [Bia73], but this can be seen directly in the case of linear actions) and the kernel of the map $P_\lambda \to G^\lambda$ is the unipotent radical of P_λ. □

Lemma 2.2. *The map $\imath_\lambda$ is a finite map.*

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

Proof. The arguments are parallel to that of the proof [MR19, Lemma 2.1]. Namely, we consider the commutative diagram



where $\widetilde{V}^{\lambda\geqslant 0} \coloneqq V^{\lambda\geqslant 0}\times^{P_\lambda} G$ (see §1.4), so that we have an equivalence of stacks $V^{\lambda\geqslant 0}/P_\lambda \simeq \widetilde{V}^{\lambda\geqslant 0}/G$. We now explain the definitions of the maps in this diagram.

The map q_λ is given by the equivariant map $(V^{\lambda\geqslant 0},P_\lambda)\to(V^\lambda,G^\lambda)$, $(v,g)\mapsto(\lim\limits_{t\to 0}\lambda(t)\cdot v, \lim\limits_{t\to 0}\lambda(t)g\lambda(t)^{-1})$. Therefore, the pullback of functions by q_λ induces a morphism between rings of invariant functions

$$\overline{q}_\lambda^*\colon \mathbf{C}[V^\lambda]^{G^\lambda}\to\mathbf{C}[\widetilde{V}^{\lambda\geqslant 0}]^G$$

whose dual is the morphism between affine schemes

$$\overline{q}_\lambda\colon \mathrm{Spec}(\mathbf{C}[\widetilde{V}^{\lambda\geqslant 0}]^G)\to\mathrm{Spec}(\mathbf{C}[V^\lambda]^{G^\lambda}) = V^\lambda/\!\!/G^\lambda\,.$$

The map $\jmath_\lambda$ is given by the equivariant morphism $(V^\lambda,G^\lambda)\to(V^{\lambda\geqslant 0},P_\lambda)$, $(v,g)\mapsto(v,g)$ which induces the morphism of algebras

$$\overline{\jmath}_\lambda^*\colon \mathbf{C}[\widetilde{V}^{\lambda\geqslant 0}]^G\to\mathbf{C}[V^\lambda]^{G^\lambda}$$

and dually the morphism of schemes

$$\overline{\jmath}_\lambda\colon \mathrm{Spec}(\mathbf{C}[V^\lambda]^{G^\lambda})\to\mathrm{Spec}(\mathbf{C}[\widetilde{V}^{\lambda\geqslant 0}]^G)\,.$$

Since $q_\lambda\circ\jmath_\lambda=\mathrm{id}$, we have $\overline{q}_\lambda\circ\overline{\jmath}_\lambda=\mathrm{id}$. On the ring of functions, we have $\overline{\jmath}_\lambda^*\circ\overline{q}_\lambda^*=\mathrm{id}$ and so $\overline{\jmath}_\lambda^*$ is surjective. This tells us that $\overline{\jmath}_\lambda$ is a closed immersion. Moreover, $\imath_\lambda=\overline{p}_\lambda\circ\overline{\jmath}_\lambda$ and so it suffices to prove that $\overline{p}_\lambda$ is a finite map. Since it is a morphism of finite type complex schemes, it suffices to prove that it is an integral morphism [Sta24, Lemma 01WJ].

Since $\widetilde{p}_\lambda\colon \widetilde{V}^{\lambda\geqslant 0}\to V$ is a projective morphism (Proof of Lemma 2.1), $(\widetilde{p}_\lambda)_*\mathcal{O}_{\widetilde{V}^{\lambda\geqslant 0}}$ is a coherent $\mathcal{O}_V$-module. Therefore, the map $\mathbf{C}[V]\to\mathbf{C}[\widetilde{V}^{\lambda\geqslant 0}]$ is finite and hence integral [Sta24, Lemma 01WJ]. We deduce that the map

$$\overline{p}_\lambda^*\colon \mathbf{C}[V]^G\to\mathbf{C}[\widetilde{V}^{\lambda\geqslant 0}]^G$$

is integral, and hence finite, as follows. Let $a\in\mathbf{C}[\widetilde{V}^{\lambda\geqslant 0}]^G$. It is a solution of a monic polynomial with coefficients in $\mathbf{C}[V]$. By applying the Reynolds operator to the equation, we obtain a monic equation with coefficients in $\mathbf{C}[V]^G$ of which a is a zero. This concludes the proof. □

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

We define the equivalence relation $\equiv$ on $X_*(T)$ by $\alpha \equiv \beta \iff \exists a,b \in \mathbf{Z}_{>0}, \quad a\alpha = b\beta$. The quotient set is denoted by $\overline{X_*(T)} \coloneqq X_*(T)/\equiv$. If $S \subseteq X_*(T)$, we denote by $\overline{S}$ its image in $\overline{X_*(T)}$. Similarly, if S is a collection of characters (with multiplicities allowed), we denote by $\overline{S}$ the corresponding collection of elements of $\overline{X_*(T)}$ (with multiplicities taken into account).

Definition 2.3. We say that a representation V of a reductive group G is

(1) *symmetric* if V and its dual V^* have the same sets of weights counted with multiplicities: $\mathcal{W}(V) = \mathcal{W}(V^*)$.
(2) *weakly symmetric* if $\overline{\mathcal{W}(V)} = \overline{\mathcal{W}(V^*)}$.

Since a representation of a reductive group is determined by its character, a representation V of G is symmetric if and only if it is self-dual, that is $V \cong V^*$ as representations of G.

Example 2.4. The following are symmetric representations:

(1) The trivial representation $\{0\}$ of a reductive group G,
(2) The adjoint representation $\mathfrak{g} = \mathrm{Lie}(G)$ of a reductive group G,
(3) The cotangent representation T^*V of any representation V of a reductive group G,
(4) The representation space of $\mathbf{d} \in \mathbf{C}^{Q_0}$-dimensional representations of a *symmetric* quiver $Q = (Q_0, Q_1)$, for $G = \prod_{i \in Q_0} \mathrm{GL}_{\mathbf{d}_i}$.
(5) Any representations of products of simple groups of types $B_n, C_n, D_n (n \text{ even}), E_7, E_8, F_4, G_2$. Indeed, for such a group G, $w_0 = -1 \in W$ and for any $w \in W$, $\dim V_\alpha = \dim V_{w\cdot\alpha}$ [BB07, Planches I–IX], [Hum90, Exercise and Table 1 page 59].

Example 2.5. The two-dimensional representation of $\mathbf{C}^*$ given by $t \cdot (u,v) = (tu, t^{-2}v)$ is weakly symmetric but not symmetric.

For $\lambda \in X_*(T)$, we define $d_\lambda \coloneqq \#\mathcal{W}^{\lambda=0}(V) - \#\mathcal{W}^{\lambda=0}(\mathfrak{g}) = \dim V^\lambda / G^\lambda$. The quantity r_λ is defined in Lemma 2.1. For later use, we give the following easy lemma.

Lemma 2.6. *Let V be a weakly symmetric representation of G. Then,*

$$d_\lambda + 2r_\lambda = d.$$

Proof. We have

$$\mathcal{W}(V) = \mathcal{W}^{\lambda>0}(V) \sqcup \mathcal{W}^{\lambda=0}(V) \sqcup \mathcal{W}^{\lambda<0}(V)$$

and by weak symmetry (Definition 2.3), $\#\mathcal{W}^{\lambda>0}(V) = \#\mathcal{W}^{\lambda<0}(V)$. Therefore, $\#\mathcal{W}(V) = 2\#\mathcal{W}^{\lambda>0}(V) + \#\mathcal{W}^{\lambda=0}(V)$. These equalities are also valid for $V = \mathfrak{g}$ under the adjoint action of G (which is a symmetric representation, Example 2.4). We obtain the lemma by combining them together. □

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

2.2. Parabolic induction

For $\lambda \in X_*(T)$, we let $\mathbf{Q}^{\mathrm{vir}}_{V^\lambda/G^\lambda} \coloneqq \mathbf{Q}_{V^\lambda/G^\lambda} \otimes \mathscr{L}^{-d_\lambda/2} = \mathcal{IC}(V^\lambda/G^\lambda)$ be the intersection complex monodromic mixed Hodge module of the smooth stack V^λ/G^λ. We refer to [DM20a] for the necessary background regarding monodromic mixed Hodge modules, and in particular regarding the square-root $\mathscr{L}^{1/2}$ of the Tate twist $\mathscr{L}$. The Tate twist $\mathscr{L} \coloneqq \mathrm{H}^*_{\mathrm{c}}(\mathbf{A}^1,\mathbf{Q})$ is a pure weight 0 complex of mixed Hodge structures, concentrated in degree 2. Alternatively, for the purposes of this paper, the reader may prefer to consider instead constructible sheaves. The Tate twist $-\otimes\mathscr{L}$ is then replaced by the shift $[-2]$.

By smoothness of q_λ (Lemma 2.1), we have $q_\lambda^* \cong q_\lambda^! \otimes \mathscr{L}^{r_\lambda}$ and thus we obtain

$$q_\lambda^* \mathbb{D}\mathbf{Q}^{\mathrm{vir}}_{V^\lambda/G^\lambda} \cong \mathbb{D}\mathbf{Q}_{V^{\lambda\geqslant 0}/P_\lambda} \otimes \mathscr{L}^{r_\lambda + d_\lambda/2}$$

which by adjunction provides us with the map

$$\mathbb{D}\mathbf{Q}^{\mathrm{vir}}_{V^\lambda/G^\lambda} \to (q_\lambda)_* \mathbb{D}\mathbf{Q}_{V^{\lambda\geqslant 0}/P_\lambda} \otimes \mathscr{L}^{r_\lambda + d_\lambda/2}. \tag{2.2}$$

Moreover, the map p_λ is proper and representable (Lemma 2.1), and so by dualising the adjunction map

$$\mathbf{Q}_{V/G} \to (p_\lambda)_* \mathbf{Q}_{V^{\lambda\geqslant 0}/P_\lambda},$$

and using $\mathbb{D}(p_\lambda)_* = (p_\lambda)_! \mathbb{D}$ and $(p_\lambda)_* = (p_\lambda)_!$, we obtain

$$(p_\lambda)_* \mathbb{D}\mathbf{Q}_{V^{\lambda\geqslant 0}/P_\lambda} \to \mathbb{D}\mathbf{Q}_{V/G}. \tag{2.3}$$

We eventually obtain the map

$$\mathrm{Ind}_\lambda \colon (\imath_\lambda)_*(\pi_\lambda)_* \mathbb{D}\mathbf{Q}^{\mathrm{vir}}_{V^\lambda/G^\lambda} \to \pi_* \mathbb{D}\mathbf{Q}_{V/G} \otimes \mathscr{L}^{r_\lambda + d_\lambda/2} \tag{2.4}$$

by composing $(\imath_\lambda)_*(\pi_\lambda)_*$ applied to (2.2) with π_* applied to (2.3) twisted by $\mathscr{L}^{r_\lambda + d_\lambda/2}$. We used the commutativity of (2.1) to identify $\pi_*(p_\lambda)_* \mathbb{D}\mathbf{Q}_{V^{\lambda\geqslant 0}/P_\lambda}$ and $(\imath_\lambda)_*(\pi_\lambda)_*(q_\lambda)_* \mathbb{D}\mathbf{Q}_{V^{\lambda\geqslant 0}/P_\lambda}$.

Lemma 2.7. *Let V be a weakly symmetric representation of G. Then, the induction morphism $\mathrm{Ind}_\lambda \colon \mathrm{H}^{*+d_\lambda}(V^\lambda/G^\lambda) \to \mathrm{H}^{*+d}(V/G)$ preserves shifted cohomological degrees, that is $\mathrm{Ind}_\lambda(\mathrm{H}^{i+d_\lambda}(V^\lambda/G^\lambda)) \subseteq \mathrm{H}^{i+d}(V/G)$ for any $i \in \mathbf{Z}$.*

Proof. The induction at the level of cohomology vector spaces is given by applying the derived global section functor to the morphism of complexes (2.4) (the sheafified induction). Moreover, by Lemma 2.6, $d/2 = r_\lambda + d_\lambda/2$. Therefore, in the weakly symmetric case, the right-hand side of (2.4) is precisely $\pi_* \mathbb{D}\mathbf{Q}^{\mathrm{vir}}_{V/G}$. Moreover, $\mathbb{D}\mathbf{Q}^{\mathrm{vir}}_{V/G} \cong \mathbf{Q}^{\mathrm{vir}}_{V/G}$ and $\mathbb{D}\mathbf{Q}^{\mathrm{vir}}_{V^\lambda/G^\lambda} \cong \mathbf{Q}^{\mathrm{vir}}_{V^\lambda/G^\lambda}$ since V/G and V^λ/G^λ are smooth and the chosen shifts/Tate twists (by the dimension) make the considered constant sheaves Verdier self-dual. □

2.3. Augmentation

Let $\lambda \in X_*(T)$. We defined a representation V^λ of G^λ in §2.1. The induction morphisms, which we have defined for (V,G), can also be defined in the exact same way for (V^λ,G^λ). We can see T as a maximal torus of G^λ. If $\mu \in X_*(T)$, we have $(V^\lambda)^\mu = V^\lambda \cap V^\mu$, $(G^\lambda)^\mu = G^\lambda \cap G^\mu$, $(V^\lambda)^{\mu\geqslant 0} = V^\lambda \cap V^{\mu\geqslant 0}$ and $P_{\mu,\lambda} \coloneqq P_\mu \cap G^\lambda$.

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

In this case, the induction diagram (2.1) becomes

$$\begin{array}{ccc} & (V^\lambda)^{\mu\geqslant 0}/P_{\mu,\lambda} & \\ {\scriptstyle q_{\lambda,\mu}}\swarrow & & \searrow{\scriptstyle p_{\lambda,\mu}} \\ (V^\lambda)^\mu/(G^\lambda)^\mu & & V^\lambda/G^\lambda \\ {\scriptstyle \pi_{\lambda,\mu}}\downarrow & & \downarrow{\scriptstyle \pi_\lambda} \\ (V^\lambda)^\mu/\!/(G^\lambda)^\mu & \xrightarrow{\imath_{\lambda,\mu}} & V^\lambda/\!/G^\lambda \end{array}$$

The definition of the induction map associated with this datum following §2.2 is

$$\mathrm{Ind}_{\mu,\lambda}\colon \mathrm{H}^*((V^\lambda)^\mu/(G^\lambda)^\mu)\to \mathrm{H}^*(V^\lambda/G^\lambda)\,.$$

If $\mu\preceq\lambda$ (1.2), then $(V^\lambda)^\mu = V^\mu$ and $(G^\lambda)^\mu = G^\mu$, so that the induction $\mathrm{Ind}_{\mu,\lambda}$ is

$$\mathrm{Ind}_{\mu,\lambda}\colon \mathrm{H}^*(V^\mu/G^\mu)\to \mathrm{H}^*(V^\lambda/G^\lambda)\,.$$

Lemma 2.8. *For any $\lambda,\mu\in X_*(T)$, one can find $\nu\in X_*(T)$ such that $V^\nu = V^\mu\cap V^\lambda$, $G^\nu = G^\mu\cap G^\lambda$, $(V^\lambda)^{\nu\geqslant 0} = (V^\lambda)^{\mu\geqslant 0}$ and $P_{\mu,\lambda} = P_{\nu,\lambda}$.*

Proof. The cocharacters λ,μ define a morphism

$$\lambda\times\mu\colon \mathbf{G}_{\mathrm{m}}^{\times 2}\to T\,.$$

We obtain ν by composing a general one-parameter subgroup $\zeta\colon \mathbf{G}_{\mathrm{m}}\to\mathbf{G}_{\mathrm{m}}^{\times 2}$, $t\mapsto(t^a,t^b)$ with $\lambda\times\mu$. This guarantees the first two equalities of the Lemma. To guarantee the last two, it suffices to take a general one-parameter subgroup with the constraint $b>0$ by replacing ζ by ζ^{-1} if necessary. □

2.4. Associativity

Proposition 2.9. *Let $\lambda,\mu,\nu\in X_*(T)$ be such that $V^\lambda\subseteq V^\mu\subseteq V^\nu$, $G^\lambda\subseteq G^\mu\subseteq G^\nu$ and $(V^\nu)^{\mu\geqslant 0}\subseteq(V^\nu)^{\lambda\geqslant 0}$, $P_{\mu,\nu}\subseteq P_{\lambda,\nu}$. Then,*

$$\mathrm{Ind}_{\lambda,\nu} = \mathrm{Ind}_{\mu,\nu}\circ\mathrm{Ind}_{\lambda,\mu}\,.$$

Proof. The equality classically follows by base-change in the commutative diagram

$$\begin{array}{ccccccc} & & & (V^\nu)^{\lambda\geqslant 0}/P_{\lambda,\nu} & & & \\ & {\scriptstyle q_{\nu,\lambda}}\swarrow & & \cup & & \searrow{\scriptstyle p_{\nu,\lambda}} & \\ & & (V^\mu)^{\lambda\geqslant 0}/P_{\lambda,\mu} & & (V^\nu)^{\mu\geqslant 0}/P_{\mu,\nu} & & \\ & {\scriptstyle q_{\mu,\lambda}}\swarrow & & {\scriptstyle p_{\mu,\lambda}}\searrow\quad {\scriptstyle q_{\nu,\mu}}\swarrow & & {\scriptstyle p_{\nu,\mu}}\searrow & \\ V^\lambda/G^\lambda & & & V^\mu/G^\mu & & & V^\nu/G^\nu\,. \end{array} \tag{2.5}$$

□

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

2.5. Explicit formula for the parabolic induction

If $\mathfrak{t}_{\mathbf{Q}}$ is a rational form of the Lie algebra of T, we have $\mathrm{H}_T^*(\mathrm{pt}) \cong \mathrm{Sym}(\mathfrak{t}^*)$. Any character $\alpha \in X_*(T)$ may be identified with an element of $\mathfrak{t}^*$ and hence $\mathrm{H}_T^*(\mathrm{pt})$ via $\alpha \mapsto \mathrm{d}\alpha(1)$ (see §1.4). By abuse, we will still denote by α the corresponding element of $\mathrm{H}_T^*(\mathrm{pt})$.

