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Topological weak-mixing of interval exchange maps

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Abstract. An interval map with only one discontinuity is isomorphic to a rotation of the circle, and has continuous eigenfunctions. What we show here is that for almost every choice of lengths of the intervals, this is the only way an irreducible interval exchange can have a somewhere continuous eigenfunction. We show slightly more, considering certain towers over the interval exchange, showing that outside of a set of choices for interval lengths of measure zero these have a somewhere continuous eigenfunction only if they are isomorphic to either a rotation, or a tower of constant height over an interval exchange.

1. *Interval exchanges*

Virtually all the material we develop here is taken from the papers of Kerckhoff [**4**] and Veech [**8**]. We include many arguments not just for completeness but because later we will need the particular structures developed.

Fix a value $n \in \mathbb{N}$ and let $\pi \in S(n)$. Taking an interval $[0, a)$ and a vector $\vec{\lambda} \in \mathbb{R}^n_+$ with $\sum \lambda = a$, we build the interval exchange $T_{\pi,\vec{\lambda}}; [0, a)$, where

$$T_{\pi,\vec{\lambda}}(x) = x + \left[\sum_{\pi(j)<\pi(i)} \lambda_j - \sum_{j<i} \lambda_j \right) \quad \text{if } x \in I_i = \left[\sum_{j<i} \lambda_j, \sum_{j\le i} \lambda_j \right).$$

We now describe a measurable family of interval exchanges, indexed by the choice of $\vec{\lambda}$. Let $\Lambda(\vec{t})$ be a simplex in the positive cone of $\mathbb{R}^n$, i.e. the convex hull of a set of vectors $\{t_i \vec{e}_i\}$, $t_i > 0$, $\{\vec{e}_i\}$ the standard basis. Let $\mu_{\vec{t}}$ be the normalized $(n-1)$-dimensional Lebesgue measure on $\Lambda(\vec{t})$. Set $\Lambda_0 = \Lambda(\vec{1})$, the set of probability vectors.

Setting $p(\vec{v}) = \vec{v}/\|\vec{v}\|_1$, $P : \Lambda(\vec{t}) \to \Lambda(\vec{1})$ and is non-singular. In particular,

$$\frac{\sup(dp(\mu_{\vec{t}})/d\mu_{\vec{1}})}{\inf(dp(\mu_{\vec{t}})/d\mu_{\vec{1}})} = \left(\frac{\sup_i(t_1)}{\inf_i(t_1)} \right) n. \tag{1}$$

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For each $\vec{\lambda} \in \Lambda(\vec{t})$, there corresponds an interval exchange $T_{\pi,\vec{\lambda}}(x)$. These form a measurable family of transformations. We can define

$$T_\pi(\vec{\lambda}, x) = (\vec{\lambda}, T_{\pi\vec{\lambda}}(x))$$

acting on

$$\overline{\Lambda}(\vec{t}) = \{(\vec{\lambda}, \vec{x}); \vec{\lambda}, \in \Lambda(\vec{t}), X \in [0, \|\vec{\lambda}\|_1)\}.$$

Of course

$$T_\pi\left(P(\vec{\lambda}), \frac{x}{\|\lambda\|_1}\right) = \left(P(\vec{\lambda}), \frac{T_{\pi,\vec{\lambda}}(x)}{\|\vec{\lambda}\|_1}\right),$$

i.e. the map

$$(\vec{\lambda}, x) \longrightarrow \left(\frac{\vec{\lambda}}{\|\lambda\|_1}, \frac{x}{\|\lambda\|_1}\right)$$

conjugates T_π on $\overline{\Lambda}(\vec{t})$ to T_π on $\overline{\Lambda}(\vec{l})$.

We concern ourselves with the $\mu_{\vec{t}}$ -a.s. dynamics of $T_{\pi,\vec{\lambda}}$. Our results are definitely only true at this level. Arnoux [**1**] has shown that particular choices of non-trivial $\vec{\lambda}$ can possess somewhere continuous eigenfunctions. Obviously one need ask this only on $\overline{\Lambda}_0$, but the argument structure developed to date involves the dynamics of Rauzy induction which progress from one simplex $\overline{\Lambda}(\vec{t})$ to another. In fact, we want to complicate the picture just a bit more by considering a tower construction $T_{\pi,\vec{\lambda}}$ of a special sort.

Definition 1.1. For $\vec{v} \in \mathbb{R}^n, \vec{\lambda} \in \Lambda(\vec{t})$ we define a function expressed as

$$\vec{v}_{\vec{\lambda}}(x) = v_i \quad \text{if } x \in \left[\sum_{j<i} \lambda_j, \sum_{j\leq i} \lambda_j\right),$$

i.e. a step function taking value v_i on the interval I_i of the exchange. This is perhaps an odd vocabulary, using the name of the vector $(\vec{v})$ as the first part of the name of the step function $(\vec{v}_{\vec{\lambda}})$ whose values are the components of $\vec{v}$. However, this correspondence of vectors to functions, for λ fixed, is a linear map, and so it is quite natural to identify the vector with the function.

For $\vec{H} = (H_1, \ldots, H_n) \in \mathbb{N}^n$ we define a tower over $[0, \|\lambda\|_1)$ using the height function $\vec{H}_{\vec{\lambda}}(x)$ and the space

$$\hat{\Lambda}(\vec{t}, \vec{H}) = \{(\vec{\lambda}, x, h); (\vec{\lambda}, x) \in \overline{\Lambda}(\vec{t}), 0 \leq h < \vec{H}_{\vec{\lambda}}(x), h \in \mathbb{Z}\}.$$

We get a natural extension of T_π to this tower:

$$T_{\pi,\vec{H}}(\vec{\lambda}, x, h) = \begin{cases} (\vec{\lambda}, x, h+1) & \text{if } h < \vec{H}_\lambda(x) - 1 \\ (\vec{\lambda}, T_{\pi,\vec{\lambda}}(x), 0) & \text{otherwise.} \end{cases}$$

We express the restriction of this to a particular $\vec{\lambda}$ as

$$T_{\pi,\vec{\lambda},\vec{H}}(x, h) = \begin{cases} (x, h+1) & \text{if } h < \vec{H}_\lambda(x) - 1 \\ (T_{\pi,\vec{\lambda}}(x), 0) & \text{otherwise.} \end{cases}$$

We will identify the subset $X \times \{0\}$ with X itself.

