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TESTING HYPOTHESES ABOUT ABSOLUTE CONCENTRATION CURVES AND MARGINAL CONDITIONAL STOCHASTIC DOMINANCE

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We consider statistical tests concerning various relationships between two absolute concentration curves (ACCs). In particular, we consider tests for determining if the two ACCs coincide, if one is above another in a specified order, or if they do not intersect without specifying which one is above/below the other one. These problems are of interest in the context of marginal conditional stochastic dominance (MCSD). Constructing statistical tests for the MCSD relies on ideas and also on their modifications developed by Linton, Maasoumi, and Whang (2005, *Review of Economic Studies* 72, 735–765) in the context of stochastic dominance for distribution functions. Our theoretical considerations are supplemented with a simulation study.

1. INTRODUCTION

Testing relationships between various population functions is a rapidly developing research area. For up-to-date results and a literature review, we refer to the paper by Linton, Maasoumi, and Whang (2005). As we shall see from our

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limited literature review here, such tests have not been fully explored in the case of absolute concentration curves (ACCs) and hence in the case of marginal conditional stochastic dominance (MCSD). The present paper is a step in this direction.

MCSD was introduced and discussed by Shalit and Yitzhaki (1994) (see also Shalit and Yitzhaki, 2003) as a way of using the stochastic dominance concept in portfolio theory. A typical problem in economics deals with optimization under constraints. Consider a decision maker that maximizes the expected utility $\mathrm{E}[u(X)]$, where u is a utility function, X is wealth, and E is the expectation operator. The utility function u is usually unknown. However, the user is ready to make certain assumptions on its properties. Recall that first-order stochastic dominance (FSD) deals with the set of functions where the first derivative u' is positive, second-order stochastic dominance (SSD) deals with the set of functions such that u' is positive and u'' is negative, and so on. The SSD rule is a very widely used and explored rule because a typical assumption in almost every economic problem is that the marginal utility of income is positive and declining. Shorrocks (1983) proved that X dominates Y according to SSD if and only if the absolute Lorenz curve (ALC) of X is not lower than the ALC of Y. As we shall see later (cf. Appendix A), the class of ALCs is a subclass of all ACCs, and this observation links the SSD concept with the MCSD concept.

Consider a decision maker who maximizes his expected utility $\mathrm{E}[u(X)]$ by choosing the share w_i of his wealth invested in asset i $(i = 1, \ldots, n)$ subject to the constraints $X = X_0 \sum_{i=1}^{n} w_i(1 + r_i)$ and $\sum_{i=1}^{n} w_i = 1$, where r_i are random rates of return. (We note in passing that if short sales are not allowed, then the additional constraints $w_i \geq 0$ should be imposed.) Two algorithms that are capable of solving a restricted version of this problem have been offered by Post (2003) and Kuosmanen (2004). Bodurtha (2006) offers evaluations and improvements of those algorithms.

MCSD rules are intended to solve a simpler problem than the preceding one. Namely, the goal is to see if a given portfolio, say, $\boldsymbol{w}_0 := (w_{01}, \ldots, w_{0n})$, is a candidate to belong to the SSD efficient set. (Throughout the paper, we use $:=$ to denote equality by definition.) Assume that the investor who holds portfolio $\boldsymbol{w}_0$ considers increasing the share of asset k on behalf of asset m by an epsilon. Is there a portfolio $\boldsymbol{w}_1$ in the neighborhood of $\boldsymbol{w}_0$ that dominates $\boldsymbol{w}_0$ according to SSD? Shalit and Yitzhaki (1994) show that such $\boldsymbol{w}_1$ exists if and only if the ACC of asset k with respect to the portfolio $\boldsymbol{w}_0$ is not lower than the ACC of asset m on the entire domain. Mayshar and Yitzhaki (1995) and Shalit and Yitzhaki (2003) extend these rules to include changes in the shares of more than two assets, whereas Yitzhaki and Mayshar (1997) prove that if there is a portfolio $\boldsymbol{w}_2$ not in the neighborhood of $\boldsymbol{w}_0$ that dominates $\boldsymbol{w}_0$, then there must be a portfolio $\boldsymbol{w}_1$ in the neighborhood of $\boldsymbol{w}_0$ that dominates $\boldsymbol{w}_0$. This means that if $\boldsymbol{w}_0$ is found as belonging to the SSD efficient set according to the MCSD, it also belongs to the SSD efficient set when the potential set of portfolios considered is not restricted to the neighborhood of $\boldsymbol{w}_0$.

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In addition to the applications discussed in the aforementioned papers, by now there are several other applications of the MCSD rules in various fields of economics. They include, for example, evaluating tax reforms (Mayshar and Yitzhaki, 1995) where the rules are referred to as Dalton improvement and when externalities are present (Lundin, 2001). Another application is in examining whether a monotonic transformation can change the sign of a regression coefficient (Yitzhaki, 1990) and of the instrumental variable estimator (Yitzhaki and Schechtman, 2004).

In the previously noted papers on the MCSD rules and their applications, essentially no statistical tests have been offered or used, even though in their concluding remarks Shalit and Yitzhaki (1994) express the need for such research. Note that tests for MCSD should, naturally, be derived based on comparing two ACCs. Establishing such a statistical inferential theory for the curves is a challenging task that requires the use of stochastic processes and establishing convergence results for them. Fortunately, the problem has been successfully resolved by Rao and Zhao (1995) and Davydov and Egorov (2000). For details on the topic, refer to Appendix 7 at the end of this paper and also to the references in the aforementioned two papers for related results and discussions.

To discuss several contributions that are available in the literature and that are closely related to the topic of the present paper, we start by describing the main contributions of Shalit and Yitzhaki (1994). Specifically, the authors provide two main results. The first one (see Shalit and Yitzhaki, 1994, Thm. 2) gives a necessary and sufficient condition for MCSD in terms of ACCs. The second result (see Shalit and Yitzhaki, 1994, Thm. 3) gives two necessary conditions for the MCSD in terms of means and so-called co-Ginis. Based on these results, several statistical tests in the area of the MCSD have been constructed in the literature. We discuss them next.

Chow (2001) presents a test for the MCSD based on the necessary and sufficient condition of Shalit and Yitzhaki (1994). Loosely speaking, Chow (2001) bases his test on the differences between the corresponding two empirical ACCs at a finite number of fixed points. Seiler (2001) considers a statistical test for the "second" necessary condition for the MCSD, which is presented in Theorem 3 of Shalit and Yitzhaki (1994). His test is based on estimating the means and the co-Ginis by U-statistics and then obtaining estimates of the asymptotic variances using the jackknife method. Schechtman and Shalit (2006) develop a statistical procedure for simultaneous testing of *both* necessary conditions for the MCSD. They reformulate the problem into the one concerning statistical inference about a two-dimensional vector whose coordinates represent the aforementioned two necessary conditions and then use multivariate statistical tools for further statistical analyses.