For $\lambda \in X_*(T)$, we define the induction kernel

$$k_\lambda := \frac{\prod_{\alpha \in \mathcal{W}^{\lambda<0}(V)} \alpha}{\prod_{\alpha \in \mathcal{W}^{\lambda<0}(\mathfrak{g})} \alpha} \in \mathrm{Frac}(\mathrm{H}^*(\mathrm{pt}/T)).$$

We let $W^\lambda := N_{G^\lambda}(T)/T$ be the Weyl group of G^λ.

Lemma 2.10. *For any $\lambda \in X_*(T)$, k_λ is W^λ-invariant.*

Proof. It suffices to prove that W^λ sends $\mathcal{W}^{\lambda<0}(V) \subseteq \mathcal{W}(V)$ to itself, and similarly for $\mathcal{W}^{\lambda<0}(\mathfrak{g})$. This comes from the invariance of the pairing between characters and cocharacters: for any $w \in W$, $\langle w\cdot\lambda, w\cdot\alpha\rangle = \langle\lambda,\alpha\rangle$ and if $w \in W^\lambda$, $w\cdot\lambda = \lambda$. Therefore, for any $\alpha \in X^*(T)$, $\langle\lambda,\alpha\rangle < 0$ if and only if $\langle\lambda, w\cdot\alpha\rangle < 0$. □

Proposition 2.11. *Let $\lambda\colon \mathbf{G}_\mathrm{m} \to T$ be a cocharacter and $f \in \mathrm{H}^*(V^\lambda/G^\lambda) \cong \mathrm{H}^*(\mathrm{pt}/T)^{W^\lambda}$. We have*

$$\mathrm{Ind}_\lambda(f) = \sum_{w \in W/W^\lambda} w\cdot(fk_\lambda) = \frac{1}{\#W^\lambda}\sum_{w\in W} w\cdot(fk_\lambda).$$

Proof. The second equality follows from the W^λ-invariance of both f (since $f \in \mathrm{H}^*(V^\lambda/G^\lambda) = \mathrm{H}^*(V^\lambda/T)^{W^\lambda}$) and k_λ (Lemma 2.10). The first equality is a computation of Euler classes exactly as in the proof of [KS11, Theorem 2] in the case of quiver representations. □

More generally, let $\lambda,\mu \in X_*(T)$. We let

$$k_{\lambda,\mu} := \frac{\prod_{\alpha \in \mathcal{W}^{\lambda<0}(V^\mu)} \alpha}{\prod_{\alpha \in \mathcal{W}^{\lambda<0}(\mathfrak{g}^\mu)} \alpha} \in \mathrm{Frac}(\mathrm{H}_T^*(\mathrm{pt})).$$

Proposition 2.12 (Augmentation of Proposition 2.11). *For $f \in \mathrm{H}^*((V^\mu)^\lambda/(G^\mu)^\lambda) \cong \mathrm{H}^*(\mathrm{pt}/T)^{W^\lambda\cap W^\mu}$, we have*

$$\mathrm{Ind}_{\lambda,\mu}(f) = \sum_{w \in W^\mu/(W^\lambda\cap W^\mu)} w(fk_{\lambda,\mu}).$$

Proof. This follows immediately from Proposition 2.11 applied to V^μ as a representation of G^μ and the cocharacter λ of T, seen as a maximal torus of G^μ, and the identification of the Weyl group of $(G^\mu)^\lambda = G^\lambda \cap G^\mu$ with $W^\lambda \cap W^\mu$. □

2.6. Parabolic induction for weakly symmetric representations

Proposition 2.13. *Let V be a (weakly) symmetric representation of a reductive group G and $\lambda \in X_*(T)$ a cocharacter of the maximal torus of T. Then, the representation V^λ of G^λ is also (weakly) symmetric.*

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

Proof. It is immediate to check that $\mathcal{W}'(V^\lambda) = \{\alpha \in \mathcal{W}'(V) \mid \langle \lambda, \alpha \rangle = 0\}$ and for any $\alpha \in \mathcal{W}'(V^\lambda)$, we can identify the weight spaces $V^\lambda_\alpha = V_\alpha$. Therefore, the (weak) symmetry of V implies that of V^λ. □

Recall the order relation on $X_*(T)$ defined in the introduction (1.2)

$$\lambda \preceq \mu \overset{\text{def.}}{\iff} \begin{cases} V^\lambda \subseteq V^\mu \\ \mathfrak{g}^\lambda \subseteq \mathfrak{g}^\mu . \end{cases}$$

for any $\lambda, \mu \in X_*(T)$.

We obtain an equivalence relation on the set $X_*(T)$

$$\lambda \sim \mu \overset{\text{def.}}{\iff} \lambda \preceq \mu \text{ and } \mu \preceq \lambda .$$

We let $\mathscr{P}_V \coloneqq X_*(T)/\sim$ be the quotient of $X_*(T)$ by this equivalence relation. If $\lambda \in X_*(T)$, $\overline{\lambda}$ denotes its class in $\mathscr{P}_V$. It has an induced order relation still denoted by $\preceq$:

$$\overline{\lambda} \preceq \overline{\mu} \iff \lambda \preceq \mu .$$

It is immediate that this is well-defined, i.e. the definition does not depend on the representatives λ, μ of $\overline{\lambda}, \overline{\mu}$.

Lemma 2.14. *Let V be a weakly symmetric representation of a reductive group G. If $\lambda, \mu \in X_*(T)$ are such that $\lambda \sim \mu$, then*

(1) $\#\mathcal{W}^{\lambda>0}(V) = \#\mathcal{W}^{\mu>0}(V) = \#\mathcal{W}^{\lambda<0}(V) = \#\mathcal{W}^{\mu<0}(V)$,
(2) $\#\mathcal{W}^{\lambda>0}(\mathfrak{g}) = \#\mathcal{W}^{\mu>0}(\mathfrak{g}) = \#\mathcal{W}^{\lambda<0}(\mathfrak{g}) = \#\mathcal{W}^{\mu<0}(\mathfrak{g})$.

Proof. We prove the statement for V. The statement for $\mathfrak{g}$ is obtained by specialising $V = \mathfrak{g}$, since $\mathfrak{g}$ under the adjoint action of G is a symmetric representation of G (by the existence of invariant pairings, nondegenerate up to the center of $\mathfrak{g}$, Example 2.4).

Since V is weakly symmetric, we have $\#\mathcal{W}^{\lambda>0}(V) = \#\mathcal{W}^{\lambda<0}(V)$ and moreover, $\#\mathcal{W}(V) = \#\mathcal{W}^{\lambda>0}(V) + \#\mathcal{W}^{\lambda<0}(V) + \mathcal{W}(V^\lambda)$. Since we have the same equalities with λ replaced by μ and $\mathcal{W}^{\lambda=0}(V) = \mathcal{W}^{\mu=0}(V)$, we obtain $\#\mathcal{W}^{\lambda>0}(V) = \#\mathcal{W}^{\mu>0}(V)$. The other equalities follow. □

Lemma 2.15. *If $\lambda, \mu \in X_*(T)$ are such that $\lambda \sim \mu$, then $k_\lambda = \xi k_\mu$ for some $\xi \in \mathbf{Q}^*$.*

Proof. If $\lambda \sim \mu$, we have $\mathcal{W}^{\lambda>0}(V) \sqcup \mathcal{W}^{\lambda<0}(V) = \mathcal{W}^{\mu>0}(V) \sqcup \mathcal{W}^{\mu<0}(V)$ and $\mathcal{W}^{\lambda>0}(\mathfrak{g}) \sqcup \mathcal{W}^{\lambda<0}(\mathfrak{g}) = \mathcal{W}^{\mu>0}(\mathfrak{g}) \sqcup \mathcal{W}^{\mu<0}(\mathfrak{g})$ and since V and $\mathfrak{g}$ are weakly symmetric representations of G, $\overline{\mathcal{W}^{\lambda>0}(V)} = -\mathcal{W}^{\lambda<0}(V)$, $\overline{\mathcal{W}^{\mu>0}(V)} = -\overline{\mathcal{W}^{\mu<0}(V)}$ and similarly for $\mathfrak{g}$. Therefore, for any $\alpha \in \mathcal{W}^{\lambda<0}(V)$ (resp. $\mathcal{W}^{\lambda<0}(\mathfrak{g})$), there exists $\xi_\alpha \in \mathbf{Q}^*$ (resp. $\zeta_\alpha \in \mathbf{Q}^*$) such that

$$\begin{array}{ccc} \mathcal{W}^{\lambda<0}(V) & \to & \mathcal{W}^{\mu<0}(V) \\ \alpha & \mapsto & \xi_\alpha \alpha \end{array}$$

and

$$\begin{array}{ccc} \mathcal{W}^{\lambda<0}(\mathfrak{g}) & \to & \mathcal{W}^{\mu<0}(\mathfrak{g}) \\ \alpha & \mapsto & \zeta_\alpha \alpha \end{array}$$

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

are bijections. Then,

$$\frac{\prod_{\alpha\in\mathcal{W}^{\lambda<0}(V)}\xi_\alpha}{\prod_{\alpha\in\mathcal{W}^{\lambda<0}(\mathfrak{g})}\zeta_\alpha}k_\lambda = k_\mu\,.$$

We may therefore set $\xi := \frac{\prod_{\alpha\in\mathcal{W}^{\lambda<0}(\mathfrak{g})}\zeta_\alpha}{\prod_{\alpha\in\mathcal{W}^{\lambda<0}(V)}\xi_\alpha}$. □

Lemma 2.16. *We assume that V is a weakly symmetric representation of G. Then, for any $\lambda,\mu\in X_*(T)$ such that $\overline{\lambda}=\overline{\mu}$, the induction maps Ind_λ and Ind_μ differ by a non-zero scalar. In particular, the images of Ind_λ and Ind_μ inside $\mathrm{H}^{*+d}(V/G)$ coincide.*

Proof. We use the explicit formula for the induction (Proposition 2.11). Let $f\in\mathrm{H}^*(V^\lambda/G^\lambda)=\mathrm{H}^*(V^\mu/G^\mu)$. We have

$$\mathrm{Ind}_\lambda(f)=\sum_{w\in W/W^\lambda} w\cdot(fk_\lambda),\quad \mathrm{Ind}_\mu(f)=\sum_{w\in W/W^\mu} w\cdot(fk_\mu)\,.$$

Moreover, $G^\lambda=G^\mu$, $W^\lambda=W^\mu$ and by Lemma 2.15, $k_\lambda=\xi k_\mu$. This concludes the proof. □

The W-action on $X_*(T)$ induces a W-action on $\mathscr{P}_V$. For $\lambda\in X_*(T)$, we let $\widetilde{\lambda}\in\mathscr{P}_V/W$ be its W-orbit.

Lemma 2.17. *Let $\lambda,\mu\in X_*(T)$ be such that $\widetilde{\lambda}=\widetilde{\mu}$. Then, the images of Ind_λ and Ind_μ coincide.*

Proof. By Lemma 2.16, we may assume that for some $w\in W$, $\mu=w\cdot\lambda$. Then, $\mathcal{W}^{\mu<0}(V)=w\cdot\mathcal{W}^{\lambda<0}(V)$ and $\mathcal{W}^{\mu<0}(\mathfrak{g})=w\cdot\mathcal{W}^{\lambda<0}(\mathfrak{g})$. Therefore, $w(k_\lambda)=k_\mu$. Using the explicit formula for the induction product (Proposition 2.11), we obtain

$$\mathrm{Ind}_\mu(f)=\mathrm{Ind}_\lambda(w^{-1}(f)),$$

and therefore, since w^{-1} induces an isomorphism $\mathrm{H}^*(V^\mu/G^\mu)=\mathrm{H}^*(V^{w\cdot\lambda}/G^{w\cdot\lambda})\cong\mathrm{H}^*(V^\lambda/G^\lambda)$, the images of Ind_λ and Ind_μ coincide. □

3. Action of tautological classes on the cohomology of the stack

Let G be a reductive group and V a representation of G. We let $\pi\colon V/G\to V/\!/G$ be the good moduli space map. We let $\kappa\colon V/\!/G\to\mathrm{pt}$ be the projection to the point.

Then, for any line bundle $\mathcal{L}$ over V/G, we have an operation $\pi_*\mathbf{Q}_{V/G}\to\pi_*\mathbf{Q}_{V/G}[-2]$ of multiplication by the first Chern class $c_1(\mathcal{L})$. More precisely, we have the following lemma.

Lemma 3.1. *There is a morphism of graded rings*

$$\mathrm{H}^*(\mathrm{pt}/G)\xrightarrow{a}\mathrm{RHom}(\pi_*\mathbf{Q}_{V/G},\pi_*\mathbf{Q}_{V/G})$$

such that the composition $\kappa_*\circ a$ where $\kappa_*\colon\mathrm{RHom}(\pi_*\mathbf{Q}_{V/G},\pi_*\mathbf{Q}_{V/G})\to\mathrm{RHom}(\kappa_*\pi_*\mathbf{Q}_{V/G},\kappa_*\pi_*\mathbf{Q}_{V/G})\cong\mathrm{RHom}(\mathrm{H}^*(\mathrm{pt}/G),\mathrm{H}^*(\mathrm{pt}/G))$, sends $a\in\mathrm{H}^*(\mathrm{pt}/G)$ to the endomorphism of $\mathrm{H}^*(\mathrm{pt}/G)$ given by the cup product with a.

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

Proof. We have a canonical adjunction isomorphism

$$\mathrm{RHom}(\pi_*\mathbf{Q}_{V/G},\pi_*\mathbf{Q}_{V/G}) \cong \mathrm{RHom}(\pi^*\pi_*\mathbf{Q}_{V/G},\mathbf{Q}_{V/G})\,.$$

By precomposing with the counit map $\pi^*\pi_*\mathbf{Q}_{V/G} \to \mathbf{Q}_{V/G}$, we obtain the morphism

$$\mathrm{RHom}(\mathbf{Q}_{V/G},\mathbf{Q}_{V/G}) \cong \mathrm{H}^*(V/G) \cong \mathrm{H}^*(\mathrm{pt}/G) \to \mathrm{RHom}(\pi_*\mathbf{Q}_{V/G},\pi_*\mathbf{Q}_{V/G}).$$

The last statement of the theorem follows from the functoriality of adjunctions. □

We now state and prove an elementary lemma regarding Cartan subalgebras of reductive Lie algebras.

Lemma 3.2. *Let $\mathfrak{g}$ be a reductive Lie algebra, $\mathfrak{h} \subseteq \mathfrak{g}$ a Lie ideal and $\mathfrak{t}$ a Cartan subalgebra. Then, $\mathfrak{h} \cap \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{h}$.*

Proof. Let $\pi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ be the projection. Since $\mathfrak{g}$ is reductive, we have a (noncanonical) decomposition $\mathfrak{g} \cong \mathfrak{h} \oplus (\mathfrak{g}/\mathfrak{h})$ as Lie algebras. If $\mathfrak{t}'$ is a Cartan subalgebra of $\mathfrak{h}$ and $\mathfrak{t}''$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{h}$, then $\mathfrak{t}' \oplus \mathfrak{t}''$ is a Cartan subalgebra of $\mathfrak{g}$. Moreover, $\pi(\mathfrak{t}) \cong \mathfrak{t}/\mathfrak{t}\cap\mathfrak{h}$ is an Abelian subalgebra of $\mathfrak{g}/\mathfrak{h}$ and $\mathfrak{t}\cap\mathfrak{h}$ is an Abelian subalgebra of $\mathfrak{h}$. By maximality of $\dim \mathfrak{t}'$, we have

$$\dim \pi(\mathfrak{t}) = \dim \mathfrak{t} - \dim(\mathfrak{t}\cap\mathfrak{h}) \geqslant \dim \mathfrak{t} - \dim \mathfrak{t}' = \dim \mathfrak{t}''\,.$$

By maximality of $\dim \mathfrak{t}''$, we have equality in this chain of inequalities. This means that $\dim(\mathfrak{t}\cap\mathfrak{h}) = \dim \mathfrak{t}'$ and therefore, $\mathfrak{t}\cap\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{h}$. □

We have the group version of Lemma 3.2.