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Notice $p(\vec{\lambda}, x, h) \longrightarrow (\vec{\lambda}/\|\vec{\lambda}\|_1, x/\|\vec{\lambda}\|_1, h)$ conjugates $(T_{\pi,\vec{H}}, \hat{\Lambda}(\vec{t}, \vec{H}))$ to the standard system

$$(T_{\pi,\vec{H}}, \hat{\Lambda}(\vec{1}, \vec{H})).$$

We say π is reducible if some initial subset $(1, 2, \ldots, m), m < n$, is π invariant. In this case $T_{\pi,\vec{\lambda}}$ is clearly never ergodic or minimal. In fact a tower system $(T_{\pi\vec{H}}, \hat{\Lambda}(\vec{l}, \vec{H}))$ would split as a disjoint union of invariant systems. Irreducibility will always be assumed.

The work from which we will build began with a result of Keane [**3**] who showed that if π is irreducible, and $T_{\pi,\vec{\lambda}}$ satisfies the 'infinite distinct orbit condition' (that two discontinuities of $T_{\pi,\vec{\lambda}}$ never lie on the same orbit), then $T_{\pi,\vec{\lambda}}$ is minimal. This i.d.o.c. condition is obviously a.s. true. Veech [**8**], and independently Masur [**6**], showed that in fact $T_{\pi,\vec{\lambda}}$ is a.s. uniquely ergodic.

Definition 1.2. We say π is a rotation if $T_{\pi,\vec{\lambda}}$ always has exactly one discontinuity, i.e. $\pi(i) = (i + k - 1) \pmod{n+1}$, for some k. If π is a rotation, $T_{\pi,\vec{\lambda}}$ is conjugate to a rotation by an angle $2\pi(\sum_{j\le k} \lambda_j/\|\vec{\lambda}\|_1)$, and so possesses continuous eigenfunctions.

What we will consider is when π is not a rotation. Basically what we will see is that for a.e. λ, $T_{\pi,\vec{\lambda}}$, if it has an eigenfunction, that eigenfunction must be everywhere discontinuous. This is a slightly unnatural result, but, as we will see, has more natural forms in Theorem 4.1 and 6.5. In fact topological weak-mixing of such $T_{\pi,\vec{\lambda}}$ is very reminiscent of Keane's result of minimality. We fully expect the stronger fact, that a.s. $T_{\pi,\vec{\lambda}}$ is weakly mixing, to also be true.

Veech has shown that some π a.s. cannot possess measurable eigenfunctions. Many cases, the smallest being $(1, 4, 2, 3)$ on four intervals, remain open. For continuous eigenfunctions we will give the complete (a.s.) answer.

2. *Rauzy induction*

Our task now is to describe the structure of Rauzy induction on a tower system $(T_{\pi,\vec{H}}, \hat{\Lambda}(\vec{l}, \vec{H}))$.

This will be a map R which will split the simplex $\Lambda(\vec{t})$ into two subsets, $\Lambda^{(1)}(\vec{t})$, and $\Lambda^{(2)}(\vec{t})$, mapping each linearly to another simplex $\Lambda(\vec{t}^{(1)})$ and $\Lambda(\vec{t}^{(2)})$. On each of these will be defined new permutations $\pi^{(1)}$ and $\pi^{(2)}$ and towers $\vec{H}^{(1)}$ and $\vec{H}^{(2)}$ so that $T_{\pi^{(i)},\vec{\lambda}^{(i)},\vec{H}^{(i)}}$ is canonically isomorphic to its preimage $T_{\pi,\vec{\lambda},\vec{H}}$. Let

$$\begin{aligned}\Lambda^{(1)}(\vec{t}) &= \{\vec{\lambda} \in \Lambda(\vec{t}); \lambda_n \ge \lambda_{\pi^{-1}(n)}\}\\ \Lambda^{(2)}(\vec{t}) &= \{\vec{\lambda} \in \Lambda(\vec{t}); \lambda_n \le \lambda_{\pi^{-1}(n)}\}.\end{aligned}$$

This is a p-equivariant splitting, $p(\Lambda^{(i)}(t)) = \Lambda^{(i)}(\vec{l})$. These subsimplexes intersect on the boundary section where $\lambda_n = \lambda_{\pi^{-1}(n)}$. In this situation, when we induce, these two intervals will coalesce into one. There will be a choice as to what we label this one interval, and the label will correspond to whether we regard it as in $\Lambda^{(1)}(\vec{t})$ or $\Lambda^{(2)}(\vec{t})$.

Discussing $\Lambda^{(1)}(\vec{t})$, define

$$T^{(1)}_{\pi,\vec{\lambda}}: \left[0, \sum_{j\ne\pi^{-1}(n)} \lambda_j\right) \to \left[0, \sum_{j\ne\pi^{-1}(n)} \lambda_j\right)$$

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to be the induced return-time map on this interval. A computation shows that $T^{(1)}_{\pi,\vec{\lambda}}$ is again an interval exchange on n-intervals. The map $T_{\pi,\vec{\lambda},\vec{H}}$ was a tower over $T_{\pi,\vec{\lambda}}$, hence it is canonically isomorphic to a tower over $T^{(1)}_{\pi,\vec{\lambda}}$, i.e. to a map of the form $T_{\pi^{(1)},\vec{\lambda}^{(1)},\vec{H}^{(1)}}$.

We wish to identify explicitly the parameters $\pi^{(1)}, \vec{\lambda}^{(1)}$ and $\vec{H}^{(1)}$. Consider the $n \times n$ matrices

$$A^{(1)}_{\pi} = \begin{bmatrix} 1 & & & & \\ & 1 & & & 0 \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & 0 & & & \\ & & -1 & & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \\ \\ \uparrow \\ \pi^{-1}(n) \end{matrix}$$

and

$$B^{(1)}_{\pi} = (A^{(1)T}_{\pi})^{-1} = \begin{bmatrix} 1 & & & & \\ & 1 & & & 0 \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & 0 & & & \\ & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \\ \\ \longleftarrow \pi^{-1}(n). \end{matrix}$$

One computes

$$\vec{\lambda}^{(1)} = A^{(1)}_{\pi}\vec{\lambda} \tag{2}$$

and

$$\vec{H}^{(1)} = B^{(1)}_{\pi}\vec{H}.$$

We can also consider the translation vector

$$\vec{\tau}(\vec{\lambda}), \quad \text{where } T^{(x)}_{\pi,\vec{\lambda}} = x + \tau_i(\vec{\lambda}), x \in T_i$$

i.e.