The goal of the current paper is to develop statistical procedures for testing the necessary and sufficient conditions for the MCSD as they are formulated in Theorem 2 of Shalit and Yitzhaki (1994). Analogously to Chow (2001), we

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base our tests on functionals of two ACCs. It is worthwhile to note that our functionals are taken over the *entire* domain of definition of the ACCs, whereas Chow (2001) considers different functionals at a *finite* number of (arbitrarily chosen) fixed points. Note in this regard that the latter approach faces consistency problems, whereas the approach presented in the current paper gives consistent tests simultaneously over all the points where the ACCs are defined. Naturally, our approach involves some elements of stochastic and in particular empirical processes, and it therefore requires additional arguments and refinements of the earlier used techniques.

The procedure presented in the current paper can easily be adapted for testing analogous hypotheses about two arbitrary curves for which empirical estimators can be constructed and weak convergence of the appropriately normalized differences between the empirical and the theoretical curves established. In this sense, our paper can be put into a broad context of testing for stochastic dominance in econometrics and actuarial science, whether it is the MCSD as in Shalit and Yitzhaki (1994), or the classical FSD, SSD, etc. (cf., e.g., Barrett and Donald, 2003; Davidson and Duclos, 2000; Eubank, Schechtman, and Yitzhaki, 1993; Horváth, Kokoszka, and Zitikis, 2006; Linton et al., 2005, and references therein).

The rest of the paper is organized as follows. In Section 2 we introduce the main hypotheses and reformulate them in terms of one-dimensional parameters. In Section 3 we construct estimators for the parameters and analyze their asymptotic behavior. Section 4 contains simulation results. Section 5 summarizes the main contributions of the present paper and gives concluding remarks. Results of a technical flavor but essential for the present paper are given in two Appendixes at the end of the paper.

2. HYPOTHESES AND PARAMETERS

Consider a decision maker who owns a portfolio $\boldsymbol{w}_0$, whose value is X. For convenience, in what follows we shall use the same letter X to denote the portfolio and its value, which should not cause any confusion. Let Y and Z represent the rates of return of two assets in the portfolio X. The decision maker is interested in maximizing the expected utility $\mathrm{E}[u(X)]$ by choosing the share w_Y of his wealth invested in the asset with the rate of return Y, the share w_Z of his wealth invested in the asset with the rate of return Z, and the share w_i invested in the asset with the rate of return r_i. This is subjected to the constraint $X = X_0[w_Y(1+Y) + w_Z(1+Z) + \sum_i w_i(1+r_i)]$ with $w_Y + w_Z + \sum_i w_i = 1$. We are interested in testing hypotheses about various relationships between the two ACCs corresponding to the pairs (Y, X) and (Z, X). Rigorous definitions and formulations now follow.

Denote the cumulative distribution function of X by F. Let F^{-1} denote the corresponding quantile function, defined on the interval $[0,1]$ by the equation $F^{-1}(t) := \inf\{x : F(x) \geq t\}$, with $F^{-1}(0) = -\infty$ by definition. Assume that the

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random variables Y and Z have finite first moments. Then the corresponding two ACCs are defined on the interval $[0,1]$ by

$$A(t) := \mathrm{E}[Y 1_{[X \le F^{-1}(t)]}] \quad \text{and} \quad B(t) := \mathrm{E}[Z 1_{[X \le F^{-1}(t)]}],$$

where $1_{[X \le F^{-1}(t)]}$ equals 1 if $X \le F^{-1}(t)$ and 0 otherwise. Despite simple mathematical expressions, the ACCs are nevertheless complex objects in terms of their geometry and econometric interpretations. Some of the most important properties are discussed in Appendix A at the end of this paper. There we also find relationships of the ACCs with the ALCs, which make up an important class of curves widely used in measuring economic inequality and also in other areas of application.

We may want to know if the two curves coincide (i.e., $A = B$) or if there is dominance in a particular direction, say, $A \le B$. We might also be interested in testing if there is dominance in any direction, without specifying the direction explicitly. Note that rejecting the latter (two-sided) null hypothesis means accepting the claim that the ACCs A and B intersect. By their nature, such hypotheses are complex, because they involve comparisons of $A(t)$ and $B(t)$ over the values $t \in [0,1]$. It is therefore natural to aim at reformulating the hypotheses in a simpler way, using a single parameter τ such that the null hypothesis would become $\tau = 0$ and the alternative $\tau > 0$. The construction of τ, naturally, depends on the hypotheses considered, and it may not be unique. For helpful ideas on the topic we refer to the paper by Linton et al. (2005), which also influenced the present paper. Details on the topic follow next.

With the help of the parameter

$$s(h) := \sup_{0 \le t \le 1} h(t),$$

we obtain four mutually exclusive cases, written as (i)–(iv) in Table 1, that we use for constructing the aforementioned parameter τ, depending on the hypotheses.

For example, suppose that we are interested in testing the hypotheses

$$\begin{cases} H_{01}: A(t) = B(t) \quad \forall\, t \in [0,1], \\ H_{11}: \exists\, t \in [0,1]: A(t) \ne B(t). \end{cases} \tag{1}$$

Table 1. Analyzing hypotheses with the help of $s(h)$

Cases	$s(A-B)$	$s(B-A)$
(i) $A(t) = B(t)\ \forall\ t \in [0,1]$	$= 0$	$= 0$
(ii) $A(t) \le B(t)\ \forall\ t \in [0,1]$ and $\exists\ t_1 : A(t_1) < B(t_1)$	$= 0$	> 0
(iii) $A(t) \ge B(t)\ \forall\ t \in [0,1]$ and $\exists\ t_2 : A(t_2) > B(t_2)$	> 0	$= 0$
(iv) $\exists\ t_1, t_2 \in [0,1]: A(t_1) < B(t_1)$ and $A(t_2) > B(t_2)$	> 0	> 0

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(We simply write H_{01} as $A = B$.) Table 1 suggests using the parameter $\tau_1 := s(A - B) \vee s(B - A)$ and reformulating H_{01} and H_{11} as $H_{01}: \tau_1 = 0$ and $H_{11}: \tau_1 > 0$. (We use $x \vee y$ to denote the maximum between two numbers x and y.) Note the equality $\tau_1 = \|A - B\|_\infty$, where the right-hand side denotes the supremum of $|A(t) - B(t)|$ over all $0 \leq t \leq 1$. Hence, as we shall explicitly see in the next section, the therein constructed estimator of τ_1 is a Kolmogorov–Smirnov type statistic.

We might be interested in testing hypotheses concerning one-sided dominance:

$$\begin{cases} H_{02}: A(t) \leq B(t) \quad \forall\, t \in [0,1], \\ H_{12}: \exists\, t_0 \in [0,1]: A(t_0) > B(t_0). \end{cases} \tag{2}$$

(We simply write H_{02} as $A \leq B$.) In this case, Table 1 suggests using the parameter $\tau_2 := s(A - B)$, which in turn leads to the following reformulation of the hypotheses in (2) as $H_{02}: \tau_2 = 0$ and $H_{12}: \tau_2 > 0$. Of course, we might also be interested in testing hypotheses such as

$$\begin{cases} H_{01}: A(t) = B(t) \quad \forall\, t \in [0,1], \\ H_{12}: \exists\, t_0 \in [0,1]: A(t_0) > B(t_0). \end{cases} \tag{3}$$

(Note the inclusion $H_{01} \subseteq H_{02}$.) In this case we still use the parameter τ_2 to reformulate the hypotheses in (3) as $H_{01}: \tau_2 = 0$ and $H_{12}: \tau_2 > 0$.