Lemma 3.3. *Let G be a reductive group, $H \subseteq G$ a normal subgroup and $T \subseteq G$ a maximal torus. Then, the neutral component of $T \cap H$ is a maximal torus of H.*

Proof. This follows from Lemma 3.2 by taking the Lie algebras of all groups appearing. □

Example 3.4. In the group situation, it is necessary to consider the neutral component of $T \cap H$ as shown by the elementary case of $H = \{\pm 1\} \subseteq \mathrm{SL}_2(\mathbf{C})$, and where $T \subseteq \mathrm{SL}_2(\mathbf{C})$ is the standard torus of diagonal matrices.

Proposition 3.5. *Let G be a reductive group. For any normal subgroup $H \subseteq G$ acting trivially on V, we have (non-canonical) actions of $\mathrm{H}^*(\mathrm{pt}/H)$ on $\mathrm{H}^*(\mathrm{pt}/G) \cong \mathrm{H}^*(V/G)$ and $\pi_*\mathbf{Q}_{V/G}$.*

Proof. By choosing a decomposition $\mathrm{Lie}(G) \cong \mathrm{Lie}(H) \oplus \mathrm{Lie}(G/H)$, we obtain a decomposition $\mathrm{H}^*(\mathrm{pt}/G) \cong \mathrm{H}^*(\mathrm{pt}/(G/H)) \otimes \mathrm{H}^*(\mathrm{pt}/H)$, giving an action of $\mathrm{H}^*(\mathrm{pt}/H)$ on $\mathrm{H}^*(\mathrm{pt}/G)$ by cup-product on the second summand. At the sheaf level, the action of the lemma is obtained by pre-composing the morphism a of Lemma 3.1 with the inclusion $\mathrm{H}^*(\mathrm{pt}/H) \to \mathrm{H}^*(\mathrm{pt}/G)$. □

Lemma 3.6. *Let G be an algebraic group and $H \subseteq G$ a normal subgroup. Then, we have a natural inclusion $\mathrm{H}^*(\mathrm{pt}/(G/H)) \to \mathrm{H}^*(\mathrm{pt}/G)$.*

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

Proof. The map $\mathrm{H}^*(\mathrm{pt}/(G/H)) \to \mathrm{H}^*(\mathrm{pt}/G)$ is the pullback in cohomology for the morphism of stacks $\mathrm{pt}/G \to \mathrm{pt}/(G/H)$ induced by the surjective morphism of groups $G \to G/H$. □

Proposition 3.7. *Let G be a reductive algebraic group, and V a finite-dimensional representation of G. Let H be a normal subgroup of G acting trivially on V. Then, the morphism*

$$\mathrm{H}^*(\mathrm{pt}/H) \otimes \mathrm{H}^*(V/(G/H)) \to \mathrm{H}^*(V/G)$$

obtained by combining the natural pullback map $\mathrm{H}^*(V/(G/H)) \to \mathrm{H}^*(V/G)$ coming from the morphism of stacks $V/G \to V/(G/H)$ (Lemma 3.6) with the $\mathrm{H}^*(\mathrm{pt}/H)$-action on $\mathrm{H}^*(V/G)$ (Proposition 3.5) is an isomorphism.

Proof. We let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. We let $\mathfrak{t} \subseteq \mathfrak{g}$ be a Cartan subalgebra. We have $\mathfrak{g} \cong \mathfrak{h} \oplus (\mathfrak{g}/\mathfrak{h})$ and therefore a decomposition of the Weyl group $W_{\mathfrak{g}} \cong W_{\mathfrak{h}} \times W_{\mathfrak{g}/\mathfrak{h}}$. We let $\pi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ be the projection. The morphism of the proposition can then be identified with the isomorphism

$$\mathrm{Sym}((\mathfrak{t} \cap \mathfrak{h})^*)^{W_{\mathfrak{h}}} \otimes \mathrm{Sym}(\pi(\mathfrak{t})^*)^{W_{\mathfrak{g}/\mathfrak{h}}} \to \mathrm{Sym}(\mathfrak{t}^*)^W,$$

see also Lemma 3.2. □

4. Cohomological integrality for weakly symmetric representations of reductive groups

We fix a weakly symmetric representation V (Definition 2.3) of a connected reductive group G and a maximal torus $T \subseteq G$.

4.1. Weyl groups

Let $\lambda\colon \mathbf{G}_{\mathrm{m}} \to T$. We let $G^\lambda \subseteq G$ be the corresponding Levi subgroup (§2.1) and $G_\lambda := Z(G^\lambda) \cap \ker(G^\lambda \to \mathrm{GL}(V^\lambda))$ the intersection of the kernel of the action of G^λ on V^λ and the center of G^λ. We let $W^\lambda = N_{G^\lambda}(T)$ be the Weyl group of G^λ and $W_\lambda := \{w \in W \mid V^{w\cdot\lambda} = V^\lambda \text{ and } G^{w\cdot\lambda} = G^\lambda\} = \{w \in W \mid w \cdot \overline{\lambda} = \overline{\lambda}\}$. The group W_λ may be thought of as a kind of relative Weyl group.

Lemma 4.1. *Let G_λ° be the neutral component of G_λ. Let $k = \dim G_\lambda$. Then, $G_\lambda^\circ \cong (\mathbf{C}^*)^k$ for some $k \geqslant 0$ and the natural pullback map $\mathrm{H}^*(\mathrm{pt}/G_\lambda) \to \mathrm{H}^*(\mathrm{pt}/G_\lambda^\circ)$ is an isomorphism.*

Proof. The group G_λ is a subgroup of the center of G^λ, and so it is a diagonalisable algebraic group. Therefore, its neutral component is a torus. We have a splitting $G \cong G_\lambda^\circ \times \pi_0(G_\lambda)$ [Hum12, Theorem p.104] and so, by the Künneth formula,

$$\mathrm{H}^*(\mathrm{pt}/G_\lambda) \cong \mathrm{H}^*(\mathrm{pt}/G_\lambda^\circ) \otimes \mathrm{H}^*(\mathrm{pt}/\pi_0(G_\lambda)).$$

Since $\pi_0(G_\lambda)$ is a finite group, $\mathrm{H}^*(\mathrm{pt}/\pi_0(G_\lambda)) \cong \mathrm{H}^*(\mathrm{pt})^{\pi_0(G_\lambda)} \cong \mathbf{Q}$, which concludes. □

Lemma 4.2. *The group W_λ is a subgroup of W and it contains W^λ as a normal subgroup.*

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

Proof. We let $\dot{w} \in N_G(T)$ be a lift of $w \in W$. Then, $V^{w\cdot\lambda} = \dot{w}V^{\lambda}$. The first statement of the lemma follows then immediately from the fact that if $w, w' \in W$, and $\dot{w}, \dot{w'} \in N_G(T)$ are respective lifts, then, $\dot{w}\dot{w'}$ is a lift of ww' and $\dot{w}^{-1}$ a lift of w^{-1}.

We prove that $W^{\lambda} \subseteq W_{\lambda}$. Let $w \in W^{\lambda}$ and $\dot{w} \in N_{G^{\lambda}}(T)$ be a lift of w. Then, $\dot{w}G^{\lambda}\dot{w}^{-1} = G^{w\cdot\lambda} = G^{\lambda}$ and $\dot{w}V^{\lambda} = V^{w\cdot\lambda} = V^{\lambda}$ and so $w \in W_{\lambda}$. This proves the inclusion.

We now show that W^{λ} is a normal subgroup of W_{λ}. It suffices to check that if $w \in W_{\lambda}$ and $w' \in W^{\lambda}$, and $\dot{w} \in N_G(T)$, $\dot{w'} \in N_{G^{\lambda}}(T)$ are any lifts, then $\dot{w}\dot{w'}\dot{w}^{-1} \in G^{\lambda}$. We have $\dot{w}\dot{w'}\dot{w}^{-1} \in \dot{w}G^{\lambda}\dot{w}^{-1} = G^{w\cdot\lambda} = G^{\lambda}$. This concludes the proof. □

Corollary 4.3. *The group W_{λ} acts on $\mathrm{H}^*(\mathrm{pt}/T)^{W^{\lambda}} \cong \mathrm{H}^*(\mathrm{pt}/G^{\lambda})$.*

Proof. The Weyl group W of G acts on $\mathrm{H}^*(\mathrm{pt}/T)$. We just have to check that the induced action of the subgroup W_{λ} preserves W^{λ}-invariants. This is a consequence of Lemma 4.2. □

Lemma 4.4. *The group W_{λ} acts naturally on $\mathrm{H}^*(\mathrm{pt}/G_{\lambda})$. The subgroup $W^{\lambda} \subseteq W_{\lambda}$ acts trivially on $\mathrm{H}^*(\mathrm{pt}/G_{\lambda})$.*

Proof. By definition, G_{λ} is preserved by W_{λ}, as G_{λ} only depends on $\overline{\lambda} \in \mathscr{P}_V$ and W_{λ} is the stabiliser of $\overline{\lambda}$. The second statement follows from the fact that G_{λ} is contained in the center of G^{λ}. □

We let $\overline{G^{\lambda}} \coloneqq G^{\lambda}/G_{\lambda}$. Since G_{λ} is a normal subgroup of G^{λ}, $\overline{G^{\lambda}}$ is a reductive group. Since G_{λ} is contained in the center of G^{λ}, the Weyl group of $\overline{G^{\lambda}}$ is isomorphic to W^{λ}.

Lemma 4.5. *The group W_{λ} acts naturally on $\mathrm{H}^*(V^{\lambda}/\overline{G^{\lambda}})$. Moreover, the W_{λ}-action on $\mathrm{H}^*(\mathrm{pt}/G^{\lambda}) \cong \mathrm{H}^*(V^{\lambda}/G^{\lambda})$ obtained in Corollary 4.3 and the W_{λ}-action on $\mathrm{H}^*(V^{\lambda}/\overline{G^{\lambda}}) \otimes \mathrm{H}^*(\mathrm{pt}/G_{\lambda})$ obtained by the tensor product of the actions given in Lemmas 4.4 and 4.5 coincide via the isomorphism given in Proposition 3.7.*

Proof. We have $\mathrm{H}^*(V^{\lambda}/\overline{G^{\lambda}}) \cong \mathrm{H}^*(\mathrm{pt}/(T/G_{\lambda}^{\circ}))^{W^{\lambda}}$. The restriction to W_{λ} of the W action on T preserves G_{λ} since G_{λ} is in the center of G^{λ}. Therefore, we obtain a W_{λ}-action on T/G_{λ}°. It induces a W_{λ}-action on $\mathrm{H}^*(\mathrm{pt}/(T/G_{\lambda}^{\circ}))$. The fact that W_{λ} normalises W^{λ} (Lemma 4.2) implies that $\mathrm{H}^*(\mathrm{pt}/(T/G_{\lambda}^{\circ}))^{W^{\lambda}}$ is preserved by the W_{λ}-action.

The tensor product decomposition of the action follows immediately from this and Lemma 4.4. □

Proposition 4.6. *There exists a character $\varepsilon_{V,\lambda} \colon W_{\lambda} \to \mathbf{Q}^*$ such that for any $w \in W_{\lambda}$, $w(k_{\lambda}) = \varepsilon_{V,\lambda}(w)^{-1}k_{\lambda} \in \mathrm{H}^*_T(\mathrm{pt})$.*

Proof. It suffices to prove the existence and unicity of the rational number $\varepsilon_{V,\lambda}(w) \in \mathbf{Q}^*$ satisfying the equality of the proposition, for any $w \in W_{\lambda}$. The multiplicativity then follows from the unicity of this sign.

For $w \in W_{\lambda}$, the cocharacters $\lambda, w^{-1}\cdot\lambda$ of T are such that $\overline{\lambda} = \overline{w^{-1}\cdot\lambda}$. Therefore, by Lemma 2.15, there exists $\xi \in \mathbf{Q}^*$ such that $w(k_{\lambda}) = k_{w^{-1}\cdot\lambda} = \xi k_{\lambda}$. We set $\varepsilon_{V,\lambda}(w) \coloneqq \xi^{-1}$. This proves the existence. The unicity of ξ is straightforward. □

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

Proposition 4.7. *Let $\lambda \in X_*(T)$, $w \in W_\lambda$, and $f \in \mathrm{H}^*(V^\lambda/G^\lambda)$. Then,* $\mathrm{Ind}_\lambda(w \cdot f) = \varepsilon_{V,\lambda}(w)\mathrm{Ind}_\lambda(f)$.

Proof. By Proposition 2.11, we have

$$\begin{aligned}\mathrm{Ind}_\lambda(w \cdot f) &= \frac{1}{\#W^\lambda} \sum_{w' \in W} w'(wf)w'(k_\lambda) \\ &= \frac{1}{\#W^\lambda} \sum_{w'' \in W} (w''f)w''(w^{-1}k_\lambda) \\ &= \frac{1}{\#W^\lambda} \sum_{w'' \in W} (w''f)w''(\varepsilon_{V,\lambda}(w)k_\lambda) \quad \text{by Proposition 4.6}\end{aligned}$$

which concludes. □

We now state an elementary result from the representation theory of finite groups for later use.

Proposition 4.8. *Let G be a finite group $\chi\colon G \to \mathbf{C}^*$ a character of G and V a finite-dimensional representation of G. Then, the formula*

$$p_\chi(v) = \frac{1}{\#G} \sum_{g \in G} \chi(g)^{-1} g \cdot v$$

is the projector onto the χ-isotypic component V^χ of V.

Proof. This is [Ser71, §2.6, Théorème 8]. □

4.2. Proof of the cohomological integrality theorem

We let V be a weakly symmetric representation of a reductive group G. In the rest of this section, we prove Theorem 1.1.

For $\lambda \in X_*(T)$, we define the $\mathbf{Z}$-graded vector space

$$\mathcal{H}_\lambda^* := \mathrm{H}^*(V^\lambda/G^\lambda).$$

When $\lambda = 0$ is the trivial cocharacter, we write $\mathcal{H} := \mathcal{H}_0 = \mathrm{H}^*(V/G)$. More generally, for any pair λ,μ, we define $\mathcal{H}_{\lambda,\mu} := \mathrm{H}^*(V^\lambda \cap V^\mu/G^\lambda \cap G^\mu)$. By §2.3, we have induction maps $\mathrm{Ind}_\lambda\colon \mathcal{H}_\lambda \to \mathcal{H}$ for $\lambda \in X_*(T)$ and more generally, $\mathrm{Ind}_{\mu,\lambda}\colon \mathcal{H}_{\lambda,\mu} \to \mathcal{H}_\lambda$ for $\lambda,\mu \in X_*(T)$.

For $\lambda \in X_*(T)$, we choose a W_λ-invariant splitting $\mathfrak{g}^\lambda = \mathfrak{g}_\lambda \oplus (\mathfrak{g}^\lambda/\mathfrak{g}_\lambda)$, where $\mathfrak{g}_\lambda := \mathrm{Lie}(G_\lambda)$ is the Lie algebra of G_λ (§4.1). It induces a splitting of the maximal torus $\mathfrak{t} = \mathfrak{g}_\lambda \oplus (\mathfrak{t}/\mathfrak{g}_\lambda)$ (Lemma 3.2). We let

$$\mathcal{H}_\lambda^{\mathrm{prim}} := \mathrm{H}^*(V^\lambda/(G^\lambda/G_\lambda^\circ)),$$

so that $\mathcal{H}_\lambda \cong \mathcal{H}_\lambda^{\mathrm{prim}} \otimes \mathrm{H}^*(\mathrm{pt}/G_\lambda)$ (Proposition 3.7). We let

$$\mathcal{A}_\lambda := \mathrm{H}^*(V^\lambda/T) \quad \text{and} \quad \mathcal{A}_\lambda^{\mathrm{prim}} := \mathrm{H}^*(V^\lambda/(T/G_\lambda^\circ))$$

so that $\mathcal{A}_\lambda \cong \mathcal{A}_\lambda^{\mathrm{prim}} \otimes \mathrm{H}^*(\mathrm{pt}/G_\lambda)$, $\mathcal{H}_\lambda = \mathcal{A}_\lambda^{W^\lambda}$ and $\mathcal{H}_\lambda^{\mathrm{prim}} \cong (\mathcal{A}_\lambda^{\mathrm{prim}})^{W^\lambda}$.

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

For each $\lambda \in X_*(T)$, we let J_λ be the smallest W^λ-stable $\mathcal{A}_\lambda^{\mathrm{prim}}$-submodule of the localisation

$$\mathcal{A}_\lambda^{\mathrm{prim,loc}} \coloneqq \mathcal{A}_\lambda^{\mathrm{prim}}[\prod_{\alpha \in \mathcal{W}(\mathfrak{g}^\lambda)\setminus\{0\}} \alpha^{-1}]$$

containing

$$k_{\mu,\lambda} = \frac{\prod_{\alpha \in \mathcal{W}^{\mu<0}(V^\lambda)} \alpha}{\prod_{\alpha \in \mathcal{W}^{\mu<0}(\mathfrak{g}^\lambda)} \alpha}$$

for all $\mu \in X_*(T)$ such that $\lambda \not\preceq \mu$ for the order relation $\preceq$ defined in §2.6.