$$\tau_i(\vec{\lambda}) = \sum_{\pi(j)<\pi(i)} \lambda_j - \sum_{j<i} \lambda_j.$$

If we define

$$v_{i,j}(\pi) = \begin{cases} 1, & \pi(j) < \pi(i), j > i \\ -1, & \pi(j) > \pi(i), j < i \\ 0, & \text{otherwise} \end{cases}$$

and

$$V(\pi) = [v_{i,j}(\pi)],$$

then

$$\vec{\tau}(\vec{\lambda}) = V(\pi)\vec{\lambda}.$$

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One computes $\vec{\tau}(\vec{\lambda}^{(1)}) = B_\pi^{(1)}(\vec{\tau}(\vec{\lambda}))$. As this applies to an open set of $\vec{\lambda}$,

$$V(\pi^{(1)})A_\pi^{(1)} = B_\pi^{(1)}V(\pi). \tag{3}$$

Another important observation is that as $B_\pi^{(1)} = (A_\pi^{(1)^T})^{-1}$,

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \langle A_\pi^{(1)}\vec{v}_1, B_\pi^{(1)}\vec{v}_2 \rangle.$$

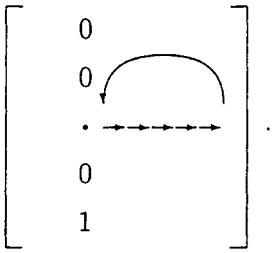
More fundamental, as we will see, is the relation of $\pi^{(1)}$ to π. Write the matrix representation $M(\pi) = (m_{i,j}), m_{i,j} = 1$ if $j = \pi(i)$ and 0 otherwise. If

$$M(\pi) = \begin{bmatrix} & & 0 & & & \\ & & 0 & & & \\ & & \cdot & & & \\ & & 0 & & & \\ 0 & \cdot & 1 & \cdot & \cdot & 0 \end{bmatrix},$$

$$\uparrow$$

$$\pi^{-1}(n)$$

then $M(\pi^{(1)})$ is obtained from $M(\pi)$ by moving the last column of $M(\pi)$ to just after column $\pi(n)$, and shifting the other columns to the right of $\pi(n)$ one place further to the right. We indicate this schematically:



$$\uparrow$$

$$\pi(n)$$

Let $\phi_{\pi,\vec{\lambda},\vec{H}}^{(1)}$ be the canonical isomorphism between $T_{\pi,\vec{\lambda},\vec{H}}$ and $T_{\pi^{(1)},\vec{\lambda}^{(1)},\vec{H}^{(1)}}$ and set $R : \hat{\Lambda}(\vec{\tau}, \vec{H}) \to \hat{\Lambda}(\vec{\tau}^{(1)}, \vec{H}^{(1)})$ to be

$$R(\vec{\lambda}, x, h) = (A_\pi^{(1)}(\vec{\lambda}), \phi_{\pi,\vec{\lambda},\vec{H}}^{(1)}(x, h)),$$

an isomorphism between $T_{\pi,\vec{H}}$ restricted to $\hat{\Lambda}^{(1)}(\vec{\tau}, \vec{H})$ and $T_{\pi^{(1)},\vec{H}^{(1)}}$. Even though $\det(A_\pi^{(1)}) = \det(B_\pi^{(1)}) = 1$, it is not necessarily measure preserving, as we use $(n-1)$-dimensional Lebesgue measure.

Carrying out the same process on $\Lambda^{(2)}(\vec{t})$, we define

$$T_{\pi,\vec{\lambda}}^{(2)}; \left[0, \sum_{j \neq n} \lambda_j\right) \to \left[0, \sum_{j \neq n} \lambda_j\right)$$

by inducing $T_{\pi,\vec{\lambda}}$ on this interval. $T_{\pi,\lambda}^{(2)}$ is an exchange on n-intervals. As $T_{\pi,\vec{\lambda},\vec{H}}$ is a

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tower over $T_{\pi,\vec{\lambda}}$, it is canonically isomorphic to a tower $T_{\pi^{(2)},\vec{\lambda}^{(2)},\vec{H}^{(2)}}$. In this case

$$A_\pi^{(2)} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & -1 \\ & & 0 & 0 \cdots & 0 & 1 \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & 1 & 0 \end{bmatrix} \longleftarrow \pi^{-1}(n)$$

and

$$B_\pi^{(2)} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & 0 & & 0 \\ & & 1 & 0 \cdots & 0 & 1 \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & 1 & 0 \end{bmatrix} \longleftarrow \pi^{-1}(n).$$

One checks that

$$\vec{\lambda}^{(2)} = A_\pi^{(2)}\vec{\lambda}, \quad \vec{N}^{(2)} = B_\pi^{(2)}\vec{H}, \quad \vec{\tau}(\vec{\lambda}^2) = B_\pi^{(2)}(\vec{\tau}(\vec{\lambda})) \tag{4}$$

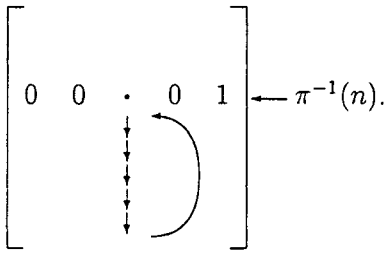
and so

$$V(\pi^{(2)})A_\pi^{(2)} = B_\pi^{(2)}V(\pi).$$

To understand the relation between $M(\pi)$ and $M(\pi^{(2)})$ write

$$M(\pi) = \begin{bmatrix} 0 & 0 & \cdot & 0 & 1 \\ & & & & \\ & & & & \end{bmatrix} \longleftarrow \pi^{-1}(n).$$

To move to $M(\pi^{(2)})$ we take the bottom row of $M(\pi)$ and place it just below row $\pi^{-1}(n)$, shifting the other rows down one place. Schematically we have:



Let $\phi_{\pi,\vec{\lambda},\vec{H}}^{(2)}$ be the canonical isomorphism between $T_{\pi,\vec{\lambda},\vec{H}}$ and let

$$R(\vec{\lambda}, x, h) = (A_\pi^{(1)}(\vec{\lambda}), \phi_{\pi,\vec{\lambda},\vec{H}}^{(2)}(x, h)),$$

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be an isomorphism between

$$T_{\vec{\pi},\vec{H}} \text{ restricted to } \hat{\Lambda}^{(2)}(\vec{\tau},\vec{H}) \text{ and } T_{\vec{\pi}^{(2)},\vec{H}^{(2)}}.$$

We want to modify our notation somewhat at this point. Since π is assumed irreducible, as long as $n > 2$, $\pi^{(1)}$ and $\pi^{(2)}$ are distinct. If $n = 2$, then $\pi = \pi^{(1)} = \pi^{(2)}$. In this case we replace π with two copies π_1, π_2 and set $\pi_i^{(1)} = \pi_i$ and to each 'permutation' in this extended sense we assign two distinct 'permutations' in this same extended sense. Now the choice of superscripts (1) or (2) can be regarded as depending rather on a choice of $\pi^{(1)}$ or $\pi^{(2)}$. Thus, rather than writing $A_\pi^{(1)}$ we will write $A^{\pi,\pi'}$, where π' is either $\pi^{(1)}$ or $\pi^{(2)}$, and similarly for $B^{\pi,\pi'}$.