The parameter τ_2 does not, of course, work when testing the hypotheses

$$\begin{cases} H_{01}: A(t) = B(t) \quad \forall\, t \in [0,1], \\ H_{13}: \exists\, t_1, t_2 \in [0,1]: A(t_1) < B(t_1) \quad \text{and} \quad A(t_2) > B(t_2). \end{cases} \tag{4}$$

Table 1 suggests using the parameter $\tau_3 := s(A - B) \wedge s(B - A)$ for the purpose (cf. Linton et al., 2005), which allows us to reformulate the hypotheses in (4) as $H_{01}: \tau_3 = 0$ and $H_{13}: \tau_3 > 0$. The reformulation is not unique, as we can also use, for example, the parameter $\tau_4 := s(A - B) \times s(B - A)$ and reformulate the hypotheses in (4) as $H_{01}: \tau_4 = 0$ and $H_{13}: \tau_4 > 0$.

Finally, the MCSD rules are directly linked to the following hypotheses:

$$\begin{cases} H_{03}: \text{either } A(t) \leq B(t) \quad \forall\, t \in [0,1] \quad \text{or} \quad A(t) \geq B(t) \quad \forall\, t \in [0,1], \\ H_{13}: \exists\, t_1, t_2 \in [0,1]: A(t_1) < B(t_1) \quad \text{and} \quad A(t_2) > B(t_2). \end{cases} \tag{5}$$

(Note the inclusion $H_{01} \subseteq H_{03}$.) A clarification of the hypotheses in (5) is needed, especially with respect to those formulated in (2). Namely, in (2) we want to know if the specified directional dominance $A \leq B$ holds (null) or not (alternative). In (5), we actually want to know if the two ACCs intersect (alternative) or not (null). In view of the fact that critical values of tests are determined by their null hypotheses, we have therefore reversed the aforementioned alternative and null hypotheses, thus arriving at the formulation in (5).

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With the help of the earlier defined parameter τ_3, we reformulate the hypotheses in (5) as $H_{03}: \tau_3 = 0$ and $H_{13}: \tau_3 > 0$. Instead of τ_3, we could have also used the parameter τ_4. Despite the equivalent reformulations of hypotheses (4) and (5) in terms of the parameter τ_3 or τ_4, we shall later see how the difference between the null hypotheses manifests itself in the definitions of critical values.

3. ESTIMATORS AND THEIR ASYMPTOTIC BEHAVIOR

The earlier introduced four parameters τ_k are not known. Hence, we estimate them empirically. To this end, we first construct two empirical ACCs, A_n and B_n, as follows. Let $(Z_1, Y_1, X_1), (Z_2, Y_2, X_2), \ldots, (Z_n, Y_n, X_n)$ be independent copies of the triplet (Z, Y, X). Then

$$A_n(t) := \frac{1}{n} \sum_{i=1}^{n} Y_i 1_{[X_i \le F_n^{-1}(t)]} \quad \text{and} \quad B_n(t) := \frac{1}{n} \sum_{i=1}^{n} Z_i 1_{[X_i \le F_n^{-1}(t)]},$$

where F_n^{-1} is the empirical quantile function based on $X_1, X_2, \ldots, X_n$. That is, we have the equation $F_n^{-1}(t) = \inf\{x : F_n(x) \ge t\}$ for all $0 < t \le 1$, where F_n is the empirical distribution function based on $X_1, X_2, \ldots, X_n$. For further properties of empirical ACCs and also for computationally convenient formulas, refer to Appendix A at the end of this paper. We are now ready to define estimators for the earlier introduced parameters τ_k:

$$\hat{\tau}_2 := s(A_n - B_n) \quad \text{and also } \hat{\tau}_{-2} := s(B_n - A_n),$$

$$\hat{\tau}_1 := \hat{\tau}_2 \vee \hat{\tau}_{-2} \quad (= \|A_n - B_n\|_\infty),$$

$$\hat{\tau}_3 := \hat{\tau}_2 \wedge \hat{\tau}_{-2},$$

$$\hat{\tau}_4 := \hat{\tau}_2 \times \hat{\tau}_{-2}.$$

The statistic $\hat{\tau}_1$ is a (two-sided) Kolmogorov–Smirnov type statistic, whereas $\hat{\tau}_2$ and $\hat{\tau}_{-2}$ are one-sided Kolmogorov–Smirnov type statistics. The statistic $\hat{\tau}_3$ is a simple adaptation of a statistic proposed and thoroughly investigated by Linton et al. (2005).

Establishing asymptotic behavior of the appropriately normalized four statistics $\hat{\tau}_k$ relies on the weak convergence result

$$D_n = \sqrt{n}((A_n - B_n) - (A - B)) \Rightarrow \Gamma, \tag{6}$$

where Γ is a Gaussian process (cf. Appendix B for details).

Statistic $\sqrt{n}\hat{\tau}_1$. Under the null hypothesis H_{01}, statement (6) implies that $\sqrt{n}\hat{\tau}_1 \to_d \|\Gamma\|_\infty$. With $x_{\alpha,1}$ such that $\mathbf{P}[\|\Gamma\|_\infty \ge x_{\alpha,1}] = \alpha$, we have that

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$$\lim_{n\to\infty} \mathbf{P}[\sqrt{n}\hat{\tau}_1 \geq x_{\alpha,1}] = \alpha. \tag{7}$$

Under the alternative H_{11}, we employ statement (6) once again and obtain that $\sqrt{n}\hat{\tau}_1 \to_{\mathbf{P}} +\infty$. Hence, the test statistic $\sqrt{n}\hat{\tau}_1$ has asymptotic power 1.

Statistic $\sqrt{n}\hat{\tau}_2$. Under the null hypothesis H_{02}, which means $A \leq B$, we have the bound $\sqrt{n}\hat{\tau}_2 \leq s(D_n)$, whose right-hand side converges in distribution to $s(\Gamma)$. In the case of H_{01}, the bound becomes equality. With $x_{\alpha,2}$ such that $\mathbf{P}[s(\Gamma) \geq x_{\alpha,2}] = \alpha$, we therefore have that

$$\limsup_{n\to\infty} \mathbf{P}[\sqrt{n}\hat{\tau}_2 \geq x_{\alpha,2}] \begin{cases} = \alpha & \text{in the case of } H_{01}, \\ \leq \alpha & \text{in the case of } H_{02}. \end{cases} \tag{8}$$

(Note that $H_{01} \subseteq H_{02}$.) The difference that we see between the null hypotheses H_{01} and H_{02} in statement (8) is natural. Indeed, the equality $A = B$ is the most "confusing" for the test statistic $\sqrt{n}\hat{\tau}_2$, and thus gives the approximate proportion α of Type I errors, whereas the null hypothesis H_{02} includes curves such that $A(t_1) < B(t_1)$ for at least one point t_1, in which case we naturally expect to make fewer Type I errors than in the case $A = B$.