We let

$$\mathcal{A}_\lambda^{\mathrm{loc}} \coloneqq \mathcal{A}_\lambda[\prod_{\alpha \in \mathcal{W}(\mathfrak{g}^\lambda)\setminus\{0\}} \alpha^{-1}] \cong \mathcal{A}_\lambda^{\mathrm{prim,loc}} \otimes \mathrm{H}^*(\mathrm{pt}/G_\lambda)$$

and

$$\mathcal{H}_\lambda^{\mathrm{loc}} \coloneqq (\mathcal{A}_\lambda^{\mathrm{loc}})^{W^\lambda} \cong \mathcal{H}_\lambda^{\mathrm{prim}}[\prod_{\alpha \in \mathcal{W}(\mathfrak{g}^\lambda)\setminus\{0\}} \alpha^{-1}] \otimes \mathrm{H}^*(\mathrm{pt}/G_\lambda)$$

(for this last isomorphism, we use the fact that W^λ acts trivially on $\mathrm{H}^*(\mathrm{pt}/G_\lambda)$ as G_λ is in the center of G^λ, Lemma 4.4).

Lemma 4.9. *We can characterise J_λ as the smallest W^λ-stable $\mathcal{A}_\lambda^{\mathrm{prim}}$-submodule of $\mathcal{A}^{\mathrm{prim,loc}}$ containing*

$$k_{\mu,\lambda} = \frac{\prod_{\alpha \in \mathcal{W}^{\mu<0}(V^\lambda)} \alpha}{\prod_{\alpha \in \mathcal{W}^{\mu<0}(\mathfrak{g}^\lambda)} \alpha}$$

for all $\mu \in X_*(T)$ such that $\mu \prec \lambda$.

Proof. If $\mu \prec \lambda$, then we have $\dim V^\mu \leqslant \dim V^\lambda$ and $\dim \mathfrak{g}^\mu \leqslant \dim \mathfrak{g}^\lambda$ with at least one strict inequality (where the order relation $\preceq$ is defined in §2.6). If by contradiction we have $\lambda \preceq \mu$, then we have

$$\dim V^\lambda \leqslant \dim V^\mu, \quad \dim \mathfrak{g}^\lambda \leqslant \dim \mathfrak{g}^\mu.$$

Therefore, if $\mu \prec \lambda$, then $\lambda \not\preceq \mu$ and by definition, $k_{\mu,\lambda} \in J_\lambda$.

Conversely, if $\lambda \not\preceq \mu$, we let $\nu \in X_*(T)$ be such that $V^\nu = V^\lambda \cap V^\mu$, $\mathfrak{g}^\nu = \mathfrak{g}^\lambda \cap \mathfrak{g}^\mu$, $(V^\lambda)^{\nu \geqslant 0} = (V^\lambda)^{\mu \geqslant 0}$ and $(\mathfrak{g}^\lambda)^{\nu \geqslant 0} = (\mathfrak{g}^\lambda)^{\mu \geqslant 0}$ (Lemma 2.8). We have $V^\nu \subseteq V^\lambda$ and $\mathfrak{g}^\nu \subseteq \mathfrak{g}^\lambda$ with at least one strict inclusion. Therefore, $\nu \prec \lambda$ and moreover, $k_{\nu,\lambda} = k_{\mu,\lambda}$. This proves the lemma. □

Lemma 4.10. *We have $J_\lambda^{W^\lambda} \subseteq \mathcal{H}_\lambda^{\mathrm{prim}}$. Moreover, $J_\lambda \cap \mathcal{H}_\lambda^{\mathrm{prim}} = J_\lambda^{W^\lambda}$.*

Proof. This comes from the fact that the induction product §2.2 is well-defined on the cohomology, without requiring localisation, even though the explicit formula (Proposition 2.11) involves rational fractions. More precisely, $J_\lambda^{W^\lambda}$ is linearly generated

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

by the elements

$$\sum_{w\in W^\lambda} w\cdot(fk_{\mu,\lambda})$$

for $f\in\mathcal{A}_\lambda^{\mathrm{prim}}$ and $\mu\prec\lambda$, by Lemma 4.9. These are averages over W^λ of elements of J_λ. We can rewrite this sum as

$$\sum_{\overline{w'}\in W^\lambda/W^\mu} w'\cdot\left[\left(\sum_{w\in W^\mu}(w\cdot f)\right)k_{\mu,\lambda}\right]$$

since $k_{\mu,\lambda}$ is W^μ-invariant (Lemma 2.10). This is (Proposition 2.11) the formula for

$$I:=\mathrm{Ind}_{\mu,\lambda}\left(\sum_{w\in W^\mu}(w\cdot f)\right).$$

Since $\sum_{w\in W^\mu}(w\cdot f)$ is polynomial, its induction I is also polynomial. Therefore, it belongs to $\mathcal{H}_\lambda$. Moreover, $I\in J_\lambda\subseteq\mathcal{A}_\lambda^{\mathrm{prim}}[\prod_{\alpha\in\mathcal{W}(\mathfrak{g}^\lambda)\setminus\{0\}}\alpha^{-1}]$ by W^λ-invariance of J_λ. Therefore, $I\in\mathcal{H}_\lambda\cap\mathcal{A}_\lambda^{\mathrm{prim}}[\prod_{\alpha\in\mathcal{W}(\mathfrak{g}^\lambda)\setminus\{0\}}\alpha^{-1}]=\mathcal{H}_\lambda^{\mathrm{prim}}$.

For the second claim of the lemma, we have obviously $J_\lambda^{W^\lambda}\subseteq J_\lambda$. Combined with the first part of the lemma, this gives the reverse inclusion $J_\lambda^{W^\lambda}\subseteq J_\lambda\cap\mathcal{H}_\lambda^{\mathrm{prim}}$. For the direct inclusion, if $f\in J_\lambda$ is also in $\mathcal{H}_\lambda^{\mathrm{prim}}=(\mathcal{A}_\lambda^{\mathrm{prim}})^{W^\lambda}$, it clearly is W^λ-invariant: $f\in J_\lambda^{W^\lambda}$. □

Lemma 4.11. *The submodule J_λ is W_λ-invariant.*

Proof. Let $w\in W_\lambda$. We let $J_\lambda^w:=w\cdot J_\lambda$. Then, J_λ^w is W^λ-stable by Lemma 4.2. It is an $\mathcal{A}_\lambda^{\mathrm{prim}}$-submodule of $\mathcal{A}_\lambda^{\mathrm{prim}}[\prod_{\alpha\in\mathcal{W}(\mathfrak{g}^\lambda)\setminus\{0\}}(w\cdot\alpha)^{-1}]=\mathcal{A}_\lambda^{\mathrm{prim}}[\prod_{\alpha\in\mathcal{W}(\mathfrak{g}^\lambda)\setminus\{0\}}\alpha^{-1}]$. Moreover, it contains

$$k_{w\cdot\mu,w\cdot\lambda}$$

for $\lambda\not\preceq\mu$. Now, we use that by Lemma 2.15, for $w\in W_\lambda$, $\{\pm k_{\mu,\lambda}\colon\lambda\not\preceq\mu\}=\{\pm k_{w\cdot\mu,w\cdot\lambda}\mid\lambda\not\preceq\mu\}$ since for $w\in W_\lambda$, $\overline{w\cdot\lambda}=\overline{\lambda}$. By minimality of J_λ, we deduce that $J_\lambda\subseteq J_\lambda^w$. By symmetry of the argument, we conclude $J_\lambda^w=J_\lambda$. □

Corollary 4.12. *$J_\lambda^{W^\lambda}$ is W_λ-invariant.*

Proof. This follows directly from Lemmas 4.2 and 4.11. □

We let $\mathcal{P}_\lambda\subseteq\mathcal{H}_\lambda^{\mathrm{prim}}$ be a W_λ-stable direct sum complement of $J_\lambda^{W^\lambda}$ (using Corollary 4.12):

$$\mathcal{H}_\lambda^{\mathrm{prim}}=J_\lambda^{W^\lambda}\oplus\mathcal{P}_\lambda.$$

Lemma 4.13. *For any $\lambda\in X_*(T)$, the image of*

$$\bigoplus_{\lambda\not\preceq\mu}\mathcal{H}_{\mu,\lambda}\xrightarrow{\oplus\mathrm{Ind}_{\mu,\lambda}}\mathcal{H}_\lambda$$

is $J_\lambda^{W^\lambda}\otimes\mathrm{H}^*(\mathrm{pt}/G_\lambda)$.

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

Similarly, the image of

$$\bigoplus_{\mu\prec\lambda}\mathcal{H}_{\mu,\lambda}\xrightarrow{\oplus\mathrm{Ind}_{\mu,\lambda}}\mathcal{H}_\lambda$$

is $J_\lambda^{W^\lambda}\otimes\mathrm{H}^*(\mathrm{pt}/G_\lambda)$.

Proof. We only prove the second part of the lemma, as the second part can be proved in the exact same way by using Lemma 4.9.

This is a calculation very similar to that in the proof of Lemma 4.11. Namely, if $f\in\mathcal{H}_{\mu,\lambda}$ and $g\in\mathrm{H}^*(\mathrm{pt}/G_\lambda)$, $\mathrm{Ind}_{\mu,\lambda}(fg)=\mathrm{Ind}_{\mu,\lambda}(f)g$ by W^λ-invariance of g (Lemma 4.4). Moreover, $fg\in\mathcal{H}_{\mu,\lambda}$. Therefore, a spanning subset of $J_\lambda^{W^\lambda}\otimes\mathrm{H}^*(\mathrm{pt}/G_\lambda)$ can be obtained in this way by considering all $\mu\prec\lambda$. Conversely, if $f\in\mathcal{H}_{\mu,\lambda}\subseteq\mathcal{A}_{\mu,\lambda}=\mathcal{A}_\lambda=\mathrm{H}^*(\mathrm{pt}/T)$ for $\mu\prec\lambda$, we write $f=\sum_{i=1}^N f_i\otimes g_i$ with $f_i\in\mathcal{A}_\lambda^{\mathrm{prim}}$ and $g_i\in\mathrm{H}^*(\mathrm{pt}/G_\lambda)$ so that

$$\mathrm{Ind}_{\mu,\lambda}(f)=\sum_{i=1}^N\mathrm{Ind}_{\mu,\lambda}(f_i)\otimes g_i$$

indeed belongs to $J_\lambda^{W^\lambda}\otimes\mathrm{H}^*(\mathrm{pt}/G_\lambda)$. □

Lemma 4.14. *For any* $\lambda\in X_*(T)$*, the map*

$$\bigoplus_{\substack{\mu\in X_*(T)\\ \mu\preceq\lambda}}\mathcal{P}_\mu\otimes\mathrm{H}^*(\mathrm{pt}/G_\mu)\xrightarrow{\oplus_{\mu\preceq\lambda}\mathrm{Ind}_{\mu,\lambda}}\mathcal{H}_\lambda$$

is surjective.

Proof. We proceed by induction on $\lambda\in X_*(T)$. If λ is minimal, then $J_\lambda=\{0\}$ (by Lemma 4.9) and so $\mathcal{P}_\lambda=\mathrm{H}^*(V^\lambda/(G^\lambda/G_\lambda))$. By Proposition 3.7, the map

$$\mathcal{P}_\lambda\otimes\mathrm{H}^*(\mathrm{pt}/G_\lambda)\to\mathrm{H}^*(V^\lambda/G^\lambda)$$

of the lemma is indeed surjective (as it even is an isomorphism).

Now, let $\lambda\in X_*(T)$. We assume that the map of the lemma is surjective for any $\lambda'\prec\lambda$. By associativity of the induction (Proposition 2.9), the image of the map of the lemma coincides with the image of the composition

$$\left(\bigoplus_{\mu\prec\lambda}\bigoplus_{\mu'\preceq\mu}\mathcal{P}_{\mu'}\otimes\mathrm{H}^*(\mathrm{pt}/G_{\mu'})\right)\oplus(\mathcal{P}_\lambda\otimes\mathrm{H}^*(\mathrm{pt}/G_\lambda))\xrightarrow{I\oplus\mathrm{id}}\left(\bigoplus_{\mu\prec\lambda}\mathcal{H}_\mu\right)\oplus(\mathcal{P}_\lambda\otimes\mathrm{H}^*(\mathrm{pt}/G_\lambda))\to\mathcal{H}_\lambda$$

where the morphism I is $\bigoplus_{\mu\prec\lambda}\bigoplus_{\mu'\preceq\mu}\mathrm{Ind}_{\mu',\mu}$. By induction hypothesis,

$$\left(\bigoplus_{\mu\prec\lambda}\bigoplus_{\mu'\preceq\mu}\mathcal{P}_{\mu'}\otimes\mathrm{H}^*(\mathrm{pt}/G_{\mu'})\right)\to\left(\bigoplus_{\mu\prec\lambda}\mathcal{H}_\mu\right)$$

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

is surjective and by Lemma 4.13, the image of

$$\left(\bigoplus_{\mu\prec\lambda}\mathcal{H}_\mu\right)\to\mathcal{H}_\lambda$$

is $J_\lambda^{W^\lambda}\otimes \mathrm{H}^*(\mathrm{pt}/G_\lambda)$, of which $\mathcal{P}_\lambda\otimes\mathrm{H}^*(\mathrm{pt}/G_\lambda)$ is a direct sum complement by definition of $\mathcal{P}_\lambda$. This proves the surjectivity. □

For $y\in\mathfrak{t}$, we let $\mathfrak{g}^y=\{\xi\in\mathfrak{g}\mid[y,\xi]=0\}$ be the centraliser of y in $\mathfrak{g}$ and $V^y=\{v\in V\mid y\cdot v=0\}$ where y acts on V via the infinitesimal action.

Lemma 4.15. *For any $y\in\mathfrak{t}$, there exists $\lambda\in X_*(T)$ such that $\mathfrak{g}^\lambda=\mathfrak{g}^y$ and $V^\lambda=V^y$.*

Proof. We may assume that $y\neq 0$. Via the natural embeddings $X_*(T)\subseteq\mathfrak{t}_{\mathbf{Q}}$ and $X^*(T)\subseteq\mathfrak{t}^*_{\mathbf{Q}}$, the natural pairing $X_*(T)\times X^*(T)\to\mathbf{Z}$ corresponds to the natural pairing $\langle-,-\rangle\colon\mathfrak{t}\times\mathfrak{t}^*\to\mathbf{C}$ (see also §1.4). We let

$$\mathcal{W}_1:=\{\alpha\in\mathcal{W}'(V)\cup\mathcal{W}'(\mathfrak{g})\mid\langle y,\alpha\rangle=0\}$$

and

$$\mathcal{W}_2:=\{\alpha\in\mathcal{W}'(V)\cup\mathcal{W}'(\mathfrak{g})\mid\langle y,\alpha\rangle\neq 0\}.$$

The rational vector space $\cap_{\alpha\in\mathcal{W}_1}\alpha^\perp\subseteq\mathfrak{t}_{\mathbf{Q}}$ is non-zero, as it is defined by the rational equations $\langle-,\alpha\rangle=0$ and its complexification contains y. The complement of the finite union of rational hyperplanes $U_y:=\cap_{\alpha\in\mathcal{W}_1}\alpha^\perp\setminus\cup_{\alpha\in\mathcal{W}_2}\alpha^\perp$ is therefore also non-empty. We let $y'\in U_y$ be a rational point ($y\in\mathfrak{t}_{\mathbf{Q}}$). For some $N\geqslant 1$, $Ny'\in X_*(T)\subseteq\mathfrak{t}$. We let $\lambda:=Ny'$. It satisfies the property of the lemma. □

Proposition 4.16. *For any $\lambda\in X_*(T)$, $\mathcal{P}_\lambda$ is a finite-dimensional vector space.*

Proof. By possibly replacing V by V^λ and G by G^λ, we may assume that $\lambda=0$. It suffices to prove that $J_\lambda^{W^\lambda}\subseteq\mathcal{H}_\lambda^{\mathrm{prim}}$ has finite codimension. If we replace V by $V\times\mathfrak{g}$, then the corresponding module $J_\lambda\subseteq\mathcal{A}_\lambda^{\mathrm{prim}}[\prod_{\alpha\in\mathcal{W}(\mathfrak{g})\setminus\{0\}}\alpha^{-1}]$ is smaller than the one for V (by the very definition of J_λ at the beginning of §4.2, since when replacing V by $V\times\mathfrak{g}$, the new k_λ is equal to the numerator of the old k_λ), and therefore so is the corresponding module of invariants $J_\lambda^{W^\lambda}$, and therefore its codimension is bigger. Moreover, in this case, $J_\lambda\subseteq\mathcal{A}_\lambda^{\mathrm{prim}}$, i.e. there is no need to localise.