We can also consider $A_\pi^{(i)}$ as depending on $\vec{\lambda}$, as long as $\vec{\lambda}$ avoids the common boundary. Hence we can also write $A^1(\vec{\lambda}), B^1(\vec{\lambda}), \pi^1(\vec{\lambda})$, the '1' indicating we have induced once.

Of course Rauzy induction can be repeated, generating a chain of interval exchange maps. As one repetitively induces, the original domain A_0 will splinter into an ever larger union of subsimplices, on each of which one follows a certain path of permutations and matrices.

We want to understand the possible sequences of permutations which would arise following such a chain of inductions.

Construct a directed graph G whose nodes are those $\pi \in S(n)$. An arrow goes from π to each of $\pi^{(1)}$ and $\pi^{(2)}$. We of course make a special case for $n = 2$, giving it two nodes.

Definition 2.1. Let $G(\pi)$ be the connected component of π in G. We write $\pi \rightsquigarrow \pi'$ if one can reach π' by following a sequence of arrows starting at π.

LEMMA 2.2. *If $\pi \rightsquigarrow \pi'$, then $\pi' \rightsquigarrow \pi$, and $G(\pi)$ is the graph of an irreducible subshift of finite type.*

Proof. It is clear from the schematic picture of the evolution of the matrices $M(\pi)$ if $\pi \rightsquigarrow \pi^{(i)}, \pi^{(i)} \rightsquigarrow \pi$. □

Definition 2.3. The nodes in $G(\pi)$ we call $R(\pi)$, the Rauzy class of π.

LEMMA 2.4. *If π is irreducible so are all $\pi' \in G(\pi)$, and each $\pi' \in G(\pi)$ has exactly two followers and two predecessors.*

Proof. If $\pi(1, \ldots, m) = (1, \ldots, m)$ then the same is true of $\pi^{(1)}$ and $\pi^{(2)}$. As long as $\pi(n) \neq n$, then $\pi^{(1)} \neq \pi^{(2)}$. The maps $\pi \to \pi^{(1)}$ and $\pi \to \pi^{(2)}$ are invertible, and again if $\pi(n) \neq n$ these predecessors are distinct. □

Starting with some $\vec{\lambda} \in \Lambda_0 = \Lambda(\vec{1})$ we can iterate Rauzy induction generating a sequence of permutations:

$$\pi = \pi_0(\vec{\lambda}), \pi_1(\vec{\lambda}), \ldots, \pi_k(\vec{\lambda}) \tag{5}$$

and matrices

$$A^{\pi_0,\pi_1}, A^{\pi_1,\pi_2}, \ldots, A^{\pi_{k-1},\pi_k} \quad \text{and} \quad B^{\pi_0,\pi_1}, B^{\pi_1\pi_2}, \ldots, B^{\pi_{k-1},\pi_k}.$$

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We will also write these as

$$A^1(\vec{\lambda}), A^2(\vec{\lambda}), \ldots, A^k(\vec{\lambda}) \quad \text{and} \quad B^1(\vec{\lambda}), B^2(\vec{\lambda}), \ldots, B^k(\vec{\lambda}), \tag{6}$$

assuming λ is never on a common boundary of subsimplices. The sequence $\vec{\pi}(\vec{\lambda}) = (\pi_0(\vec{\lambda}), \ldots)$ is an infinite path in $G(\pi)$. Conversely, to any such infinite path there will correspond some non-empty closed subset of $\vec{\lambda}$'s.

View this more explicitly as follows. Suppose $\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_k)$ is a finite path in $G(\pi)$. There is a subsimplex $\Lambda^{\vec{\pi}} \subseteq \Lambda_0$ which follows this path. Applying Rauzy induction k times,

$$A^{\pi_{k-1},\pi_k} \ldots A^{\pi_1,\pi_2} A^{\pi_0,\pi_1}(\Lambda^{\vec{\pi}}) = \Lambda(\vec{t}(\vec{\pi}))$$

will be a full simplex. It splits non-trivially into two pieces on which $\pi_{k+1} = \pi_k^{(1)}$ or $\pi_k^{(2)}$. The pull-backs of these two to Λ_0 form $\Lambda^{\vec{\pi},\pi_k^{(1)}}$ and $\Lambda^{\vec{\pi},\pi_k^{(2)}}$, splitting $\Lambda^{\vec{\pi}}$ into two pieces.

Intersecting a chain of such closed sets leads to a set of $\vec{\lambda}$'s on which this path in $G(\pi)$ is obtained. For $\vec{\lambda}$ never on a common boundary, the sequence $\vec{\pi}$ is unique and we write

$$\Lambda^k(\vec{\lambda}) = \Lambda^{(\pi_0(\vec{\lambda}),\ldots,\vec{\pi}_k(\vec{\lambda}))}$$

and we will have

$$\Lambda(\vec{\lambda}) = A^{\pi_{k-1},\pi_k} \ldots A^{\pi_{0-1},\pi}(\Lambda^k(\vec{\lambda})),$$

with $\mu_k(\vec{\lambda})$ as its normalized Lebesque measure.

We are now ready to state the basic recurrence results of Veech, Masur and Kerckhoff for Rauzy induction.

Notice that the B-matrices are $(0, 1)$-matrices, and setting

$$\vec{N}^k(\vec{\lambda}) = B^k(\vec{\lambda}) \ldots B^1(\vec{\lambda})(\vec{1}), \tag{7}$$

the terms of $\vec{N}^k(\vec{\lambda})$ represent the tower heights, after inducing k-times, starting from $T_{\pi,\vec{\lambda}}$ itself.