To show that the asymptotic power of the test statistic $\sqrt{n}\hat{\tau}_2$ is 1, we need to check that under the alternative H_{12}, the statistic tends to $+\infty$ in probability. To this end we first note that $\sqrt{n}\hat{\tau}_2$ can be bounded from below by $\sqrt{n}(A(t_2) - B(t_2)) + D_n(t_2)$, where the point t_2 is specified in Table 1. In view of statement (6), the random variable $D_n(t_2)$ has a nondegenerate limiting distribution, whereas the quantity $\sqrt{n}(A(t_2) - B(t_2))$ tends to $+\infty$ when $n \to \infty$, because $A(t_2) > B(t_2)$. Hence, the asymptotic power of $\sqrt{n}\hat{\tau}_2$ is 1.

Statistics $\sqrt{n}\hat{\tau}_3$ and $n\hat{\tau}_4$. When $A = B$, then $\sqrt{n}\hat{\tau}_3$ is $s(D_n) \wedge s(-D_n)$. Consequently, $\sqrt{n}\hat{\tau}_3$ converges in distribution to $s(\Gamma) \wedge s(-\Gamma)$. That is, with $x_{\alpha,3}$ such that $\mathbf{P}[s(\Gamma) \wedge s(-\Gamma) \geq x_{\alpha,3}] = \alpha$, we have that

$$\lim_{n\to\infty} \mathbf{P}[\sqrt{n}\hat{\tau}_3 > x_{\alpha,3}] = \alpha. \tag{9}$$

Similarly, assuming $A = B$ and using $x_{\alpha,4}$ such that $\mathbf{P}[s(\Gamma) \times s(-\Gamma) \geq x_{\alpha,4}] = \alpha$, we have that

$$\lim_{n\to\infty} \mathbf{P}[n\hat{\tau}_4 > x_{\alpha,4}] = \alpha.$$

Under the alternative H_{13}, the test statistics $\sqrt{n}\hat{\tau}_3$ and $n\hat{\tau}_4$ have the asymptotic power 1. Indeed, $s(-D_n - \sqrt{n}(A - B))$ tends to $+\infty$ in probability because $A(t_1) < B(t_1)$. Likewise, $s(D_n + \sqrt{n}(A - B))$ tends to $+\infty$ in probability because $A(t_2) > B(t_2)$. Hence, under the alternative hypothesis H_{13}, both $\sqrt{n}\hat{\tau}_3$ and $n\hat{\tau}_4$ tend to $+\infty$ in probability. This concludes our discussion concerning H_{01} versus H_{13}.

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The case H_{03} versus H_{13} is more complex, and we note at the outset that the equality $A = B$ (i.e., H_{01}) might not be the most "confusing" case when testing these hypotheses. We shall next clarify the situation. Let $A \leq B$ and $A(t_1) < B(t_1)$ for some t_1. Rewrite the statistic $\sqrt{n}\hat{\tau}_3$ as the minimum between $s(D_n + \sqrt{n}(A - B))$ and $s(-D_n - \sqrt{n}(A - B))$. Note that $s(D_n + \sqrt{n}(A - B))$ does not exceed $s(D_n)$ and the latter converges in distribution to $s(\Gamma)$. Furthermore, $s(-D_n - \sqrt{n}(A - B))$ converges to $+\infty$ in probability because of the existence of t_1 as specified previously. Hence, with $y_{\alpha,3}$ such that $\mathbf{P}[s(\Gamma) \geq y_{\alpha,3}] = \alpha$, we have that

$$\limsup_{n\to\infty} \mathbf{P}[\sqrt{n}\hat{\tau}_3 > y_{\alpha,3}] \leq \alpha. \tag{10}$$

Analogous considerations show that when $A \geq B$ and $A(t_2) > B(t_2)$ for some t_2, then

$$\limsup_{n\to\infty} \mathbf{P}[\sqrt{n}\hat{\tau}_3 > z_{\alpha,3}] \leq \alpha, \tag{11}$$

where $z_{\alpha,3}$ is such that $\mathbf{P}[s(-\Gamma) \geq z_{\alpha,3}] = \alpha$. The three bounds in (9)–(11) cover all the cases under the null hypothesis H_{03}. Because we do not know which of the three cases may actually happen in practice, we thus combine the three cases into one, which results in the earlier defined critical value $x_{\alpha,1}$. Indeed, this is a consequence of the fact that $\|\Gamma\|_\infty$ is the smallest upper bound for the three random variables $s(\Gamma) \wedge s(-\Gamma)$, $s(\Gamma)$, and $s(-\Gamma)$. Hence, $x_{\alpha,1}$ estimates $x_{\alpha,3}$, $y_{\alpha,3}$, and $z_{\alpha,3}$ from before. In view of this, it is only natural that for any combination of the ACCs A and B from the alternative H_{03}, the critical value $x_{\alpha,1}$ will likely be an "overestimation," which makes the corresponding test conservative in the sense that the proportion of Type I errors will be smaller than the specified α. The test, of course, has the asymptotic power 1.

Estimating critical values. The critical values defined previously are not distribution free and therefore need to be estimated empirically. Recall that they are defined in terms of functionals of the process Γ, which looks like D_n in view of the weak convergence result in (6). We shall now discuss how to imitate the process Γ using bootstrap, which we shall in turn use for estimating the critical values of the tests discussed earlier. We start with the note that the (unknown) distribution of the triplet (Z, Y, X) can be approximated by the (known) empirical distribution of the data $(Z_1, Y_1, X_1), (Z_2, Y_2, X_2), \ldots, (Z_n, Y_n, X_n)$. Hence, we imitate the process D_n by artificially simulating n new triplets (Z_1^*, Y_1^*, X_1^*), $(Z_2^*, Y_2^*, X_2^*), \ldots, (Z_n^*, Y_n^*, X_n^*)$ from the original triplets $(Z_1, Y_1, X_1), (Z_2, Y_2, X_2), \ldots, (Z_n, Y_n, X_n)$ using sampling with replacement. Then we construct the corresponding "asterisked" ACCs:

$$A_n^*(t) := \frac{1}{n}\sum_{i=1}^{n} Y_i^* 1_{[X_i^* \leq (F_n^*)^{-1}(t)]} \quad \text{and} \quad B_n^*(t) := \frac{1}{n}\sum_{i=1}^{n} Z_i^* 1_{[X_i^* \leq (F_n^*)^{-1}(t)]}.$$

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The $(F_n^*)^{-1}$ in the preceding formulas is the empirical quantile function corresponding to the distribution function F_n^*, which is based on $X_1^*, X_2^*, \ldots, X_n^*$. It is now natural to expect that the (unobservable) process D_n looks like the (observable) process

$$D_n^* = \sqrt{n}((A_n^* - B_n^*) - (A_n - B_n)). \tag{12}$$

We use the process D_n^* to estimate the desired critical values as follows. Suppose that we want to estimate $x_{\alpha,1}$. We define an estimator $x_{\alpha,1}^*$ by the following procedure. We generate n asterisked triplets M times and obtain M values of $\|D_n^*\|_\infty$. Based on these values, we define $x_{\alpha,1}^*$ as the smallest x such that the proportion of those M values of $\|D_n^*\|_\infty$ that are on or above x is at most α. Using the just defined $x_{\alpha,1}^*$, we reject H_{01} if $\sqrt{n}\hat{\tau}_1 > x_{\alpha,1}^*$ and retain H_{01} otherwise. Analogous considerations are valid for the remaining three test statistics $\sqrt{n}\hat{\tau}_2$, $\sqrt{n}\hat{\tau}_3$, and $n\hat{\tau}_4$.