We have an inclusion

$$\mathcal{H}_\lambda^{\mathrm{prim}}/J_\lambda^{W^\lambda}\to\mathcal{A}_\lambda^{\mathrm{prim}}/J_\lambda,$$

because $\mathcal{H}_\lambda^{\mathrm{prim}}\cap J_\lambda=J_\lambda^{W^\lambda}$ (second part of Lemma 4.10). Therefore, it suffices to prove that the ideal $J_\lambda\subseteq\mathcal{A}_\lambda^{\mathrm{prim}}$ has finite codimension. We see $\mathcal{A}_\lambda^{\mathrm{prim}}$ as the ring of regular functions on the affine algebraic variety $\mathfrak{t}/\mathfrak{g}_\lambda$. We prove that the closed subscheme defined by J_λ is supported on $\{0\}$. If the ideal J_λ vanishes on $y\in(\mathfrak{t}/\mathfrak{g}_\lambda)_{\overline{\mathbf{Q}}}\setminus\{0\}$ (where $\overline{\mathbf{Q}}$ is the field of algebraic numbers) we let $y'\in\mathfrak{t}$ be a lift of y and ν be given by Lemma 4.15. It satisfies $\mathcal{W}^{\nu=0}(V)=\{\alpha\in\mathcal{W}(V)\mid\langle y,\alpha\rangle=0\}\subseteq\mathcal{W}(V)$ and $\mathcal{W}^{\nu=0}(\mathfrak{g})=\{\alpha\in\mathcal{W}(\mathfrak{g})\mid\langle y,\alpha\rangle=0\}$

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

$\subseteq \mathcal{W}(\mathfrak{g})$. Moreover, at least one of these inequalities is strict. Indeed, we have $y' \notin \mathfrak{g}_\lambda$, which means that either y' is not in the center of $\mathrm{Lie}(G^\lambda)$ and so ν defines a strict Levi $G^\nu = G^y$ of G^λ or y' is not in the Lie algebra of the kernel of the action $G^\lambda \to \mathrm{GL}(V^\lambda)$ in which case $V^\nu = V^y \subsetneq V^\lambda$. Therefore, $\nu \prec \lambda = 0$, i.e $\overline{\nu}$ is nontrivial. By definition, we have $\prod_{\alpha \in \mathcal{W}^{\nu<0}(V^\lambda)} \alpha \in J_\lambda$.

We also have $\prod_{\alpha \in \mathcal{W}^{\nu<0}(V^\lambda)} \langle y, \alpha \rangle \neq 0$: indeed, by choice of ν, for any $\alpha \in \mathcal{W}^{\nu<0}(V^\lambda)$, $\langle \nu, \alpha \rangle = 0$ if and only if $\langle y, \alpha \rangle = 0$. This is a contradiction since we found an element $\prod_{\alpha \in \mathcal{W}^{\nu<0}(V^\lambda)} \alpha \neq 0$ of J_λ not vanishing on y. Therefore, J_λ defines a subscheme of $\mathfrak{t}/\mathfrak{g}_\lambda$ supported on 0. □

The rest of this section is devoted to the proof of the injectivity of the cohomological integrality map in Theorem 1.1.

For $\lambda \in X_*(T)$, we let $\widetilde{J}_\lambda := J_\lambda \mathcal{A}_\lambda$. This is a $\mathcal{A}_\lambda$-submodule of $\mathcal{A}_\lambda^{\mathrm{loc}}$.

Recall the inclusion $\mathcal{H}_\lambda = \mathcal{A}_\lambda^{W^\lambda} \to \mathcal{A}_\lambda$.

Lemma 4.17. *The elements $\alpha \in \mathcal{W}^{\lambda<0}(V) \cup \mathcal{W}^{\lambda<0}(\mathfrak{g})$ (or more generally, $\alpha \in X^*(T)$ such that $\langle \lambda, \alpha \rangle \neq 0$), seen as elements of $\mathcal{A}_\lambda$, are not zero divisors in the quotient $\mathcal{A}_\lambda$-module $\mathcal{A}_\lambda^{\mathrm{loc}}/\widetilde{J}_\lambda$.*

Proof. Let $\alpha \in \mathcal{W}^{\lambda<0}(V) \cup \mathcal{W}^{\lambda<0}(\mathfrak{g})$. We see $\alpha \in \mathfrak{t}^*$ via the natural inclusion $X^*(T) \subseteq \mathfrak{t}^*$ and $\langle \lambda, \alpha \rangle \neq 0$ implies that α does not vanish on $\mathfrak{g}_\lambda$ since it takes a nonzero value on $\left(\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t)\right)_{t=1} \in \mathfrak{g}_\lambda \subseteq \mathfrak{t}$. Therefore, the projection $\alpha' \in \mathfrak{g}_\lambda^*$ of α with respect to the chosen (or any) decomposition $\mathfrak{t}^* \cong (\mathfrak{t}/\mathfrak{g}_\lambda)^* \oplus \mathfrak{g}_\lambda^*$ is nonzero.

We choose a basis $(x_1, \ldots, x_{\dim \mathfrak{g}_\lambda})$ of $\mathfrak{g}_\lambda^*$ such that $x_1 = \alpha'$ and we order the monomials by the lexicographic order so that $x_1 \geqslant x_2 \geqslant \ldots$. Let $g \in \mathcal{A}_\lambda^{\mathrm{loc}} \cong \mathcal{A}_\lambda^{\mathrm{prim,loc}} \otimes \mathrm{H}^*(\mathrm{pt}/G_\lambda)$. It may be written in a unique way (see Lemma 4.1)

$$g = \sum_{(\nu_1, \ldots, \nu_{\dim \mathfrak{g}_\lambda}) \in \mathbf{N}^{\dim \mathfrak{g}_\lambda}} g_\nu x^\nu \tag{4.1}$$

with $g_\nu \in \mathcal{A}_\lambda^{\mathrm{prim,loc}}$. We have an equivalence between the following two statements:

(1) $g \notin \widetilde{J}_\lambda$,

(2) $g_\nu \notin J_\lambda$ for some $\nu \in \mathbf{N}^{\dim \mathfrak{g}_\lambda}$.

The implication (1) $\Rightarrow$ (2) is immediate as if all g_ν belong to J_λ, then g would be in $\widetilde{J}_\lambda$. The reverse implication (2) $\Rightarrow$ (1) is also true by the unicity of the decomposition (4.1) of g.

Assume that $g \notin \widetilde{J}_\lambda$. We need to show that for $\alpha \in \mathcal{W}^{\lambda<0}(V) \cup \mathcal{W}^{\lambda<0}(\mathfrak{g})$, $\alpha g \notin \widetilde{J}_\lambda$. We let $\nu = \max\{\nu' \in \mathbf{N}^{\dim \mathfrak{g}_\lambda} : g_{\nu'} \neq 0\}$ (the maximum is taken with respect to the lexicographic order). We may assume that $g_\nu \notin J_\lambda$. We have

$$\begin{aligned} \alpha g &= (\alpha - \alpha')g + \alpha' g \\ &= (\alpha - \alpha')g + x_1 g \end{aligned}$$

and $\alpha - \alpha' \in (\mathfrak{t}/\mathfrak{g}_\lambda)^* \subseteq \mathrm{Sym}((\mathfrak{t}/\mathfrak{g}_\lambda)^*)$. Therefore, the term of highest degree in the variables $(x_1, \ldots, x_{\dim \mathfrak{g}_\lambda})$ in αg is $g_\nu x_1 x^\nu$ and $g_\nu \notin J_\lambda$. By the equivalence above, $\alpha g \notin \widetilde{J}_\lambda$. This concludes the proof. □

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

We let $\mathcal{A}'_\lambda := \mathcal{A}_\lambda[\prod_{\alpha\in\mathcal{W}^{\lambda<0}(V)\cup\mathcal{W}^{\lambda<0}(\mathfrak{g})}\alpha^{-1}]$, $\mathcal{A}'^{\mathrm{loc}}_\lambda := \mathcal{A}_\lambda[\alpha^{-1}\colon \alpha\in\mathcal{W}(\mathfrak{g})\setminus\{0\}]$ and $\mathcal{H}'_\lambda := (\mathcal{A}'_\lambda)^{W^\lambda}$. We have the localisation maps

$$L\colon \mathcal{A}_\lambda \to \mathcal{A}'_\lambda, \quad L\colon \mathcal{A}^{\mathrm{loc}}_\lambda \to \mathcal{A}'^{\mathrm{loc}}_\lambda, \quad L\colon \mathcal{H}_\lambda \to \mathcal{H}'_\lambda\,.$$

We let $\widetilde{J}'_\lambda := \mathcal{A}'_\lambda L(\widetilde{J}_\lambda)$ be the localised $\mathcal{A}'_\lambda$-module inside $\mathcal{A}'^{\mathrm{loc}}_\lambda$ (i.e. the $\mathcal{A}'_\lambda$-submodule of $\mathcal{A}'^{\mathrm{loc}}_\lambda$ generated by J_λ). We emphasised here the fact that $\widetilde{J}'_\lambda$ is *not* a $\mathcal{A}'^{\mathrm{loc}}_\lambda$-submodule.

Lemma 4.18. *The localisation maps*

$$L\colon \mathcal{A}^{\mathrm{loc}}_\lambda/\widetilde{J}_\lambda \to \mathcal{A}'^{\mathrm{loc}}_\lambda/\widetilde{J}'_\lambda, \quad L\colon \mathcal{H}_\lambda/(\widetilde{J}_\lambda)^{W^\lambda} \to \mathcal{H}'_\lambda/(\widetilde{J}'_\lambda)^{W^\lambda}$$

are injective.

Proof. This follows from Lemma 4.17 as we localise by a set of nonzero divisors of the quotient $\mathcal{A}_\lambda$-module $\mathcal{A}^{\mathrm{loc}}_\lambda/\widetilde{J}_\lambda$ in $\mathcal{A}_\lambda$. More precisely, the first localisation morphism $L\colon \mathcal{A}^{\mathrm{loc}}_\lambda/\widetilde{J}_\lambda \to \mathcal{A}'^{\mathrm{loc}}_\lambda/\widetilde{J}'_\lambda$ is the localisation with respect to $\prod_{\alpha\in\mathcal{W}^{\lambda<0}(V)\cup\mathcal{W}^{\lambda<0}(\mathfrak{g})}\alpha$ for the $\mathcal{A}_\lambda$-module $\mathcal{A}^{\mathrm{loc}}_\lambda/\widetilde{J}_\lambda$, and the localisation element is not a zero-divisor of $\mathcal{A}^{\mathrm{loc}}_\lambda/\widetilde{J}_\lambda$. Therefore, the first localisation morphism is injective.

Now, the two localisation morphisms of the lemma fit into the commutative diagram

$$\begin{array}{ccc} \mathcal{H}_\lambda/(\widetilde{J}_\lambda)^{W^\lambda} & \xrightarrow{L} & \mathcal{H}'_\lambda/(\widetilde{J}'_\lambda)^{W^\lambda} \\ {\scriptstyle a}\downarrow & & \downarrow \\ \mathcal{A}^{\mathrm{loc}}_\lambda/\widetilde{J}_\lambda & \xrightarrow{L} & \mathcal{A}'^{\mathrm{loc}}_\lambda/\widetilde{J}'_\lambda \end{array} \tag{4.2}$$

The vertical map a is induced by the inclusion $\mathcal{H}_\lambda = \mathcal{H}_\lambda^{W^\lambda} \subseteq \mathcal{A}_\lambda \subseteq \mathcal{A}^{\mathrm{loc}}_\lambda$. The map a is injective since $\mathcal{H}_\lambda \cap \widetilde{J}_\lambda = (\widetilde{J}_\lambda)^{W^\lambda}$, which follows from the arguments of the proof of Lemma 4.10. Therefore, the second localisation map (the top line of (4.2)) must be injective. □

Lemma 4.19. *Let $\lambda,\mu\in X_*(T)$ and ν as in Lemma 2.8. Then,*

$$k_\mu = k_{\nu,\lambda}\frac{\prod_{\alpha\in\mathcal{W}^{\mu<0}(V)\cap\mathcal{W}^{\lambda\neq 0}(V)}\alpha}{\prod_{\alpha\in\mathcal{W}^{\mu<0}(\mathfrak{g})\cap\mathcal{W}^{\lambda\neq 0}(\mathfrak{g})}\alpha}\,.$$

Proof. By the properties satisfied by the cocharacter ν given by Lemma 2.8, we have $\mathcal{W}^{\mu<0}(V) = \mathcal{W}^{\nu<0}(V)\cap\mathcal{W}^{\lambda=0}(V)\sqcup\mathcal{W}^{\mu<0}(V)\cap\mathcal{W}^{\lambda\neq 0}(V)$ and similarly for V and $\mathfrak{g}$ exchanged. Therefore,

$$\begin{aligned} k_\mu &= \frac{\prod_{\alpha\in\mathcal{W}^{\mu<0}(V)}\alpha}{\prod_{\alpha\in\mathcal{W}^{\mu<0}(\mathfrak{g})}\alpha} \\ &= \frac{\prod_{\alpha\in\mathcal{W}^{\nu<0}(V)\cap\mathcal{W}^{\lambda=0}(V)}\alpha}{\prod_{\alpha\in\mathcal{W}^{\nu<0}(\mathfrak{g})\cap\mathcal{W}^{\lambda=0}(\mathfrak{g})}\alpha}\frac{\prod_{\alpha\in\mathcal{W}^{\mu<0}(V)\cap\mathcal{W}^{\lambda\neq 0}(V)}\alpha}{\prod_{\alpha\in\mathcal{W}^{\mu<0}(\mathfrak{g})\cap\mathcal{W}^{\lambda\neq 0}(\mathfrak{g})}\alpha} \\ &= k_{\nu,\lambda}\frac{\prod_{\alpha\in\mathcal{W}^{\mu<0}(V)\cap\mathcal{W}^{\lambda\neq 0}(V)}\alpha}{\prod_{\alpha\in\mathcal{W}^{\mu<0}(\mathfrak{g})\cap\mathcal{W}^{\lambda\neq 0}(\mathfrak{g})}\alpha}\,. \end{aligned}$$

□

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

We define

$$k'_{\mu,\lambda} \coloneqq \frac{\prod_{\alpha\in\mathcal{W}^{\mu<0}(V)\cap\mathcal{W}^{\lambda\neq 0}(V)}\alpha}{\prod_{\alpha\in\mathcal{W}^{\mu<0}(\mathfrak{g})\cap\mathcal{W}^{\lambda\neq 0}(\mathfrak{g})}\alpha}.$$

This is an invertible element in $\mathcal{A}'_\lambda$.

Lemma 4.20. *Let $\lambda,\mu\in X_*(T)$ and w in W be such that $\lambda\not\preceq w^{-1}\cdot\mu$. Then, $w\cdot k_\mu\in\widetilde{J}'_\lambda$.*

Proof. Of course, the presence of W is superfluous since $wk_\mu = k_{w^{-1}\cdot\mu}$ for any $w\in W$ and it suffices to prove that for any $\lambda,\mu\in X_*(T)$ such that $\lambda\not\preceq\mu$, $k_\mu\in\widetilde{J}'_\lambda$.

We let $\nu\in X_*(T)$ be such that $V^\nu = V^\lambda\cap V^\mu$, $\mathfrak{g}^\nu=\mathfrak{g}^\lambda\cap\mathfrak{g}^\mu$, $(V^\lambda)^{\nu\geqslant 0} = (V^\lambda)^{\mu\geqslant 0}$ and $(\mathfrak{g}^\lambda)^{\nu\geqslant 0} = (\mathfrak{g}^\lambda)^{\mu\geqslant 0}$ (Lemma 2.8). By assumption, $\nu\prec\lambda$.

We can write, by Lemma 4.19,

$$k_\mu = k_{\nu,\lambda}\underbrace{\frac{\prod_{\alpha\in\mathcal{W}^{\mu<0}(V)\cap\mathcal{W}^{\lambda\neq 0}(V)}\alpha}{\prod_{\alpha\in\mathcal{W}^{\mu<0}(\mathfrak{g})\cap\mathcal{W}^{\lambda\neq 0}(\mathfrak{g})}\alpha}}_{=k'_{\mu,\lambda}\in\mathcal{A}'_\lambda}.$$

This element is in $\widetilde{J}'_\lambda$ since $k_{\nu,\lambda}$ is in J_λ, by Lemma 4.9. □

Lemma 4.21. *Let $\lambda\in X_*(T)$ be such that for some $w\in W$, $\lambda\preceq w\cdot\lambda$. Then, $\overline{w\cdot\lambda}=\overline{\lambda}$.*

Proof. The lemma follows from the inclusions $V^\lambda\subseteq V^{w\cdot\lambda}$ and $\mathfrak{g}^\lambda\subseteq\mathfrak{g}^{w\cdot\lambda}$ and the equalities of dimensions $\dim V^\lambda = \dim V^{w\cdot\lambda}$ and $\dim\mathfrak{g}^\lambda=\dim\mathfrak{g}^{w\cdot\lambda}$. □

For $\lambda\in X_*(T)$, we have an inclusion

$$\mathrm{Res}_\lambda\colon\mathcal{H}\cong\mathcal{A}^W\cong\mathcal{A}_\lambda^W\to\mathcal{H}_\lambda\cong\mathcal{A}_\lambda^{W^\lambda}.$$

since $\mathcal{A}\cong\mathcal{A}_\lambda=\mathrm{H}^*(\mathrm{pt}/T)$ and $W^\lambda\subseteq W$ (Lemma 4.2).