More precisely, entry i, j of $B^k(\vec{\lambda}) \circ \cdots \circ B^1(\vec{\lambda})$ is an integer which counts the number of occurrences in the tower over the ith interval of the induced interval exchange of the jth interval, I_j of the original interval exchange.

We now state a series of results from Kerckhoff [**4**]. His matrix A_m, the mth in the expansion of $\vec{\lambda}$, is our matrix $(B^m(\vec{\lambda}) \circ \cdots \circ B^1(\vec{\lambda}))^T$.

THEOREM 2.5. (Kerckhoff [**4**, Corollary 1.9]) *If π is irreducible, there is a constant $1 > c = c(\pi) > 0$ and for a.e. $\vec{\lambda} \in \Lambda_0$ there are infinitely many values $k_t(\vec{\lambda})$ with*

(a) $\pi_{k_t(\vec{\lambda})}(\vec{\lambda}) = \pi$ (8)

and

(b) $$\frac{\sup_i \vec{N}_i^{k_t(\vec{\lambda})}(\vec{\lambda})}{\inf_i \vec{N}_i^{k_t(\vec{\lambda})}(\vec{\lambda})} \leq \frac{1}{c(\pi)}.$$

Proof. The only difference here from Corollary 1.9 of [**4**] is that we claim $\pi_{k_t(\vec{\lambda})}(\vec{\lambda}) = \pi$. This extra condition can be carried through Kerckhoff's proof, or equally well follows from his Corollary 1.9. □

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Definition 2.6. We call an index $k_t(\vec{\lambda})$ satisfying the above a 'good recurrence time' for $\vec{\lambda}$.

COROLLARY 2.7. *If k is a good recurrence time for $\vec{\lambda}$, then*

$$\frac{1}{c(\pi)} > \frac{\sup_i N_i^{k_t}(\vec{\lambda})}{\inf_i N_i^{k_t}(\vec{\lambda})} = \frac{\sup\{\|\lambda'\|_1; \vec{\lambda}' \in \Lambda^k(\vec{\lambda})\}}{\inf\{\|\lambda'\|_1; \vec{\lambda}' \in \Lambda^k(\vec{\lambda})\}} = \left[\frac{\sup(dp(\mu_k(\vec{\lambda}))/d\mu_0)}{\inf(dp(\mu_k(\vec{\lambda}))/d\mu_0)}\right]^{1/n}.$$

COROLLARY 2.8. (Kerckhoff [**4**, Corollary 1.9]) *For π irreducible, if $\vec{\pi} = \pi_0, \pi_1, \ldots, \pi_t$ is some finite word in $G(\pi)$ beginning with π, then for a.e. $\vec{\lambda} \in \Lambda_0$ there are infinitely many good recurrence times k of $\vec{\lambda}$ with*

$$\pi_{k+s}^{(\vec{\lambda})} = \pi_s, \quad s = 0, \ldots, t.$$

COROLLARY 2.9. (Kerckhoff [**4**, Theorem 1.10]) *For π irreducible, for a.e. $\vec{\lambda} \in \Lambda_0$,* $\operatorname{dia}(\Lambda^k(\vec{\lambda})) \longrightarrow_k 0$.

Thus off a set of measure zero, $\cap_k \Lambda^k(\vec{\lambda}) = \vec{\lambda}$ and the sequence of refining partitions into the various $\Lambda^k(\vec{\lambda})$ a.s. separate points and hence generate the σ-algebra of measurable sets. If $\vec{R}_i^k$ is the ith row of

$$B^k(\vec{\lambda}) \circ \cdots \circ B^1(\vec{\lambda}) = ((A^k(\vec{\lambda}) \circ \cdots \circ A^1(\vec{\lambda}))^T)^{-1},$$

then the vertices of $\Lambda^k(\vec{\lambda})$ are $p(\vec{R}_1^k), i = 1, \ldots, n$. Thus we have the following.

COROLLARY 2.10. ([**8**]) *For a.e. $\lambda \in \Lambda_0, i = 1, \ldots, n$,*

$$\lim_{k\to\infty} p(\vec{R}_i^k) = \vec{\lambda}.$$

This explicitly tells us that in each tower $N_i^k(\vec{\lambda})$ the proportion of occurrences of I_j is converging in k to λ_j. This is essentially unique ergodicity of the interval exchange.

3. *The subspaces $H(\pi)$*

We now will use the recurrence information described in the previous section to develop Veech's picture of the asymptotics of the tower heights $\vec{N}^k(\vec{\lambda})$.

Definition 3.1. We say π is 'standard' if $\pi(1) = n$ and $\pi(n) = 1$.

LEMMA 3.2. *π is irreducible if there is a standard $\pi' \in R(\pi)$.*

Proof. Sufficiency is clear.

Consider the two types of transitions $\pi \to \pi^{(1)}$ and $\pi \to \pi^{(2)}$ in $G(\pi)$. Follow a path of the first type of transitions to a point which minimizes $\pi_1^{-1}(n)$ on this loop of $G(\pi)$. Thus

$$\pi_1^{-1}(n) = \inf(\pi^{-1}(j); \geq \pi(n)) \quad \text{and} \quad \pi_1(n) = \pi(n).$$

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From π_1 follow the loop of transitions of the second kind to a point π_2, minimizing $\pi_2(n)$, i.e.

$$\pi_2(n) = \inf(\pi_1(j); \pi_1^{-1}(j) \geq \pi^{-1}(n)) \quad \text{and} \quad \pi_1^{-1}(n) = \pi_2^{-1}(n).$$

Thus

$$\pi_2(n) \leq \pi_1(n) = \pi(n) \quad \text{and} \quad \pi_2^{-1}(n) = \pi_1^{-1}(n) \leq \pi^{-1}(n).$$

Iterating, we must reach a stage where neither $\pi_k(n)$ nor $\pi_k^{-1}(n)$ can be decreased. It follows that

$$\pi_k; \{j; j \geq \pi_k^{-1}(n)\} \longleftrightarrow \{j; j \geq \pi_k(n)\}$$

and unless $\pi_k^{-1}(n) = \pi_k(n) = 1$, π is reducible. □

We want to describe some special paths in $G(\pi), \pi$ standard, called $P(i,k)$, where $i \in \{1, \ldots, n\}, k \in \mathbb{N}$.