4. A SIMULATION STUDY

We start with checking empirical significance levels of the four test statistics $\sqrt{n}\hat{\tau}_k$, $k = 1,2,3$, and $n\hat{\tau}_4$. We set two values for the theoretical significance level: $\alpha = 0.05$ and 0.10. The simulation procedure has seven steps, as follows:

1. We assume that the three assets Z, Y, and W have the same normal distribution $\mathcal{N}(\mu,\sigma)$ with mean $\mu = 5$ and standard deviation $\sigma = 1$, and then we define the portfolio X by the equation $X = 0.2Y + 0.2Z + 0.6W$. (The means of the rates of return were chosen to be 5%, because this is the rate of return on a risk free asset, which is commonly used in the literature.)
2. We generate n independent copies (Z_i, Y_i, X_i) of (Z, Y, X) and produce the empirical ACCs A_n and B_n corresponding to (Z_i, X_i)'s and (Y_i, X_i)'s, respectively. We calculate the values of the four statistics $\sqrt{n}\hat{\tau}_k$, $k = 1,2,3$, and $n\hat{\tau}_4$. The sample sizes: $n = 50$, 100, 200, 500, 1,000, and 2,000.
3. We sample with replacement from the original sample of (Z_i, Y_i, X_i)'s and obtain a new sample of (Z_i^*, Y_i^*, X_i^*)'s of size n and produce the corresponding two empirical ACCs denoted by A_n^* and B_n^*. We calculate $T_{n,1}^* := \|D_n^*\|_\infty$, $T_{n,2}^* := s(D_n^*)$, $T_{n,3}^* := s(D_n^*) \wedge s(-D_n^*)$, and $T_{n,4}^* := s(D_n^*) \times s(-D_n^*)$. The sampling distribution of $T_{n,k}^*$ is expected to mimic the limiting distribution of the corresponding test statistic under $H_{0,1}$.
4. We repeat the resampling procedure $M := 500$ times and obtain M values of $T_{n,k}^*$. Then we calculate the $(1 - \alpha) \times 100$th percentile of the M values. The percentile defines the critical value, denoted by $x_{\alpha,k}^*$, of the test.
5. We check if $\sqrt{n}\hat{\tau}_k > x_{\alpha,k}^*$ when $k = 1,2,3$, or $n\hat{\tau}_4 > x_{\alpha,4}^*$.
6. We repeat the preceding steps $Q := 5{,}000$ times and calculate the proportions of those instances when the inequalities in step (5) hold.

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TABLE 2. Empirical significance levels averaged over 10 experiments, with $(\mu_Z, \mu_Y, \mu_W) = (5,5,5)$ and $(\sigma_Z, \sigma_Y, \sigma_W) = (1,1,1)$

	$\hat{\tau}_1$		$\hat{\tau}_2$		$\hat{\tau}_3$		$\hat{\tau}_4$	
	5%	10%	5%	10%	5%	10%	5%	10%
50	0.0454	0.0875	0.0456	0.0885	0.0191	0.0429	0.0192	0.0409
100	0.0442	0.0846	0.0440	0.0851	0.0181	0.0417	0.0175	0.0396
200	0.0433	0.0840	0.0439	0.0860	0.0209	0.0458	0.0203	0.0440
500	0.0456	0.0868	0.0437	0.0866	0.0218	0.0493	0.0233	0.0483
1,000	0.0445	0.0854	0.0435	0.0851	0.0257	0.0544	0.0260	0.0533
2,000	0.0440	0.0877	0.0441	0.0884	0.0297	0.0607	0.0288	0.0578

7. To achieve stability in the final results, we repeat the preceding steps $R := 10$ times and average the corresponding proportions. The findings are reported in Table 2.

We see from Table 2 that the statistics $\hat{\tau}_1$ and $\hat{\tau}_2$ exhibit a somewhat conservative nature, whereas the statistics $\hat{\tau}_3$ and $\hat{\tau}_4$ are more noticeably conservative. This pattern of "conservatism" also holds for larger sample sizes (e.g., $n = 10{,}000$) and also for larger values of M or Q. In this regard, we may recall the mathematical analyses in Section 3 where the theoretical (true) levels are sometimes equal to, and sometimes smaller than, the specified α, depending on the hypotheses considered. Specifically, when discussing the statistics $\hat{\tau}_1$ and $\hat{\tau}_2$, we noted there that the case $A = B$ is the most "confusing" for the two statistics. In the case of the statistics $\hat{\tau}_3$ and $\hat{\tau}_4$, the picture is more complex (see Section 3).

We now discuss the empirical powers of the four tests (cf. Table 3). The case when the two ACCs A and B intersect (i.e., $A(t_1) < B(t_1)$ and $A(t_2) > B(t_2)$ for some $t_1, t_2 \in [0,1]$) is a part of all the three alternatives H_{11}, H_{12}, and H_{13}. Hence, we investigate the power of the four tests by constructing intersecting ACCs with various levels of intersection, as shown in Figure 1. We see from the figure that the larger the standard deviation of the asset Z is, the more distant the case is from the equality $A = B$; hence, it becomes easier for the tests to identify the case correctly. For each test, we use two significance levels, $\alpha = 0.05$ and 0.1. The critical values of the four tests are calculated based on, respectively, $\|D_n^*\|_\infty$, $s(D_n^*)$, $s(D_n^*) \wedge s(-D_n^*)$, and $s(D_n^*) \times s(-D_n^*)$. Our findings are reported in Table 3. As expected, we see from the table that, for a given sample size and when proceeding from scenario (a) to scenario (d), the power of each test statistic generally increases. This observation is consistent with the fact that the greater the standard deviation, the easier it is to detect the intersection. When the sample size n exceeds 500, the power of each statistic becomes fairly

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TABLE 3. Empirical powers averaged over 10 experiments