Corollary 4.22. *For any $f_\mu\in\mathcal{H}_\mu$, $\widetilde{\mu}\in\mathscr{P}_V/W$ such that $f_\mu=0$ if $\lambda\prec w\cdot\mu$ for some $w\in W$, we have*

$$L\mathrm{Res}_\lambda\left(\sum_{\widetilde{\mu}\in\mathscr{P}_V/W}\mathrm{Ind}_\mu(f_\mu)\right)\equiv k_\lambda\cdot\sum_{w\in W_\lambda/W^\lambda}\varepsilon_{V,\lambda}(w)^{-1}(w\cdot f_\lambda)\pmod{\widetilde{J}'_\lambda}.$$

Proof. We calculate:

$$\begin{aligned}
&L\mathrm{Res}_\lambda\left(\sum_{\widetilde{\mu}\in\mathscr{P}_V/W}\mathrm{Ind}_\mu(f_\mu)\right)\\
&\quad= L\mathrm{Res}_\lambda\left(\sum_{\widetilde{\mu}\in\mathscr{P}_V/W}\frac{1}{\#W^\mu}\sum_{w\in W}(w\cdot f_\mu)(w\cdot k_\mu)\right)\quad\text{by Proposition 2.11}\\
&\quad\equiv\frac{1}{\#W^\lambda}\sum_{\substack{w\in W\\ w\overline{\lambda}=\overline{\lambda}}}(w\cdot f_\lambda)(w\cdot k_\lambda)\pmod{\widetilde{J}'_\lambda}\quad\text{by Lemmas 4.20 and 2.21}\\
&\quad=\frac{1}{\#W^\lambda}k_\lambda\cdot\sum_{w\in W_\lambda}\varepsilon_{V,\lambda}(w)^{-1}(w\cdot f_\lambda)\quad\text{by definition of the character}\varepsilon_{V,\lambda}\text{, Proposition 4.6.}
\end{aligned}$$

□

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

Proposition 4.23. *The map*

$$\bigoplus_{\widetilde{\mu}\in\mathscr{P}_V/W} (\mathcal{P}_\mu \otimes \mathrm{H}^*(\mathrm{pt}/G_\mu))^{\varepsilon_{V,\mu}} \xrightarrow{\bigoplus \mathrm{Ind}_\mu} \mathrm{H}^*(V/G)$$

induced by the parabolic induction morphisms is injective.

Proof. Let $f_\mu \in (\mathcal{P}_\mu \otimes \mathrm{H}^*(\mathrm{pt}/G_\mu))^{\varepsilon_{V,\mu}}$, $\widetilde{\mu} \in \mathscr{P}_V/W$. We let $U \coloneqq \sum_{\widetilde{\mu}\in\mathscr{P}_V/W} \mathrm{Ind}_\mu(f_\mu)$. We assume that not all of the f_μ's are zero, and we have to prove that $U \neq 0$. We let $\widetilde{\lambda} \in \mathscr{P}_V/W$ be such that $f_\lambda \neq 0$ and for any $\widetilde{\mu} \in \mathscr{P}_V/W$ such that $f_\mu \neq 0$, for any $w \in W$, $\lambda \not\trianglelefteq w\cdot\mu$ (otherwise, we may replace $\widetilde{\lambda}$ by $\widetilde{\mu}$, etc.).

By Corollary 4.22, we have

$$\begin{aligned} L\mathrm{Res}_\lambda\left(\sum_{\widetilde{\mu}\in\mathscr{P}_V/W} \mathrm{Ind}_\mu(f_\mu)\right) &\equiv k_\lambda \cdot \sum_{w\in W_\lambda/W^\lambda} \varepsilon_{V,\lambda}(w)^{-1}(w\cdot f_\lambda) \pmod{\widetilde{J}'_\lambda} \\ &\equiv k_\lambda \#(W_\lambda/W^\lambda) f_\lambda \pmod{\widetilde{J}'_\lambda} \text{ by Proposition 4.8.} \end{aligned}$$

and moreover, $f_\lambda \notin \widetilde{J}'_\lambda$ since $f_\lambda \in \mathcal{P}_\lambda \otimes \mathrm{H}^*(\mathrm{pt}/G_\lambda)$, and k_λ is invertible in $\mathcal{A}'_\lambda$. We obtain that $\mathrm{Res}_\lambda(U) \neq 0$. Therefore, $U \neq 0$, proving injectivity. □

We are now ready to prove Theorem 1.1.

Corollary 4.24. *The map*

$$\bigoplus_{\widetilde{\mu}\in\mathscr{P}_V/W} (\mathcal{P}_\mu \otimes \mathrm{H}^*(\mathrm{pt}/G_\mu))^{\varepsilon_{V,\mu}} \xrightarrow{\bigoplus_{\widetilde{\mu}\in\mathscr{P}_V/W} \mathrm{Ind}_\mu} \mathrm{H}^*(V/G)$$

is an isomorphism.

Proof. The injectivity is Proposition 4.23. The surjectivity comes from Lemmas 4.14 and 2.17. Indeed, by Lemma 4.14, any element $f \in \mathrm{H}^*(V/G)$ has the form $f = \sum_{\mu\preceq 0} \mathrm{Ind}_\mu(g_\mu)$ for some $g_\mu \in \mathcal{P}_\mu \otimes \mathrm{H}^*(\mathrm{pt}/G_\mu)$. By Lemma 2.17, we may rewrite $f = \sum_{\widetilde{\mu}\in\mathscr{P}_V/W} \mathrm{Ind}_\mu(\widetilde{g}_\mu)$. We have the formula

$$\mathrm{Ind}_\mu(\widetilde{g}_\mu) = \frac{1}{\#W^\mu} \sum_{\overline{w'}\in W/W_\mu} w' \cdot \left(\sum_{w\in W_\mu} \varepsilon_{V,\mu}(w)^{-1} w\cdot \widetilde{g}_\mu\right)(w'\cdot k_\mu)$$

for $\widetilde{g}_\mu \in \mathcal{H}_\mu$, deduced from Propositions 2.11 and 4.6 which implies, using Proposition 4.8, that $\mathrm{Ind}_\mu(\widetilde{g}_\mu)$ vanishes if f is not in the $\varepsilon_{V,\mu}$-isotypic component of $\mathcal{P}_\mu \otimes \mathrm{H}^*(\mathrm{pt}/G_\mu)$ for the W_μ-action. Therefore, we may assume that $\widetilde{g}_\mu$ is $\varepsilon_{V,\lambda}$-invariant. This concludes the proof of the corollary. □

4.3. The BPS subspaces

In analogy to the case of quivers [Efi12; MR19] and also [DM20a] for quivers with potential, we may call the subspaces $\mathcal{P}_\lambda \subseteq \mathrm{H}^*(V^\lambda/G^\lambda)$ the *BPS subspaces*. Although we lack both qualitative and quantitative properties of these spaces, there are cases when we can prove they vanish.

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

Proposition 4.25. *Let $\lambda \in X_*(T)$. If $\ker(G^\lambda \to \mathrm{GL}(V^\lambda))^\circ \not\subseteq Z(G^\lambda)$, then $\mathcal{P}_\lambda = \{0\}$.*

Proof. By possibly replacing V by V^λ and G by G^λ, we may assume that λ is the trivial cocharacter of T. If $\ker(G \to \mathrm{GL}(V))^\circ \not\subseteq Z(G)$, then there exists a cocharacter $\mu\colon \mathbf{G}_\mathrm{m} \to \ker(G \to \mathrm{GL}(V)) \subseteq T$ which does not factor through $Z(G)$. We have $\overline{\mu} \in \mathscr{P}_V \setminus \{\overline{0}\}$. Then, the induction diagram reads

$$\begin{array}{ccc} & V/P_\mu & \\ {\scriptstyle q_\mu}\swarrow & & \searrow{\scriptstyle p_\mu} \\ V/G^\mu & & V/G \end{array}$$

and the induction map $\mathrm{Ind}_\mu\colon \mathrm{H}^*(V/G^\mu) = \mathrm{H}^*(\mathrm{pt}/T)^{W^\mu} \to \mathrm{H}^*(V/G) = \mathrm{H}^*(\mathrm{pt}/T)$ is the map

$$f \mapsto \sum_{w \in W/W^\mu} w \cdot \left(\frac{f}{\prod_{\alpha \in \mathcal{W}^{\mu<0}(\mathfrak{g})} \alpha} \right).$$

It is surjective. Hence, $\mathcal{P}_0 = \{0\}$. □

5. Vanishing bounds for the BPS spaces

We recall that $\mathcal{H}_\lambda = \mathrm{H}^*(V^\lambda/G^\lambda)$ for $\lambda \in X_*(T)$. We recall the chosen $\mathbf{Z}$-grading on $\mathcal{H}_\lambda$: $\mathcal{H}_\lambda^k = \mathrm{H}^{k+d_\lambda}(V^\lambda/G^\lambda)$ for $k \in \mathbf{Z}$. This is just a shift of the cohomological grading, chosen so that induction morphisms preserve this grading (Lemma 2.7). In this section, we prove Proposition 1.5.

Proof of the lower bound in Proposition 1.5. First, we have an inclusion $\mathcal{P}_\lambda^k \subseteq \mathcal{H}_\lambda^k = \mathrm{H}^{k+d_\lambda}(V^\lambda/G^\lambda)$, and the latter vector space is zero for $k < -d_\lambda = \dim G^\lambda - \dim V^\lambda$. □

The proof of the upper bound uses [Efi12, Lemma 3.6] and is analogous to the proof of [Efi12, Theorem 3.5].

Lemma 5.1 [Efi12, Lemma 3.6]. *Let k be a field and let $B = k[z_1, \ldots, z_n]$ be an algebra of polynomials with $\deg(z_i) = 2$ (to match the cohomological degrees). Suppose that $\ell_1, \ldots, \ell_s \in B^1$ are pairwise linearly independent linear forms in z_i. Let $\{P_1, \ldots, P_r\} \subseteq B$ be non-empty set of polynomials of the form*

$$P_i = \prod_{j=1}^{s} \ell_j^{d_{i,s}}$$

for some $d_{i,j} \in \mathbf{Z}_{\geqslant 0}$. Let $d_j := \max_{1 \leqslant i \leqslant r} d_{i,j}$ for $1 \leqslant j \leqslant s$. Then, the following are equivalent:

(i) *$B^{2d} \subseteq (P_1, \ldots, P_r)$ for $d \geqslant \sum_{j=1}^s d_j - n + 1$,*
(ii) *the ideal $(P_1, \ldots, P_r) \subseteq B$ has finite codimension,*

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

(iii) *For any sequence* $p_1,\dots,p_r$ *of numbers in* $\{1,\dots,s\}$ *such that* $d_{i,p_i}>0$ *for* $1\leqslant i\leqslant r$, *the linear forms generate the space* B^1.

Proof of the upper bound in Proposition 1.5. As we already noticed in the proof of Proposition 4.16, if we replace V by $V\times\mathfrak{g}$, then we have inclusions $\mathcal{P}_{V,\lambda}\subseteq\mathcal{P}_{V\times\mathfrak{g},\lambda}$, and therefore, it suffices to give the upper bound for $\mathcal{P}_{V\times\mathfrak{g},\lambda}$. We set $B:=\mathrm{Sym}[(\mathfrak{t}/\mathfrak{g}_\lambda)^*]=\mathcal{A}_\lambda^{\mathrm{prim}}$, the (polynomial) algebra of functions on $\mathfrak{g}/\mathfrak{g}_\lambda$. The proof of Proposition 4.16 shows that the ideal $J_\lambda\subseteq\mathrm{Sym}[(\mathfrak{t}/\mathfrak{g}_\lambda)^*]$ has finite codimension, that is satisfies the second point of Lemma 5.1. In our case, the quantity $\sum_{j=1}^s d_j$ is equal to $\frac{\dim V^\lambda-\dim V^T}{2}$. Therefore, an upper bound for $\sum_{j=1}^s d_j-n+1$ is $\frac{\dim V^\lambda}{2}$.

Now, Lemma 5.1 gives $(\mathcal{A}_\lambda^{\mathrm{prim}})^{2d}\subseteq J_\lambda$ for $d\geqslant\frac{\dim V^\lambda}{2}$ (for the unshifted cohomological gradings), where $(\mathcal{A}_\lambda^{\mathrm{prim}})^e$ denotes degree e polynomials inside $\mathcal{A}_\lambda^{\mathrm{prim}}$. By taking into account the cohomological shift by $\dim G^\lambda-\dim V^\lambda$, we obtain the upper bound: $\mathcal{P}_\lambda^k=0$ for $k>\dim V^\lambda+\dim G^\lambda-\dim V^\lambda=\dim G^\lambda$. □

6. Examples

In this section, we give some explicit examples of cohomological integrality isomorphisms for some choices of pairs (G,V) of a reductive group G and a (weakly) symmetric representation V of G. We give each time the explicit formula for the cohomological integrality isomorphisms produced by Theorem 1.1. In each case, it is possible to verify by hand that they are indeed isomorphisms.

6.1. $(\mathbf{C}^*)^2\curvearrowright\mathbf{C}^2\oplus(\mathbf{C}^2)^\vee$

We let $G:=(\mathbf{C}^*)^2$ act on $V:=\mathbf{C}^2\oplus(\mathbf{C}^2)^\vee\cong\mathbf{C}^4$ by

$$(t,u)\cdot(a,b,c,d):=(ta,ub,t^{-1}c,u^{-1}d).$$

We can identify the set of equivalence classes of cocharacters (§2.6) $\mathscr{P}_V=\{\overline{\lambda_0},\overline{\lambda_1},\overline{\lambda_2},\overline{\lambda_3}\}$ where

$$\begin{array}{rccc}
\lambda_0 : & \mathbf{G}_{\mathrm{m}} & \to & (\mathbf{C}^*)^2\\
& v & \mapsto & (1,1),\\
\lambda_1 : & \mathbf{G}_{\mathrm{m}} & \to & (\mathbf{C}^*)^2\\
& v & \mapsto & (v^{-1},1),\\
\lambda_2 : & \mathbf{G}_{\mathrm{m}} & \to & (\mathbf{C}^*)^2\\
& v & \mapsto & (1,v^{-1}),\\
\lambda_3 : & \mathbf{G}_{\mathrm{m}} & \to & (\mathbf{C}^*)^2\\
& v & \mapsto & (v^{-1},v^{-1}).
\end{array}$$

We have $V^{\lambda_0}=V$, $V^{\lambda_1}=(0\oplus\mathbf{C})\oplus(0\oplus\mathbf{C}^\vee)$, $V^{\lambda_2}=(\mathbf{C}\oplus 0)\oplus(\mathbf{C}^\vee\oplus 0)$ and $V^{\lambda_3}=\{0\}$. We have $G^{\lambda_0}=G^{\lambda_1}=G^{\lambda_2}=G^{\lambda_3}=(\mathbf{C}^*)^2$. We can describe the induction maps as follows:

$$\begin{array}{rccc}
\mathrm{Ind}_{\lambda_1,\lambda_0} : & \overbrace{\mathbf{Q}[x_1,x_2]}^{=\mathrm{H}^*(V^{\lambda_1}/G^{\lambda_1})} & \to & \overbrace{\mathbf{Q}[x_1,x_2]}^{=\mathrm{H}^*(V/G)}\\
& f & \mapsto & x_1 f
\end{array}$$

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

$$\begin{array}{rcccc}
\mathrm{Ind}_{\lambda_2,\lambda_0} & : & \overbrace{\mathbf{Q}[x_1,x_2]}^{=\mathrm{H}^*(V^{\lambda_2}/G^{\lambda_2})} & \to & \overbrace{\mathbf{Q}[x_1,x_2]}^{=\mathrm{H}^*(V/G)} \\
 & & f & \mapsto & x_2 f \\
\mathrm{Ind}_{\lambda_3,\lambda_0} & : & \overbrace{\mathbf{Q}[x_1,x_2]}^{=\mathrm{H}^*(V^{\lambda_3}/G^{\lambda_3})} & \to & \overbrace{\mathbf{Q}[x_1,x_2]}^{\mathrm{H}^*(V/G)} \\
 & & f & \mapsto & x_1x_2 f \\
\mathrm{Ind}_{\lambda_3,\lambda_1} & : & \overbrace{\mathbf{Q}[x_1,x_2]}^{=\mathrm{H}^*(V^{\lambda_3}/G^{\lambda_3})} & \to & \overbrace{\mathbf{Q}[x_1,x_2]}^{=\mathrm{H}^*(V^{\lambda_1}/G^{\lambda_1})} \\
 & & f & \mapsto & x_2 f \\
\mathrm{Ind}_{\lambda_3,\lambda_2} & : & \overbrace{\mathbf{Q}[x_1,x_2]}^{=\mathrm{H}^*(V^{\lambda_3}/G^{\lambda_3})} & \to & \overbrace{\mathbf{Q}[x_1,x_2]}^{=\mathrm{H}^*(V^{\lambda_2}/G^{\lambda_2})} \\
 & & f & \mapsto & x_1 f
\end{array}$$

We have

$$\mathcal{P}_{\lambda_0} = \mathbf{Q} \subseteq \mathbf{Q}[x_1,x_2], \quad \mathcal{P}_{\lambda_1} = \mathbf{Q} \subseteq \mathbf{Q}[x_1,x_2], \quad \mathcal{P}_{\lambda_2} = \mathbf{Q} \subseteq \mathbf{Q}[x_1,x_2], \quad \mathcal{P}_{\lambda_3} = \mathbf{Q} \subseteq \mathbf{Q}[x_1,x_2].$$