For $i \neq n$,

$$M(\pi) = \begin{bmatrix} & & & & & 1 \\ & & \cdot & & & \\ & \cdot & 1 & \cdot & \cdot & \cdot \\ & & \cdot & & & \\ & & \cdot & & & \\ & & \cdot & & & \\ 1 & & & & & \end{bmatrix} \longleftarrow i.$$

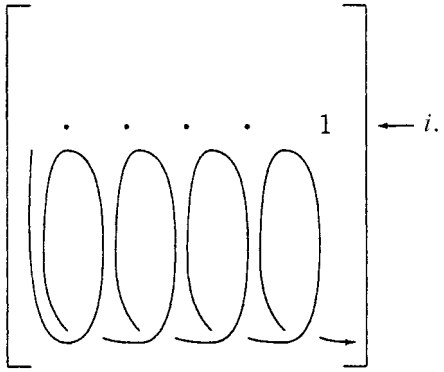
$$\uparrow$$

$$\pi(i)$$

To begin, follow a sequence of steps of type-1 $(n-\pi(i))$-times to reach π', $\pi'(i) = n$:

$$M(\pi') = \begin{bmatrix} \cdot & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & & & & & \\ \cdot & & & & & \\ 1 & & & & & \end{bmatrix} \longleftarrow i.$$

Next take type-2 steps $(n-i)$-times, looping the type-2 cycle k-times ending again at π':



Finally, take $\pi(i) - 1$ steps of type-1 to return to π. Set $P(n,k)$ to be the iteration

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of type-1 steps $k(n-1)$-times. All these are paths in $G(\pi)$ from π to π. Let $\vec{z} = (z_1, \ldots, z_n)$. Let $\vec{v}_i(\pi) = (v_{i,j}(\pi))$ be the ith row of $V(\pi)$. A computation shows that

$$B^{P(i,k)}\vec{z} = \vec{z} + kz_i(\vec{v}_i - \vec{v}_n) - z_n(\vec{v}_n), \quad i \neq n$$

and

$$B^{p(n,k)}\vec{z} = \vec{z} - kz_n(\vec{v}_n).$$

All the vectors $\vec{v}_i - \vec{v}_n, -\vec{v}_n$ are in the positive cone. Explicitly:

$$(\vec{v}_i - \vec{v}_n)_j = \begin{cases} 1 & \text{if } j \leq i, \pi(j) \leq \pi(i) \\ 1 & \text{if } j > i, \pi(j) > \pi(i) \\ 2 & \text{if } n > j > i, \pi(j) < \pi(i) \\ 0 & \text{if } j < i, \pi(j) > \pi(i) \\ 1 & \text{if } j = n \end{cases} \tag{9}$$

$$-\vec{v}_n = (1, 1, 1, \ldots, 1, 0),$$

and, for example,

$$\vec{v}_1 - \vec{v}_n = (1, 2, 2, \ldots, 2, 1). \tag{10}$$

Definition 3.3. For π irreducible, set $H(\pi) =$ linear span $(\vec{v}_1, \ldots, \vec{v}_n)$.

LEMMA 3.4. *For π' a follower of π,*

$$B^{\pi,\pi'}; H(\pi) \longrightarrow H(\pi').$$

Proof.

$$H(\pi) = \text{ Column space } (V(\pi))$$

as V is antisymmetric. Thus

$$\begin{aligned} B^{\pi,\pi'}(H(\pi)) &= \text{Column space } (V(\pi')A^{\pi,\pi'}) \\ &= \text{Column space } (V(\pi')) = H(\pi'). \end{aligned}$$

□

Extend π to be a permutation of $(0, 1, \ldots, n)$ by setting $\pi(0) = 0$ and set

$$V'(\pi) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & V(\pi) & \\ 0 & & & \end{bmatrix},$$

and $H'(\pi) =$ Column space $(V'(\pi))$.

LEMMA 3.5.

$$\begin{aligned} B &= \left\{\vec{b}; \sum_{i=0}^{n} b_i = 0 \text{ and } \vec{b} \in H'(\pi)^{\perp}\right\} \\ &= \left\{\vec{b}; \sum_{i=0}^{n} b_i = 0 \text{ and } \sum_{\pi(i)\leq\pi(k)} b_i = \sum_{i\leq k} b_i \text{ for all } k\right\}. \end{aligned}$$

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Proof.

$$(V(\pi)(\vec{b}))_k = \sum_{\substack{j\le k\\ \pi(j)\ge\pi(k)}}^{n} b_j - \sum_{\substack{j\le k\\ \pi(j)\ge\pi(k)}}^{n} b_j.$$

Adding

$$\sum_{\substack{j\le k\\ \pi(j)\le\pi(k)}}^{n} b_j$$

to both terms gives

$$\sum_{\substack{j\le k\\ \pi(j)\le\pi(k)}} b_j - \sum_{j\le k}^{n} b_j. \qquad \square$$

On $\{0,\dots,n\}$ let $+$ indicate the group operation of addition mod $(n+1)$. Set $\sigma(i) = \pi^{-1}(\pi(i)+1) - 1$.

LEMMA 3.6. *For $\vec{b} \in \mathbb{R}^{n+1}$, $\sum b_i = 0$, set $g(\vec{b},k) = \sum_{j\le k} b_j$. Now $g(b,.)$ is σ-invariant iff*

$$\sum_{j\le k} b_j = \sum_{\pi(j)\le\pi(k)} b_j$$

for all k.

Proof. We argue by induction on ℓ that

$$\sum_{j\le k} b_j = \sum_{\pi(j)\le\pi(k)} b_j \quad \text{for } k = \pi^{-1}(\ell') \text{ for all } \ell' \le \ell - 1.$$

For $\ell = 1$ this is true as $\pi(0) = 0$ so $\sum_{j\le 0} b_j = \sum_{\pi(j)\le\pi(0)} b_j$. Assume the result for $\ell - 1$. For $k = \pi^{-1}(\ell-1), k' = \pi^{-1}(\ell)$,

$$\begin{aligned}\sum_{j\le k'} b_j &= \sum_{j\le\pi^{-1}(\ell)} b_j = \sum_{j\le\pi^{-1}(\pi(k)+1)} b_j\\ &= g(\vec{b},\sigma(k)) + b_{\sigma(k)+1}\\ &= g(\vec{b},\sigma(k)) + b_{\pi^{-1}(\ell)}.\end{aligned}$$