	$\hat{\tau}_1$		$\hat{\tau}_2$		$\hat{\tau}_3$		$\hat{\tau}_4$	
	5%	10%	5%	10%	5%	10%	5%	10%
(a) $(\mu_Z, \mu_Y, \mu_W) = (5.5, 5.0, 5.0)$ and $(\sigma_Z, \sigma_Y, \sigma_W) = (2.0, 1.0, 1.0)$								
50	0.2592	0.3596	0.3380	0.4500	0.2900	0.3814	0.4461	0.5523
100	0.4758	0.5976	0.5708	0.6866	0.5331	0.6202	0.8059	0.8632
200	0.8137	0.9029	0.8676	0.9193	0.7730	0.8331	0.9800	0.9868
500	0.9992	0.9985	0.9677	1.0000	0.9999	0.9995	0.9821	1.0000
(b) $(\mu_Z, \mu_Y, \mu_W) = (5.5, 5.0, 5.0)$ and $(\sigma_Z, \sigma_Y, \sigma_W) = (2.5, 1.0, 1.0)$								
50	0.1930	0.3139	0.2333	0.3342	0.4711	0.5528	0.5987	0.6770
100	0.3985	0.5956	0.4218	0.5430	0.7638	0.8131	0.8859	0.9075
200	0.8279	0.9536	0.7186	0.8145	0.9558	0.9668	0.9859	0.9880
500	1.0000	1.0000	0.9838	0.9932	0.9996	0.9996	0.9999	0.9999
(c) $(\mu_Z, \mu_Y, \mu_W) = (5.5, 5.0, 5.0)$ and $(\sigma_Z, \sigma_Y, \sigma_W) = (3.0, 1.0, 1.0)$								
50	0.2050	0.3734	0.1767	0.2656	0.5378	0.5950	0.6440	0.6947
100	0.5122	0.7659	0.3235	0.4406	0.7663	0.7972	0.8612	0.8762
200	0.9658	0.9972	0.5826	0.7025	0.9248	0.9366	0.9690	0.9726
500	1.0000	1.0000	0.9378	0.9678	0.9967	0.9975	0.9993	0.9993
(d) $(\mu_Z, \mu_Y, \mu_W) = (5.5, 5.0, 5.0)$ and $(\sigma_Z, \sigma_Y, \sigma_W) = (3.5, 1.0, 1.0)$								
50	0.2820	0.4945	0.1411	0.2200	0.5300	0.5739	0.6312	0.6681
100	0.7137	0.9195	0.2553	0.3630	0.7155	0.7443	0.8236	0.8374
200	0.9848	1.0000	0.4761	0.5972	0.8787	0.8953	0.9470	0.9511
500	1.0000	1.0000	0.8633	0.9215	0.9907	0.9925	0.9977	0.9978

close to 1, and we therefore do not report the results for larger sample sizes in the table. It is also interesting to note (see Table 3) that in many instances the statistics $\hat{\tau}_3$ and $\hat{\tau}_4$ are more powerful in rejecting the specified intersecting cases than the statistics $\hat{\tau}_1$ and $\hat{\tau}_2$. The advantage is more noticeable at relatively small sample sizes.

We conclude the discussion with a note that the herein presented simulation study is a part of a larger study (by Shelef, 2006) that, in addition to the normal distribution, also includes beta, log-normal, and other distributions of interest. The results are in line with those presented here for the normal distribution, and we therefore do not include them in the present paper.

5. SUMMARY AND CONCLUDING REMARKS

The notion of marginal conditional stochastic dominance, as introduced and discussed by Shalit and Yitzhaki (1994), can be reformulated in terms of dom-

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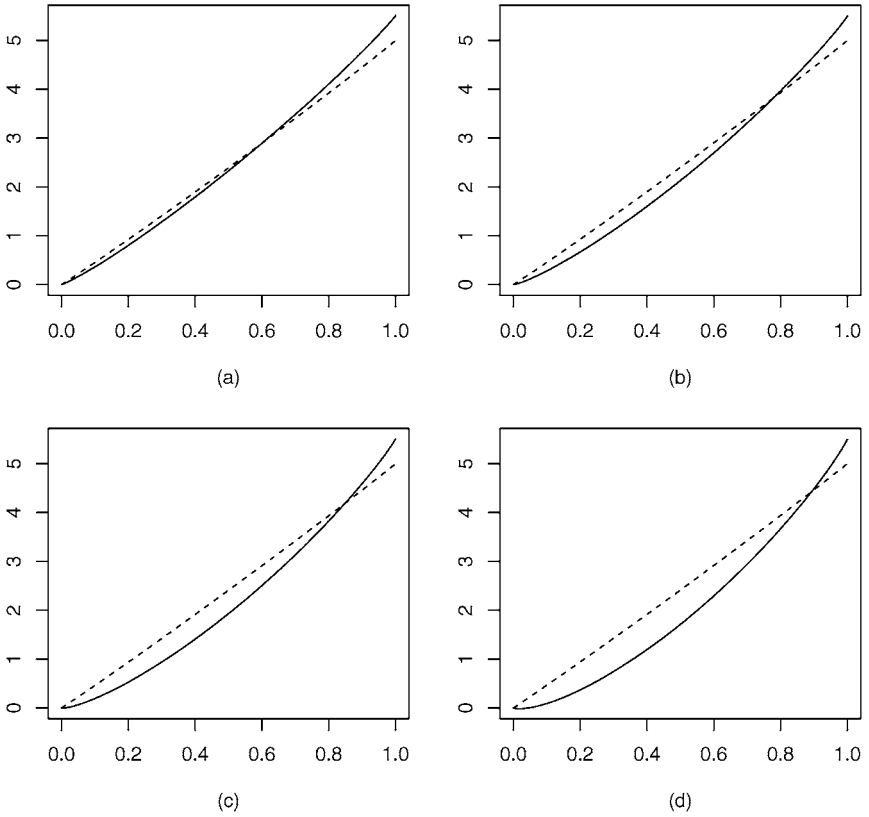


FIGURE 1. ACCs A (solid) and B (small breaks) with (μ_Z, μ_Y, μ_W) and $(\sigma_Z, \sigma_Y, \sigma_W)$ as specified by (a)–(d) in Table 3.

inance of two absolute concentration curves. In the present paper we have considered statistical procedures for testing various hypotheses about two ACCs, including statistical tests for checking if (1) the two ACCs coincide; (2) one is below another, in a specified order; or (3) they do not intersect (no specification of the order of dominance). Because the tests are based on functionals of empirical curves over the *entire* domain $[0,1]$ of their definition, the tests avoid the consistency problems that are inevitably present when using the approaches available in the literature that are based on finite fixed partitions of the interval $[0,1]$.

The herein investigated statistical tests can easily be adapted in other areas of applied and theoretical research where it is of interest to test for equality, one-sided dominance, or two-sided dominance in the earlier noted senses. For example, with some obvious modifications, the statistical procedures can be

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used for testing hypotheses about two absolute or, if desired, relative Lorenz curves.

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APPENDIX A: Properties of ACCs

In this section we have collected several notes concerning the theoretical and empirical ACCs. The notes are intended not only to clarify the geometry of the curves but also to explain the construction of, and possible differences between, expressions of the ACCs and related curves and also their estimators.

Note 1. When the distribution function F is continuous, then the ACC A is also continuous. Moreover, in this case we have (see equation (A.2), which follows) that $A(t)$ equals $\int_0^t \mu_{Y|X}(F^{-1}(u))\,du$, where $\mu_{Y|X}(x) = \mathrm{E}[Y|X = x]$.

Note 2. When the random variables Y and X are independent and the distribution function F of X is continuous, then the curve A coincides with the straight line connecting the points $(0,0)$ and $(1,\mathrm{E}[Y])$.

Note 3. If Y is nonnegative, then the function A is nondecreasing. If, however, the random variable Y can also take on negative values, then A might not be nondecreasing.

Note 4. The function A may or may not be convex. If the regression function $\mu_{Y|X}$ is nondecreasing, then A is convex.