We have

$$G_{\lambda_0} = 1 \subseteq (\mathbf{C}^*)^2, \quad G_{\lambda_1} = \mathbf{C}^* \times 1 \subseteq (\mathbf{C}^*)^2, \quad G_{\lambda_2} = 1 \times \mathbf{C}^* \subseteq (\mathbf{C}^*)^2, \quad G_{\lambda_3} = (\mathbf{C}^*)^2.$$

The cohomological integrality isomorphism is

$$\begin{array}{ccccccccc}
\overbrace{\mathbf{Q}}^{=\mathcal{P}_{\lambda_0}} & \oplus & \overbrace{\mathbf{Q}[x_1]}^{=\mathcal{P}_{\lambda_1}\otimes\mathrm{H}^*(\mathrm{pt}/G_{\lambda_1})} & \oplus & \overbrace{\mathbf{Q}[x_2]}^{=\mathcal{P}_{\lambda_2}\otimes\mathrm{H}^*(\mathrm{pt}/G_{\lambda_2})} & \oplus & \overbrace{\mathbf{Q}[x_1,x_2]}^{=\mathcal{P}_{\lambda_3}\otimes\mathrm{H}^*(\mathrm{pt}/G_{\lambda_3})} & \to & \overbrace{\mathbf{Q}[x_1,x_2]}^{=\mathrm{H}^*(V/G)} \\
 & & (a_0, f(x_1), g(x_2), h(x_1,x_2)) & & & & & \mapsto & a_0 + x_1 f(x_1) + x_2 g(x_2) + x_1 x_2 h(x_1,x_2).
\end{array}$$

6.2. $\mathrm{GL}_2(\mathbf{C}) \curvearrowright \mathbf{C}^2 \oplus (\mathbf{C}^2)^\vee$

We let $G \coloneqq \mathrm{GL}_2(\mathbf{C})$ act on $V \coloneqq \mathbf{C}^2 \oplus (\mathbf{C}^2)^\vee$ via

$$g \cdot (u,v) \coloneqq (gu, (g^{-1})^t v),$$

where the superscript t indicates the transpose. We let $T \coloneqq (\mathbf{C}^*)^2 \subseteq \mathrm{GL}_2(\mathbf{C})$ be the standard torus. The Weyl group is $W = \mathfrak{S}_2$. We have $\mathscr{P}_V/W = \{\widetilde{\lambda_0}, \widetilde{\lambda_1}, \widetilde{\lambda_2}, \widetilde{\lambda_3}\}$ where

$$\begin{array}{rcccc}
\lambda_0 & : & \mathbf{G}_\mathrm{m} & \to & (\mathbf{C}^*)^2 \\
 & & v & \mapsto & (1,1) \\
\lambda_1 & : & \mathbf{G}_\mathrm{m} & \to & (\mathbf{C}^*)^2 \\
 & & v & \mapsto & (v^{-1},1) \\
\lambda_2 & : & \mathbf{G}_\mathrm{m} & \to & (\mathbf{C}^*)^2 \\
 & & v & \mapsto & (v^{-1},v^{-2}) \\
\lambda_3 & : & \mathbf{G}_\mathrm{m} & \to & (\mathbf{C}^*)^2 \\
 & & v & \mapsto & (v^{-1},v^{-1})
\end{array}$$

Therefore, we have $V^{\lambda_0} = V$, $V^{\lambda_1} = (0 \oplus \mathbf{C}) \oplus (0 \oplus \mathbf{C}^\vee)$, $V^{\lambda_2} = \{0\}$, and $V^{\lambda_3} = \{0\}$. We have $G^{\lambda_0} = \mathrm{GL}_2(\mathbf{C}), G^{\lambda_1} = T$, $G^{\lambda_2} = T$ and $G^{\lambda_3} = G$. We can describe the induction maps

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

as follows:

$$\begin{array}{rccc} \mathrm{Ind}_{\lambda_1,\lambda_0} & : & \overbrace{\mathbf{Q}[x_1,x_2]}^{=\mathrm{H}^*(V^{\lambda_1}/G^{\lambda_1})} & \to & \overbrace{\mathbf{Q}[x_1,x_2]^{\mathfrak{S}_2}}^{=\mathrm{H}^*(V/G)} = \mathbf{Q}[x_1+x_2,x_1x_2] \\ & & f & \mapsto & \frac{x_1}{x_1-x_2}f(x_1,x_2)+\frac{x_2}{x_2-x_1}f(x_2,x_1) \end{array}$$

$$\begin{array}{rccc} \mathrm{Ind}_{\lambda_2,\lambda_0} & : & \overbrace{\mathbf{Q}[x_1,x_2]}^{=\mathrm{H}^*(V^{\lambda_2}/G^{\lambda_2})} & \to & \overbrace{\mathbf{Q}[x_1,x_2]^{\mathfrak{S}_2}}^{=\mathrm{H}^*(V/G)} = \mathbf{Q}[x_1+x_2,x_1x_2] \\ & & f & \mapsto & \frac{x_1x_2}{x_1-x_2}f(x_1,x_2)+\frac{x_1x_2}{x_2-x_1}f(x_2,x_1) \end{array}$$

$$\begin{array}{rccc} \mathrm{Ind}_{\lambda_2,\lambda_1} & : & \overbrace{\mathbf{Q}[x_1,x_2]}^{=\mathrm{H}^*(V^{\lambda_2}/G^{\lambda_2})} & \to & \overbrace{\mathbf{Q}[x_1,x_2]}^{\mathrm{H}^*(V^{\lambda_1}/G^{\lambda_1})} \\ & & f & \mapsto & x_2f \end{array}$$

$$\begin{array}{rccc} \mathrm{Ind}_{\lambda_2,\lambda_3} & : & \overbrace{\mathbf{Q}[x_1,x_2]}^{=\mathrm{H}^*(V^{\lambda_2}/G^{\lambda_2})} & \to & \overbrace{\mathbf{Q}[x_1+x_2,x_1x_2]}^{\mathrm{H}^*(V^{\lambda_3}/G^{\lambda_3})} \\ & & f & \mapsto & \frac{f(x_1,x_2)-f(x_2,x_1)}{x_1-x_2} \end{array}$$

The induction $\mathrm{Ind}_{\lambda_1,\lambda_0}$ is surjective, since

$$x_1+x_2 = \mathrm{Ind}_{\lambda_1,\lambda_0}(x_1)$$

and

$$x_1x_2 = \mathrm{Ind}_{\lambda_1,\lambda_0}\left(\frac{1}{2}x_1x_2\right).$$

The induction $\mathrm{Ind}_{\lambda_2,\lambda_3}$ is surjective as any symmetric polynomial f is the image by $\mathrm{Ind}_{\lambda_2,\lambda_3}$ of $(x_1-x_2)f$.

A direct sum complement of the image of $\mathrm{Ind}_{\lambda_2,\lambda_1}$ is $\mathbf{Q}[x_1] = \mathcal{P}_{\lambda_1}\otimes\mathbf{Q}[x_1]$ by definition of $\mathcal{P}_{\lambda_1}$.

Therefore, we have $\mathcal{P}_{\lambda_0}=0$, $\mathcal{P}_{\lambda_1}=\mathbf{Q}$, $\mathcal{P}_{\lambda_2}=\mathbf{Q}$ and $\mathcal{P}_{\lambda_3}=0$. We have $G_{\lambda_0}=1, G_{\lambda_1}=\mathbf{C}^*\times 1$, $G_{\lambda_2}=T$ and $G_{\lambda_3}=G$. The cohomological integrality isomorphism reads

$$\begin{array}{ccccc} \overbrace{\mathbf{Q}[x_1]}^{=\mathcal{P}_{\lambda_1}\otimes\mathrm{H}^*(\mathrm{pt}/G_{\lambda_1})} & \oplus & \overbrace{\mathbf{Q}[x_1,x_2]^{\mathrm{sgn}}}^{=(\mathcal{P}_{\lambda_2}\otimes\mathrm{H}^*(\mathrm{pt}/G_{\lambda_2}))^{\varepsilon_{V,\lambda_2}}} & \to & \overbrace{\mathbf{Q}[x_1,x_2]^{\mathfrak{S}_2}}^{=\mathrm{H}^*(V/G)} \\ (f(x_1),g(x_1,x_2)) & & & \mapsto & \frac{x_1f(x_1)-x_2f(x_2)}{x_1-x_2}+2x_1x_2\frac{g(x_1,x_2)}{x_1-x_2} \end{array}$$

where $\mathbf{Q}[x_1,x_2]^{\mathrm{sgn}}$ is the sign-isotypic component, since the kernel $\frac{x_1x_2}{x_1-x_2}$ changes sign when we exchange the variables x_1,x_2, and so the character $\varepsilon_{V,\lambda_2}$ of $W_{\lambda_2}=\mathfrak{S}_2$ given by Proposition 4.6 is the sign.

6.3. $\mathrm{GL}_2(\mathbf{C})\curvearrowright(\mathbf{C}^2\oplus(\mathbf{C}^2)^\vee)^g$

We generalise the previous situation by considering the diagonal action of $\mathrm{GL}_2(\mathbf{C})$ on $(\mathbf{C}^2\oplus(\mathbf{C}^2)^\vee)^g$ $(g\geqslant 0)$. The set $\mathscr{P}_V/W$ is the same as in §6.2. The fixed point sets $V^{\lambda_0}, V^{\lambda_1}$, and V^{λ_2} can be computed similarly and the Levi subgroups $G^{\lambda_0}, G^{\lambda_1}$ and G^{λ_2} are the ones given in §6.2. We can describe the induction maps as follows, analogously

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

to §6.2.

$$\begin{array}{rccc}
\mathrm{Ind}_{\lambda_1,\lambda_0} : & \mathbf{Q}[x_1,x_2] & \to & \mathbf{Q}[x_1,x_2]^{\mathfrak{S}_2} = \mathbf{Q}[x_1+x_2,x_1x_2] \\
& f & \mapsto & \frac{x_1^g}{x_1-x_2}f(x_1,x_2)+\frac{x_2^g}{x_2-x_1}f(x_2,x_1) \\
\mathrm{Ind}_{\lambda_2,\lambda_0} : & \mathbf{Q}[x_1,x_2] & \to & \mathbf{Q}[x_1,x_2]^{\mathfrak{S}_2} = \mathbf{Q}[x_1+x_2,x_1x_2] \\
& f & \mapsto & \frac{x_1^gx_2^g}{x_1-x_2}f(x_1,x_2)+\frac{x_1^gx_2^g}{x_2-x_1}f(x_2,x_1)
\end{array}$$

$$\begin{array}{rccc}
\mathrm{Ind}_{\lambda_2,\lambda_1} : & \mathbf{Q}[x_1,x_2] & \to & \mathbf{Q}[x_1,x_2] \\
& f & \mapsto & x_2^g f
\end{array}$$

$$\begin{array}{rccc}
\mathrm{Ind}_{\lambda_2,\lambda_3} : & \mathbf{Q}[x_1,x_2] & \to & \mathbf{Q}[x_1+x_2,x_1x_2] \\
& f & \mapsto & \frac{f(x_1,x_2)-f(x_2,x_1)}{x_1-x_2}
\end{array}$$

We have $\mathcal{P}_{\lambda_0} = \bigoplus_{j=0}^{g-2}\mathbf{Q}(x_1+x_2)^j$, $\mathcal{P}_{\lambda_1} = \bigoplus_{j=0}^{g-1}\mathbf{Q}x_2^j$, $\mathcal{P}_{\lambda_2} = \mathbf{Q}$ and $\mathcal{P}_{\lambda_3} = 0$ and the cohomological integrality isomorphism reads

$$\begin{array}{ccc}
\mathcal{P}_{\lambda_0}\oplus(\mathcal{P}_{\lambda_1}\otimes\mathbf{Q}[x_1])\oplus\mathbf{Q}[x_1,x_2]^{\mathrm{sgn}} & \to & \mathbf{Q}[x_1,x_2]^{\mathfrak{S}_2} \\
(f(x_1,x_2),g(x_1,x_2),h(x_1,x_2)) & \mapsto & f(x_1,x_2)+\frac{x_1^gg(x_1,x_2)-x_2^gg(x_2,x_1)}{x_1-x_2}+2x_1^gx_2^g\frac{h(x_1,x_2)}{x_1-x_2}.
\end{array}$$

6.4. $\mathrm{SL}_2(\mathbf{C})\curvearrowright\mathbf{C}^2$

We let $G \coloneqq \mathrm{SL}_2(\mathbf{C})$ act on $\mathbf{C}^2$ via the natural action. It is a symmetric representation. We have $W = \mathfrak{S}_2$. We let $T \subseteq \mathrm{SL}_2(\mathbf{C})$ be the maximal torus of diagonal matrices. We have $\mathscr{P}_V/W = \{\widetilde{\lambda_0},\widetilde{\lambda_1}\}$ where

$$\begin{array}{rccc}
\lambda_0 : & \mathbf{G}_{\mathrm{m}} & \to & T \\
& v & \mapsto & (1,1) \\
\lambda_1 : & \mathbf{G}_{\mathrm{m}} & \to & T \\
& v & \mapsto & (v^{-1},v).
\end{array}$$

We have $V^{\lambda_0} = V$, $V^{\lambda_1} = \{0\}$. We can describe the induction map as follows.

$$\begin{array}{rccc}
\mathrm{Ind}_{\lambda_1,\lambda_0} : & \mathbf{Q}[x] & \to & \mathbf{Q}[x^2] \\
& f & \mapsto & \frac{1}{2}(f(x)+f(-x)).
\end{array}$$

The factor $\frac{1}{2}$ comes from the fact that the T-weight of $\mathfrak{g}^{\lambda_1>0} = \begin{pmatrix}0 & *\\ 0 & 0\end{pmatrix}$ is 2. We have $\mathcal{P}_{\lambda_1} = \mathbf{Q}$ and $\mathcal{P}_{\lambda_0} = \{0\}$ and the cohomological integrality isomorphism is

$$\begin{array}{ccc}
\mathbf{Q}[x^2] & \to & \mathbf{Q}[x^2] \\
f & \mapsto & f.
\end{array}$$

6.5. $\mathrm{SL}_2(\mathbf{C})\curvearrowright\mathbf{C}^d$

Let V_d be the d-dimensional irreducible representation of $\mathrm{SL}_2(\mathbf{C})$. Up to a scaling factor C_d, the induction morphism is

$$\begin{array}{rccc}
\mathrm{Ind}_{\lambda_1,\lambda_0} : & \mathbf{Q}[x] & \to & \mathbf{Q}[x^2] \\
& f & \mapsto & C_dx^{\lfloor\frac{d}{2}\rfloor-1}f(x)+C_d(-x)^{\lfloor\frac{d}{2}\rfloor-1}f(-x).
\end{array}$$

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

We have $\mathcal{P}_{\lambda_1} = \mathbf{Q}$ and $\mathcal{P}_{\lambda_0} = \mathbf{Q}[x^2]_{\deg<2(\lfloor \frac{d}{2} \rfloor -1)}$ where $C_d \in \mathbf{Q}$ is some constant depending only on d (note that $\deg(x) = 2$ in the cohomological grading).

The cohomological integrality isomorphism is

$$\begin{array}{ccc} \mathcal{P}_{\lambda_0} \oplus \mathbf{Q}[x]^{\varepsilon_d} & \to & \mathbf{Q}[x^2] \\ (f,g) & \mapsto & f + 2C_d x^{\lfloor \frac{d}{2} \rfloor -1} g \end{array}$$

where $\mathbf{Q}[x]^{\varepsilon_d} = \mathbf{Q}[x^2]$ if $\lfloor \frac{d}{2} \rfloor - 1$ is even, and $\mathbf{Q}[x]^{\varepsilon_d} = x\mathbf{Q}[x^2]$ if $\lfloor \frac{d}{2} \rfloor - 1$ is odd.

Remark 6.1. We note, for example, in §6.2 and also §6.4, that the integrality morphisms are not isomorphisms when we consider cohomology with integral coefficients instead of rational coefficients. In §6.2, this comes from the presence of the factor 2 in the cohomological integrality morphism.

6.6. $\mathrm{SL}_2(\mathbf{C}) \curvearrowright (\mathfrak{sl}_2(\mathbf{C}))^g$

Let $g \geqslant 0$. We consider the diagonal adjoint action of $\mathrm{SL}_2(\mathbf{C})$ on $V \coloneqq (\mathfrak{sl}_2(\mathbf{C}))^g$. Again, $\mathscr{P}_V = \{\overline{\lambda_0}, \overline{\lambda_1}\}$ with λ_0, λ_1 as in §6.4. The induction map reads

$$\begin{array}{cccc} \mathrm{Ind}_{\lambda_1,\lambda_0} & : & \mathbf{Q}[x] & \to & \mathbf{Q}[x^2] \\ & & f & \mapsto & (2x)^{g-1} f(x) + (-2x)^{g-1} f(-x). \end{array}$$

We have $V^{\lambda_1} \cong \mathbf{C}^g$ with the trivial action of T. Therefore, $\mathcal{P}_{\lambda_1} = \mathbf{Q}$ and $\mathcal{P}_{\lambda_0} = \mathbf{Q}[x^2]_{\deg<2(g-1)}$. The cohomological integrality isomorphism is

$$\begin{array}{ccc} \mathbf{Q}[x]^{\varepsilon_g} \oplus \mathcal{P}_0 & \to & \mathbf{Q}[x^2] \\ (f,g) & \mapsto & g(x) + 2^g x^{g-1} f(x) \end{array}$$

where $\mathbf{Q}[x]^{\varepsilon_g} = \mathbf{Q}[x^2]$ if $g-1$ is even and $\mathbf{Q}[x]^{\varepsilon_g} = x\mathbf{Q}[x^2]$ is $g-1$ is odd.