On the other hand,

$$\sum_{\pi(j)\le\pi(k')} b_j = \sum_{\pi(j)\le\ell} b_j = \sum_{\pi(j)\le\ell-1} b_j + b_{\pi^{-1}(\ell)} = g(\vec{b},k) + b_{\pi^{-1}(\ell)}.$$

Thus the result extends to $k' = \pi^{-1}(\ell)$ if and only if $g(\vec{b},\sigma(k)) = g(\vec{b},k)$. $\square$

COROLLARY 3.7. (Veech [**8**, Proposition 3.3])

$$\dim(H(\pi)) = n - (\#\ \textit{cycles in}\ \sigma) + 1.$$

Proof.

$$\begin{aligned}\dim(H'(\pi)^{\perp}) &= \dim(H(\pi)^{\perp}) = \dim\left\{\vec{b}\in\mathbb{R}^n;\ \sum b_i = 0,\ g(\vec{b},\cdot)\ \text{is } \sigma\text{-invariant}\right\}\\ &= (\#\text{ cycles in } \sigma) - 1,\end{aligned}$$

as $g(\vec{b},\cdot)$ is linear and of determinant 1. $\square$

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Definition 3.8. We say π is of rotation class if $R(\pi)$ contains a rotation.

LEMMA 3.9. *π is of rotation class iff $T_{\pi,\lambda}$ has either one or two discontinuities, i.e. π looks like one of the following:*

$$M(\pi) = \begin{bmatrix} & 0 & 1 & & & 0 \\ & \vdots & & \ddots & & \\ 0 & \vdots & & & \ddots & \\ & \vdots & & & & \\ & \vdots & 0 & & & 1 \\ 1 & \cdots & 0 & \cdots & \cdots & 0 \\ & \ddots & \vdots & & & \\ & & \vdots & & 0 & \\ 0 & & 1 & & & \end{bmatrix}$$

or

$$M(\pi) = \begin{bmatrix} & & & & 0 & 1 & \cdots & 0 \\ 0 & & 0 & & \vdots & & \ddots & \vdots \\ & & & & \vdots & & & \\ & & & & \vdots & 0 & & 1 \\ & 0 & 1 & \cdots & 0 & \cdots & & 0 \\ & \vdots & & \ddots & \vdots & & & \\ 0 & \vdots & & & \vdots & & 0 & \\ & \vdots & & \ddots & \vdots & & & \\ & \vdots & 0 & & 1 & & & \\ 1 & \cdots & 0 & \cdots & 0 & & & \\ & \ddots & \vdots & & & & & \\ & & \vdots & 0 & & & 0 & \\ 0 & & 1 & & & & & \end{bmatrix}.$$

Proof. The collection of such π's is induction invariant. The first type is a rotation and the second is equivalent to a rotation. □

COROLLARY 3.10. *If π is irreducible, then* $\dim(H(\pi)) = 2$ *iff π is of rotation class.*

Proof. It is a simple computation that if π is of rotation class then (# cycles of σ) $= n-1$.

In order for (# cycles of σ) $= n-1$, the cycles of σ must be either one 3-cycle, all others singletons, or two 2-cycles and all others singletons. Now i is a singleton cycle of σ iff $i \neq n$ and $\pi(i+1) = \pi(i)+1$, i.e. there is no discontinuity of $T_{\pi,\vec{\lambda}}$ between I_i and I_{i+1}.

If σ has $(n-2)$ singletons, then $T_{\pi,\vec{\lambda}}$ has one discontinuity and is a rotation. If σ has $(n-3)$ singletons then $T_{\pi,\vec{\lambda}}$ has two discontinuities and π is of rotation class. □

Permutations of rotation class, which are essentially exchanges of three intervals, are well understood. We will discuss them separately later. It is the non-rotation class π's which interest us most.

We are now going to prove some basic lemmas from Veech [**8**]. Our purpose is mainly to establish a certain picture of the dynamics of the sequence $\vec{N}^K(\vec{\lambda})$.

Definition 3.11. For $\vec{u}$ and $\vec{v}$ both in the positive cone of $\mathbb{R}^n$ we say

$$\vec{u} = \vec{v}(1 \pm \epsilon) \quad \text{if } v_i(1-\varepsilon) \leq u_i \leq v_i(1+\varepsilon), i = 1, \ldots, n.$$

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Definition 3.12. We say $\vec{v} \in \Lambda_0$ is recurrent for $(\pi, \vec{\lambda})$ if for any $\vec{u} \in \Lambda_0$ there exist $k_i \to \infty$ with $\pi_{ki}(\vec{\lambda}) = \pi$ and

$$\lim_{i\to\infty} p(B^{k_i}(\vec{\lambda}) \circ \cdots \circ B^1(\vec{\lambda}) \circ \vec{u}) = \vec{v}.$$

LEMMA 3.13. *Suppose* $\vec{v} \in H(\pi) \cap \Lambda_0$ *and* $\vec{u} = \vec{v}(1 \pm \varepsilon)$. *Then*

$$B^{\pi,\pi'}(\vec{u}) = B^{\pi,\pi'}(\vec{v})(1 \pm \varepsilon) = \vec{v}'(1 \pm \varepsilon),$$

where $\vec{v}' \in H(\pi')$.

Proof. Just notice that $B^{\pi,\pi'}$ has only 0's and 1's as entries. □

LEMMA 3.14. *For* π *standard, for a.e.* $\vec{\lambda} \in \Lambda_0$ *and* $\varepsilon > 0$, *for any* $\vec{u} \in \Lambda_0$, *there is an* m *with*

$$p(B^m(\vec{\lambda}) \circ \cdots \circ B^1(\vec{\lambda})\vec{u}) = \frac{\vec{v}_1 - \vec{v}_n}{\|\vec{v}_1 - \vec{v}_n\|_1}(1 \pm \varepsilon).$$