Note 5. Because $A(t)$ equals $\mathrm{E}[\mu_{Y|X}(X)1_{[X\le F^{-1}(t)]}]$, we have the equation

$$A(t) = \int_{(-\infty, F^{-1}(t)]} \mu_{Y|X}(x)\,dF(x). \tag{A.1}$$

On the other hand, if we assume that the distribution function F is *continuous*, then $A(t)$ equals $\mathrm{E}[\mu_{Y|X}(F^{-1}(U))1\{U \le t\}]$, where $U = F(X)$ is a uniform on $[0,1]$ random variable. Hence, $A(t) = A^{\circ}(t)$, where

$$A^{\circ}(t) = \int_0^t \mu_{Y|X}(F^{-1}(u))\,du. \tag{A.2}$$

Formulas (A.1) and (A.2) give two representations for ACCs. As we shall see in the next note, the continuity assumption is necessary for the equality $A = A^{\circ}$.

Note 6. When $Y = X$, then $\mu_{Y|X}(x) = x$, and so we have that $A(t)$ equals $\int_{(-\infty, F^{-1}(t)]} x dF(x)$. The latter integral defines a curve on $[0,1]$ that is sometimes called the absolute Lorenz curve (ALC). It should be noted, however, that the usual definition of the ALC—more frequently known as the *generalized* Lorenz curve (cf. Shorrocks, 1983; Sen, 1997, pp. 135–138)—is as follows:

$$ALC(t) = \int_0^t F^{-1}(u)\,du. \tag{A.3}$$

This shows that the results concerning ACCs given in the present paper can easily be adjusted for constructing tests about two ALCs. The aforementioned continuity assump-

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tion on F is necessary for having the equality $A = A^{\circ}$. To show this, it is sufficient to consider the case when the portfolio has only one asset. Namely, let $Y_i = X_i$ for all i. Then $A_n(t)$ equals $(1/n)\sum_{i=1}^{k} X_{i:n}$ for all $t \in ((k-1)/n, k/n]$, where $X_{1:n} \leq \cdots \leq X_{n:n}$ are the order statistics of $X_1, \ldots, X_n$. In other words, the integral $\int_{(-\infty, F_n^{-1}(t)]} x dF_n(x)$ equals $(1/n)\sum_{i=1}^{k} X_{i:n}$ for the specified values of t. On the other hand, the integral $\int_0^t F_n^{-1}(s)\, ds$ equals $(1/n)\sum_{i=1}^{k-1} X_{i:n} + (t - ((k-1)/n))X_{k:n}$. Taking these facts together, we arrive at the equation

$$\int_{(-\infty, F_n^{-1}(t)]} x dF_n(x) = \int_0^t F_n^{-1}(s)\, ds + \left(\frac{k}{n} - t\right) X_{k:n}.$$

Hence, the difference between the two integrals in the preceding equation is not, generally, equal to zero, which in turn shows that the difference between A and A° may not be negligible for small sample sizes n. These issues have been discussed in detail by Schechtman and Zitikis (2006).

Note 7. When working with empirical ACCs in a simulation study or practice, it is convenient to rewrite them in terms of so-called induced order statistics. Consider the curve A_n. For every $k = 1, \ldots, n$ and $t \in ((k-1)/n, k/n]$, we have the equations

$$A_n(t) = \frac{1}{n}\sum_{i=1}^{n} Y_i 1_{[X_i \leq X_{k:n}]} = \frac{1}{n}\sum_{i=1}^{n} Y_{(i)} 1_{[X_{i:n} \leq X_{k:n}]} = \frac{1}{n}\sum_{i=1}^{k} Y_{(i)}, \tag{A.4}$$

where $Y_{(1)}, \ldots, Y_{(n)}$ are the values of $Y_1, \ldots, Y_n$ corresponding to the order statistics $X_{1:n}, \ldots, X_{n:n}$, respectively. The random variables $Y_{(1)}, \ldots, Y_{(n)}$ are called induced order statistics. The right-hand side of (A.4) gives a computationally convenient formula for $A_n(t)$.

Note 8. The difference $A - B$ is the ACC A_{Y-Z} corresponding to the difference $Y - Z$, that is, we have the equation $A_{Y-Z}(t) = \mathrm{E}[(Y - Z)1_{[X \leq F^{-1}(t)]}]$. Hence, one may equivalently think about the tests discussed in this paper as those for testing whether the ACC A_{Y-Z} coincides with the horizontal line passing 0, is below or above it, intersects it, and so on. The empirical estimator $\hat{A}_{Y-Z}$ of A_{Y-Z} is defined by

$$\hat{A}_{Y-Z}(t) := \frac{1}{n}\sum_{i=1}^{n} (Y_i - Z_i) 1_{[X_i \leq F_n^{-1}(t)]}.$$

Note 9. The preceding arguments in the case of two ACCs can be generalized to several ACCs. For example, when testing for the optimality of portfolios we might be interested in whether the relationship

$$\sum_{k=1}^{K} d_k A_{Y_k} \geq 0$$

holds, where d_i are certain parameters. Because the relationship is equivalent to $A_V \geq 0$ with $V = \sum_{k=1}^{K} d_k Y_k$, the statistical inferential tools developed in the present paper are applicable for testing $A_V \geq 0$. Indeed, let the earlier considered ACC A be identically zero and let B be A_V. Hence, testing $A_V \geq 0$ is equivalent to testing $A \leq B$. The test statistic is $\sqrt{n}s(-A_{V,n})$, where $A_{V,n} = \sum_{k=1}^{K} d_k \hat{A}_{Y_k}$ with $\hat{A}_{Y_k}$ denoting the empirical ACC

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corresponding to the n pairs $(Y_{k,1}, X_1), \ldots, (Y_{k,n}, X_n)$, where the first coordinates $Y_{k,1}, \ldots, Y_{k,n}$ are independent copies of Y_k. Note also that in the current situation, the earlier used D_n and D_n^* are, respectively, $\sqrt{n}(A_{V,n} - A_V)$ and $\sqrt{n}(A_{V,n}^* - A_{V,n})$, where $A_{V,n}^*$ is the empirical ACC corresponding to the pairs $(Y_{k,1}^*, X_1^*), \ldots, (Y_{k,n}^*, X_n^*)$ obtained from $\boldsymbol{\xi}_1^* := (Y_{1,1}^*, \ldots, Y_{K,1}^*, X_1^*), \ldots, \boldsymbol{\xi}_n^* := (Y_{1,n}^*, \ldots, Y_{K,n}^*, X_n^*)$ by choosing the kth coordinate (i.e., asset) of each vector $\boldsymbol{\xi}_i^*$ and also the last coordinate (i.e., the portfolio value). The vectors $\boldsymbol{\xi}_1^*, \ldots, \boldsymbol{\xi}_n^*$ are obtained by simple random sampling from the original data $\boldsymbol{\xi}_1 := (Y_{1,1}, \ldots, Y_{K,1}, X_1), \ldots, \boldsymbol{\xi}_n := (Y_{1,n}, \ldots, Y_{K,n}, X_n)$. Under the null hypothesis $A_V \geq 0$, we have that

$$\limsup_{n\to\infty} \mathbf{P}[\sqrt{n}s(-A_{V,n}) \geq x] \leq \lim_{n\to\infty} \mathbf{P}[\sqrt{n}s(-(A_{V,n} - A_V)) \geq x] \leq \alpha, \tag{A.5}$$

where x_α is such that $\mathbf{P}[s(-\Gamma_0) \geq x_\alpha] = \alpha$ for a nondegenerate Gaussian process Γ_0. (If desired, the process Γ_0 can be specified explicitly, but we do not actually need to do so at the moment.) The critical value x_α can be estimated using the earlier described bootstrap procedure, with obvious modifications. Namely, we generate n asterisked triplets M times and obtain M values of $s(-\sqrt{n}(A_{V,n}^* - A_{V,n}))$. Based on these values we define x_α^* as the smallest x such that the proportion of those M values of $s(-\sqrt{n}(A_{V,n}^* - A_{V,n}))$ that are on or above x is at most α.