Remark 6.2. In this example, $G_{\lambda_0} = \{\pm 1\}$ is a finite group. We see that it does not affect the cohomological integrality isomorphism.

6.7. $G \curvearrowright \{0\}$

In this section, we present the cohomological integrality isomorphism for the trivial representation of a reductive group G. Then, $\mathscr{P}_{\{0\}}/W$ is in bijection with the set of conjugacy classes of Levi subgroups of G. If $\lambda \in X_*(T)$ is a general cocharacter, then the induction morphism

$$\mathrm{Ind}_\lambda : \mathrm{H}^*(\mathrm{pt}/T) \to \mathrm{H}^*(\mathrm{pt}/G)$$

is surjective. Indeed, we have

$$\mathrm{Ind}_\lambda(f) = \sum_{w \in W} w \left(\frac{f}{\prod_{\substack{\alpha \in \mathcal{W}(\mathfrak{g}) \\ \langle \lambda, \alpha \rangle < 0}} \alpha} \right).$$

Therefore, $f = \mathrm{Ind}_\lambda \left(\frac{1}{\#W} f \prod_{\substack{\alpha \in \mathcal{W}(\mathfrak{g}) \\ \langle \lambda, \alpha \rangle < 0}} \alpha \right)$.

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

This implies that $\mathcal{P}_\lambda = \{0\}$ if G^λ is not the maximal torus of G and $\mathcal{P}_\lambda = \mathbf{Q}$ if $G^\lambda = T$. In particular, the cohomological integrality reads (for a choice of generic $\lambda \in X_*(T)$):

$$\begin{array}{ccc} \mathbf{Q}[\mathfrak{t}]^{\varepsilon_W} & \to & \mathbf{Q}[\mathfrak{t}]^W \\ f & \mapsto & \dfrac{f}{\prod_{\substack{\alpha\in\mathcal{W}(\mathfrak{g})\,\alpha \\ \langle\lambda,\alpha\rangle<0}}}, \end{array}$$

where $\varepsilon_W(w) = (-1)^{\ell(w)}$ is the sign character of the Coxeter group W; $\ell(w)$ denotes the length of $w \in W$. This morphism is indeed an isomorphism since $\prod_{\substack{\alpha\in\mathcal{W}(\mathfrak{g})\\ \langle\lambda,\alpha\rangle<0}} \alpha$is a generator of the $\mathbf{Q}[\mathfrak{t}]^W$-module of anti-invariant polynomials.

6.8. $G \curvearrowright \mathfrak{g}$

Let G be a reductive group and $\mathfrak{g}$ its Lie algebra with the adjoint action. Then, $\mathscr{P}_\mathfrak{g}$ is in one-to-one correspondence with the set of standard Levi subgroups of G and $\mathscr{P}_\mathfrak{g}/W$ is in bijection with conjugacy classes of Levi subgroups of $\mathfrak{g}$. In this case, $\mathbf{Q}[\mathfrak{t}]^W$ is a polynomial algebra and for $\lambda \preceq \mu \in X_*(T)$, the induction map is

$$\begin{array}{rccc} \mathrm{Ind}_{\lambda,\mu} \quad : & \mathbf{Q}[\mathfrak{t}]^{W^\lambda} & \to & \mathbf{Q}[\mathfrak{t}]^{W^\mu} \\ & f & \mapsto & \sum_{w\in W^\mu/W^\lambda} w\cdot f\,. \end{array}$$

Therefore, the induction maps are surjective. This implies that $\mathcal{P}_\lambda = 0$ except if λ is a generic cocharacter (i.e. $\mathfrak{g}^\lambda$ is a torus), in which case $\mathcal{P}_\lambda = \mathbf{Q}$. The cohomological integrality isomorphism is then

$$\begin{array}{ccc} \mathbf{Q}[\mathfrak{t}]^W & \to & \mathbf{Q}[\mathfrak{t}]^W \\ f & \mapsto & (\#W)\cdot f\,. \end{array}$$

6.9. Topology of the algebra of invariants

Computing the polynomial invariants of binary forms is a long-standing problem in invariant theory, solved for binary forms with small degrees, and still challenging in higher degrees.

The strong version of the sheafified upgrade of our cohomological integrality theorem, which is the subject of a forthcoming work, would give an algorithm to compute the intersection cohomology of the GIT quotients $V/\!/G$ for symmetric representations V of a reductive group G.

Conjecture 6.3 (Strong integrality conjecture). *Let $\lambda \in X_*(T)$. Then, there is a canonical identification $\mathcal{P}_\lambda \cong \mathrm{IH}^*(V^\lambda/\!/G^\lambda)$ if a general closed orbit of G^λ/G_λ inside V^λ has finite stabiliser in G^λ/G_λ, and $\mathcal{P}_\lambda = 0$ otherwise.*

In particular, for $G = \mathrm{SL}_2(\mathbf{C})$, it gives a computation of the intersection cohomology of $V_d/\!/\mathrm{SL}_2$, where V_d is the d-dimensional irreducible representation of $\mathrm{SL}_2(\mathbf{C})$. In this section, we verify this strong integrality conjecture for some representations of $\mathrm{SL}_2(\mathbf{C})$. More precisely, we check that for small values of d, when a general closed orbit in V_d has finite stabilisers, then $\dim \mathrm{IH}(V_d/\!/\mathrm{SL}_2(\mathbf{C})) = \dim \mathcal{P}_0$, where $\mathcal{P}_0$ is the vector space given by Theorem 1.1 for the trivial cocharacter $\lambda = 0 = \lambda_0$.

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

In our situation, motivated by integrality results, we also want to determine the *stable* locus inside V_d, that is, the open subset of closed $\mathrm{SL}_2(\mathbf{C})$-orbits with finite stabiliser.

Remark 6.4. While this paper was under refereeing process, the work [Bu+25] appears. The authors solve Conjecture 6.3 under an orthogonality assumption for the representation V of G: they assume that there is a nondegenerate symmetric G-invariant bilinear form on V. This condition is stronger than symmetricity (and a fortiori, weak symmetricity) we assume in the present paper, as demonstrated for example by the natural representation $\mathbf{C}^{2n}$ of the symplectic group $\mathrm{Sp}(2n)$. This example can be treated by a direct approach. In combination with [Kno07], it is possible to prove Conjecture 6.3 for any symmetric representation V of G such that $\mathbf{C}[V]^G = \mathbf{C}$, that is, $V/\!/G = \mathrm{pt}$.

6.9.1. Invariants of linear binary forms. Let $V_2 = \mathbf{C}^2$ be the natural representation of $\mathrm{SL}_2(\mathbf{C})$. Then, we calculate easily $V_2/\!/\mathrm{SL}_2(\mathbf{C}) = \mathrm{pt}$ since the action of $\mathrm{SL}_2(\mathbf{C})$ on V_2 has two orbits, $\{0\}$ and $V_2 \setminus \{0\}$, of those only $\{0\}$ is closed. The stabiliser of the closed orbit $\{0\}$ is $\mathrm{SL}_2(\mathbf{C})$, which is not finite. Therefore, the stable locus is empty. One expects that $\mathcal{P}_{\lambda_0} = 0$. This is confirmed by §6.4.

6.9.2. Invariants of binary quadratic forms. We let V_3 be the 3-dimensional irreducible representation of $\mathrm{SL}_2(\mathbf{C})$. There is a single polynomial invariant for binary quadratic forms $ax^2 + bxy + cy^2$, which is the discriminant

$$D = b^2 - 4ac,$$

see [Sch13, Satz 1.9, 2.8]. Therefore, we have $V_2/\!/\mathrm{SL}_2(\mathbf{C}) \cong \mathbf{C}$. Since $\dim V_3/\mathrm{SL}_2(\mathbf{C}) = 0$ and $\dim \mathbf{C} = 1$, the stable locus is empty. Therefore, we expect that $\mathcal{P}_{\lambda_0} = 0$, and this is confirmed by §6.5.

6.9.3. Invariants of binary cubic forms. We let V_4 be the 4-dimensional irreducible representation of $\mathrm{SL}_2(\mathbf{C})$. There is one polynomial invariant for binary cubic forms $ax^3 + 3bx^2y + 3cxy^2 + dy^3$, which is the discriminant

$$D = 3b^2c^2 + 6abcd - 4b^3d - 4c^3a - a^2d^2,$$

see [Sch13, Satz 2.8]. Therefore, $V_4/\!/\mathrm{SL}_2(\mathbf{C}) \cong \mathbf{C}$. Moreover, since $\dim V_4/\mathrm{SL}_2(\mathbf{C}) = 1 = \dim \mathbf{C}$, the stable locus is non-empty. By $\mathbf{C}^*$-equivariance, the stable locus is the open locus of binary cubic forms of non-zero discriminant.

We notice that $\dim \mathrm{IH}(\mathbf{C}) = 1 = \dim \mathcal{P}_{\lambda_0} = 1$ where $\mathcal{P}_{\lambda_0}$ is given in §6.5 for $d = 4$.

6.9.4. Invariants of binary quartic forms. Let V_5 be the irreducible 5-dimensional representation of $\mathrm{SL}_2(\mathbf{C})$. The ring of invariants of binary quartic forms has two algebraically independent generators i, j [Sch13, Satz 2.9]. Therefore, $V_5/\!/\mathrm{SL}_2(\mathbf{C}) \cong \mathbf{C}^2$. Moreover, $\dim V_5/\mathrm{SL}_2(\mathbf{C}) = 2 = \dim \mathbf{C}^2$ and therefore, the stable locus is non-empty.

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

In this case, we expect that P_{λ_0} is of dimension $1 = \dim \mathrm{IH}^2(\mathbf{C}^2)$, and this is again confirmed by §6.5 for $d = 5$.

Acknowledgements. At various stages of the preparation of this work, the author was supported by the Royal Society, by the National Science Foundation under Grant No. DMS-1928930 and by the Alfred P. Sloan Foundation under grant G-2021-16778, while the author was in residence at the Simons Laufer Mathematical Sciences Institute (formerly MSRI) in Berkeley, California, during the Spring 2024 semester. The author thanks the SLMath Sciences Institute and the University of Edinburgh for the excellent working conditions. The author is grateful to Ben Davison for the support provided, in particular via postdoctoral fellowships, to Woonam Lim for pointing out a mistake in the example section, to Dragoş Frăţilă for interesting comments, to Sebastian Schlegel Mejia for suggesting improvements and corrections, to Dimitri Wyss for interesting related discussions, and to Andrei Neguţ for suggesting me to investigate weakenings of the symmetricity condition, which led to the current version involving *weakly symmetric* representations (Definition 2.3). I am grateful to Daniel Juteau for discussions on perverse sheaves. I would also like to thank the anonymous referee for allowing me to improve this paper.

Competing interests. The author has no competing interest to declare.

Data availability statement. No data are associated with this study.

References

[AHR20] Alper J, Hall J and Rydh D (2020) A Luna étale slice theorem for algebraic stacks. *Ann Math* **191**(3), 675–738.

[Alp13] Alper J (2013) Good moduli spaces for Artin stacks. *Annales de l'Institut Fourier* **63**(6) 2349–2402.

[BB07] Bourbaki N (2007) *Groupes de Coxeter et systèmes de Tits*. Springer.

[Bia73] Białynicki-Birula A (1973) Some theorems on actions of algebraic groups. *Ann Math* **98**(3), 480–497.

[BJM19] Bussi V, Joyce D and Meinhardt S (2019) On motivic vanishing cycles of critical loci. *J Algebr Geom* **28**, pp. 405–438.

[Bu+25] Bu C, Davison B, Ibáñez Núñez A, Kinjo T and Păduraru T (2025) Cohomology of symmetric stacks. arXiv preprint 2502.04253.

[Bu24] Bu C (2024) A motivic integral identity for (-1)-shifted symplectic stacks. arXiv preprint 2405.10092.

[Dav23] Davison B (2023) A boson-fermion correspondence in cohomological Donaldson–Thomas theory. *Glasg Math J* **65**(S1), S28–S52.

[DC70] Dieudonné JA and Carrell JB (1970) Invariant theory, old and new. *Adv Math* **4**(1), 1–80.

[DHM22] Davison B, Hennecart L and Schlegel Mejia S (2022) BPS Lie algebras for totally negative 2-Calabi-Yau categories and nonabelian Hodge theory for stacks. arXiv preprint 2212.07668.

[DHM23] Davison B, Hennecart L and Schlegel Mejia S (2023) BPS algebras and generalised Kac-Moody algebras from 2-Calabi-Yau categories. arXiv preprint 2303.12592.

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

[DM20a] Davison B and Meinhardt S (2020) Cohomological Donaldson–Thomas theory of a quiver with potential and quantum enveloping algebras. *Invent Math* **221**(3), 777–871.

[DM20b] Digne F and Michel J (2020) *Representations of Finite Groups of Lie Type*, vol. 95. Cambridge University Press.

[Efi12] Efimov AI (2012) Cohomological Hall algebra of a symmetric quiver. *Compos Math* **148**(4), 1133–1146.

[Hal15] Halpern-Leistner D (2015) The derived category of a GIT quotient. *J Am Math Soc* **28**(3), 871–912.

[Hen24] Hennecart L (2024) Cohomological integrality for symmetric quotient stacks. arXiv preprint 2408.15786.

[Hum12] Humphreys JE (2012) *Linear Algebraic Groups*. 4th edition, vol. 21. Springer Science & Business Media.

[Hum90] Humphreys JE (1990) Reflection groups and Coxeter groups, vol. 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press.

[KL15] Kuznetsov A and Lunts VA (2015) Categorical resolutions of irrational singularities. *Int Math Res Notices* **2015**(13), 4536–4625.

[Kno07] Knop F (2007) Invariant functions on symplectic representations. *J Algebra* **313**(1), 223–251.

[KS08] Kontsevich M and Soibelman Y (2008) Stability structures, motivic Donaldson-Thomas invariants and cluster transformations. arXiv preprint 0811.2435.

[KS10] Kontsevich M and Soibelman Y (2010) Motivic Donaldson-Thomas invariants: summary of results. *Contemporary Mathematics*, **527**, American Mathematical Society, 55–89.

[KS11] Kontsevich M and Soibelman Y (2011) Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson–Thomas invariants. *Commun Number Theory Phys* **5**(2), 231–252.

[KV22] Kapranov M and Vasserot E (2022) The cohomological Hall algebra of a surface and factorization cohomology. *J Eur Math Soc* **25**(11), 4221–4289.

[Lun10] Lunts VA (2010) Categorical resolution of singularities. *J Algebra* **323**(10), 2977–3003.

[Lus84] Lusztig G (1984) Intersection cohomology complexes on a reductive group. *Invent Math* **75**(2), 205–272.

[Mei15] Meinhardt S (2015) Donaldson-Thomas invariants vs. intersection cohomology for categories of homological dimension one. arXiv preprint 1512.03343.

[Mei17] Meinhardt S (2017) An introduction to (Motivic) Donaldson-Thomas theory. *Conflu Math* **9**(2), 101–158.

[MFK94] Mumford D, Fogarty J and Kirwan F (1994) *Geometric Invariant Theory*, vol. 34. Springer Science & Business Media.

[MO19] Maulik D and Okounkov A (2019) Quantum groups and quantum cohomology. *Astérisque* **408**, ix+209.

[MR19] Meinhardt S and Reineke M (2019) Donaldson–Thomas invariants versus intersection cohomology of quiver moduli. *J Reine Angew Math* **2019**(754), 143–178.

[Nak94] Nakajima H (1994) Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras. *Duke Math J* **76**(1), 365–416.

[PV94] Popov VL and Vinberg EB (1994) Invariant theory. In: *Algebraic Geometry IV: Linear Algebraic Groups Invariant Theory*. Springer, 123–278.

[Rei08] Reineke M (2008) Moduli of representations of quivers. arXiv preprint 0802.2147.

[Sch13] Schur I (2013) *Vorlesungen über Invariantentheorie*. Vol. 143. Springer-Verlag.

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press

[Ser71] Serre J-P (1971) Représentation linéaire des groupes finis. In: *Hermann, Paris.*

[Sta24] The Stacks Project authors. (2024) *The Stacks Project.* https://stacks.math.columbia.edu.

[ŠV17] Špenko Š and den Bergh MV (2017) Non-commutative resolutions of quotient singularities for reductive groups. *Invent Math* **210**, 3–67.

[ŠV21] Špenko Š and den Bergh MV (2021) Semi-orthogonal decompositions of GIT quotient stacks. *Selecta Math* **27**, 1–43.

[Sze14] Szendroi B (2014) Cohomological Donaldson–Thomas theory. *Proc String Math* 2015, pp. 363–396.

[Van04] Van den Bergh M (2004) Non-commutative crepant resolutions. In: *The Legacy of Niels Henrik Abel: The Abel Bicentennial, Oslo, 2002.* Springer, pp. 749–770.

https://doi.org/10.1017/S1474748025101357 Published online by Cambridge University Press