APPENDIX B: Asymptotic Behavior of Empirical ACCs

In this section we present technical details concerning statement (6), upon which our previous considerations are crucially based. The problem is complex from the theoretical point of view. Fortunately, it has been resolved by Rao and Zhao (1995) and Davydov and Egorov (2000). For the sake of completeness of the present paper, we discuss the results in detail.

Because statement (6) is a "two-sample" problem, it is natural to start considerations with a simpler "one-sample" problem, which is establishing the asymptotic behavior of $\sqrt{n}(A_n - A)$. To simplify the problem even further, we first look at the convergence in distribution of $\sqrt{n}(A_n(t) - A(t))$ at any given point $t \in (0,1)$.

THEOREM 1. *Let the distribution function F be continuous. Then, for every continuity point $t \in (0,1)$ of the regression function $\mu_{Y|X}(F^{-1}(t))$, we have that*

$$\sqrt{n}(A_n(t) - A(t)) \to_d \mathcal{N}(0, v_{Y,X}^2(t)),$$

where $\mathcal{N}(0, v_{Y,X}^2(t))$ is a mean zero Gaussian random variable with the variance

$$v_{Y,X}^2(t) = \mu_{Y|X}^2(F^{-1}(t))t(1-t) + \int_0^t \sigma_{Y|X}^2(F^{-1}(u))\,du + \int_0^t \mu_{Y|X}^2(F^{-1}(u))\,du - \left(\int_0^t \mu_{Y|X}(F^{-1}(u))\,du\right)^2,$$

$$\mu_{Y|X}(x) = \mathrm{E}[Y|X = x] \quad \textit{and} \quad \sigma_{Y|X}^2(x) = \mathrm{Var}[Y|X = x].$$

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Theorem 1, even though it was not explicitly formulated by Rao and Zhao (1995) and Davydov and Egorov (2000), can nevertheless be easily derived following the arguments in those papers.

Theorem 1 is useful if one desires to have a pointwise $(1-\alpha)100\%$ confidence interval for $A(t)$, which is

$$A_n(t) \pm z_{\alpha/2} \frac{v_{Y,X}(t)}{\sqrt{n}}, \tag{B.1}$$

where $z_{\alpha/2} = \Phi^{-1}(1-\alpha/2)$ and Φ is the standard Gaussian distribution function. The confidence interval in (B.1) depends on the (unknown) standard deviation $v_{Y,X}(t)$, which has to be estimated empirically. This can be done using either a plug-in estimator or a bootstrap technique.

Theorem 1 also allows us to construct confidence intervals and test hypotheses in the two-sample case. A useful role in accomplishing this task is played by the earlier noted fact that $A - B$ is the ACC corresponding to the pair $(Y - Z, X)$. Likewise, $A_n - B_n$ is the empirical ACC based on the pairs $(Y_i - Z_i, X_i)$.

Now we formulate a technical result crucial for the present paper.

THEOREM 2 (see Rao and Zhao, 1995; Davydov and Egorov, 2000). *Let the distribution function F be continuous. Furthermore, let the function $\mu_{Y|X}(F^{-1}(t))$ be continuous on the interval $(0,1)$ and let there exist $0 < \gamma < \frac{1}{2}$ such that*

$$\sup_{0<t<1} t^{\gamma}(1-t)^{\gamma}|\mu_{Y|X}(F^{-1}(t))| < \infty. \tag{B.2}$$

Then we have the weak convergence

$$\sqrt{n}(\hat{A} - A) \Rightarrow \Gamma_0,$$

where the Gaussian process Γ_0 is defined by

$$\Gamma_0(t) := -\int_{(0,t]} \mathcal{B}(u)\, d\mu_{Y|X}(F^{-1}(u)) + \int_{(0,t]} \sigma^2_{Y|X}(F^{-1}(u))\, d\mathcal{W}(u)$$

with $\mathcal{B}$ and $\mathcal{W}$ denoting independent Brownian and Wiener processes, respectively.

A comment about condition (B.2) is worthwhile. Let $Y = X$, which gives the equality $\mu_{Y|X}(x) = x$ for all x. In this case, condition (B.2) is equivalent to

$$\sup_{0<t<1} t^{\gamma}(1-t)^{\gamma}|F^{-1}(t)| < \infty. \tag{B.3}$$

Because we can choose $\gamma \in (0,\frac{1}{2})$ as close to $\frac{1}{2}$ as desired, condition (B.3) means (roughly speaking) that the moment $\mathrm{E}[|X|^{2+\epsilon}]$ or, equivalently, $\mathrm{E}[|Y|^{2+\epsilon}]$ is finite for some $\epsilon > 0$. Because the finiteness of the second moment is necessary for establishing confidence intervals for the mean of Y, the preceding discussion suggests that condition (B.2) is close to the necessary one in the context of the present paper.

Now we are ready to formulate Corollary 1, which establishes statement (6) and thus, in turn, justifies our earlier considered statistical tests.

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COROLLARY 1. *Let the distribution function F be continuous. Furthermore, let the regression functions $\mu_{Y|X}(F^{-1}(t))$ and $\mu_{Z|X}(F^{-1}(t))$ be continuous on the interval $(0,1)$ and let there exist $0 < \gamma < \frac{1}{2}$ such that*

$$\sup_{0<t<1} t^{\gamma}(1-t)^{\gamma}(|\mu_{Y|X}(F^{-1}(t))| + |\mu_{Z|X}(F^{-1}(t))|) < \infty.$$

Then we have that

$$\sqrt{n}((A_n - B_n) - (A - B)) \Rightarrow \Gamma,$$

where the Gaussian process Γ is defined by

$$\Gamma(t) := -\int_{(0,t]} \mathcal{B}(u)\, d\mu_{(Y-Z)|X}(F^{-1}(u)) + \int_{(0,t]} \sigma^2_{(Y-Z)|X}(F^{-1}(u))\, d\mathcal{W}(u)$$

with $\mathcal{B}$ and $\mathcal{W}$ denoting independent Brownian and Wiener processes, respectively.

Proof. Use Note 8 of Appendix A together with Theorem 2 of the present Appendix. The corollary follows.

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