Proof. By Corollary 2.8, for a.e. $\vec{\lambda} \in \Lambda_0$ there are good recurrence times m' for $\vec{\lambda}$ with $\pi_{m'}(\vec{\lambda}), \ldots, \pi_{m'+(k+1)(n-1)}(\vec{\lambda}) = P(1, k)$. Let $m = m' + (k + 1)(n - 1)$, and $\vec{z} = B^{m'}(\vec{\lambda}) \circ \cdots \circ B^1(\vec{\lambda})\vec{u}$. By Corollary 2.10, once m' is large enough, $\vec{R}_i^{m'} = N_i^{m'}(\vec{\lambda})\vec{\lambda}(1 \pm \varepsilon)$. As m' is a good recurrence time for $\vec{\lambda}$,

$$\frac{\sup_i z_i}{\inf_i z_i} \le \frac{\sup_i N_i^{m'}(\vec{\lambda}) \sup_i (\lambda_i u_i)(1 + \varepsilon)}{\inf_i N_i^{m'}(\vec{\lambda}) \inf_i (\lambda_i u_i)(1 - \varepsilon)} < b < \infty,$$

where b is independent of m', a sufficiently large good recurrence time. Now

$$B^m(\vec{\lambda}) \circ \cdots \circ B^{m'+1}(\vec{\lambda})\vec{z} = B^{p(1,k)}(\vec{z}) = \vec{z} + kz_1(\vec{v} - \vec{v}_n) + z_n(-\vec{v}_n).$$

As $\vec{v}_1 - \vec{v}_n = (1, 2, \ldots, 2, 1)$, if $k > 2b/\varepsilon$, then

$$B^{p(1,k)}(\vec{z}) = kz_1(\vec{v}_1 - \vec{v}_n)(1 \pm \varepsilon).$$ □

COROLLARY 3.15. *If* π *is irreducible, for a.e.* $\vec{\lambda} \in \Lambda_0$, *if* $\vec{v}$ *is recurrent for* $(\pi, \vec{\lambda})$ *then* $\vec{v} \in H(\pi)$.

Proof. As there must be a $\pi' \in R(\pi)$ which is standard, combining Lemma 3.8 and 3.9 gives the result. □

The same trick we have just used, using the words $p(j, k)$, $j = 1, \ldots, n$, gives us the following.

THEOREM 3.16. *For* π *standard, the vectors* $p(\vec{v}_j - \vec{v}_n)$, $j \ne n$, *and* $p(-\vec{v}_n)$ *are all recurrent for a.e.* (π, λ).

These results *per se* are not critical for our work. They do tell us why later we must restrict our attention to $H(\pi)$, as it is here the recurrent dynamics of the $B^k(\vec{\lambda})$ acting on tower heights lies.

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4. *Eigenfunctions and affine coboundaries*

Fix a vector $\vec{H} \in \mathbb{N}^n$ and consider the map $T_{\pi,\vec{\lambda},\vec{H}}$. If $\gcd(H_i) = p \neq 1$, then $T^p_{\pi,\vec{\lambda},\vec{H}}$ is not ergodic and hence $T_{\pi,\vec{\lambda},\vec{H}}$ has $1/p$ as an eigenvalue. We will always use additive notation for eigenfunctions:

$$f(T(x)) = (\alpha + f(x)) \pmod 1.$$

In this case notice that the eigenfunction f is constant on the base of the tower. If π is a rotation, then $T_{\pi,\lambda}$ itself has eigenfunctions which are continuous. Also notice that if $T_{\pi,\vec{\lambda},\vec{H}}$ has an eigenfunction which is continuous on the base of the tower, when we Rauzy induce we obtain $T_{\pi^1,\vec{\lambda}^1,\vec{H}^1}$ which again will possess an eigenfunction continuous on the base of the tower.

What we show in this section is that for a.e. $\vec{\lambda} \in \Lambda_0$, these are the only cases where $T_{\pi,\vec{\lambda},\vec{H}}$ can have an eigenfunction continuous on the base of the tower. Precisely, if π is not induced from a rotation and $\gcd(H_i) = 1$ then for a.e. $\vec{\lambda}$ no such eigenfunction exists.

THEOREM 4.1. *If $T_{\pi,\vec{\lambda}}$ satisfies Keane's i.d.o.c. and f is a measurable eigenfunction of $T_{\pi,\vec{\lambda},\vec{H}}$ which is somewhere continuous on the base of the tower, then f is continuous everywhere on the base of the tower, and has one-sided limits at both endpoints.*

Proof. Keane's condition says any orbit of $T_{\pi,\vec{\lambda}}$ contains at most one discontinuity. Hence for any $x \in [0,1)$, either $\{T^i_{\pi,\vec{\lambda}}(x); i \geq 0\}$ or $\{T^i_{\pi,\vec{\lambda}}(x); i < 0\}$ contains no discontinuities. Keane's condition also tells us $T_{\pi,\vec{\lambda}}$ is minimal (Keane [**3**]). Thus both $\{T^i_{\pi,\vec{\lambda}}(x); i \geq 0\}$ and $\{T^i_{\pi,\vec{\lambda}}(x); i < 0\}$ are dense. Suppose f is continuous at x_0. Given $\varepsilon > 0$, there is a δ, and for

$$x' \in (x_0 - \delta, x_0 + \delta), \quad |f(x') - f(x_0)| < \varepsilon.$$

Suppose $\{T^i_{\pi,\vec{\lambda}}(x); i \geq 0\}$ are continuous at x. For some i_0, $|T^{i_0}_{\pi,\vec{\lambda}}(x) - x_0| < \delta/2$. For some $\delta' < \delta/2$,

$$T^i_{\pi,\vec{\lambda}}|_{(x_0-\delta',x_0+\delta')} \text{ is a translation for all } 0 \leq i \leq i_0.$$

Then for any $x' \in (x - \delta', x + \delta')$,

$$|f(x) - f(x')| = \left| f(T^{i_0}_{\pi,\vec{\lambda}}(x)) - f(T^{i_0}_{\pi,\vec{\lambda}}(x')) - \alpha\left(\sum_{j=0}^{i_0-1} \vec{H}_{\vec{\lambda}}(T^j_{\pi,\vec{\lambda}}(x)) - \vec{H}_{\vec{\lambda}}(T^j_{\pi,\vec{\lambda}}(x'))\right)\right| < 2\varepsilon.$$

Analogous one-sided arguments hold at the left and right endpoints. □

Suppose f is an eigenfunction of $T_{\pi,\vec{\lambda},\vec{H}}$, and $x \in I_j$. Then

$$f(T_{\pi,\vec{\lambda}}(x)) = f(T^{\vec{H}_{\vec{\lambda}}(x)}_{\pi,\vec{\lambda},\vec{H}}(x,0)) = f(x) + H_j\alpha \pmod 1.$$

Let $\widetilde{f}$ be a continuous lift of $f : X \to S_1$ to a map $\widetilde{f} : X \to \mathbb{R}$. Then

$$\widetilde{f}(T_{\pi,\vec{\lambda}}(x)) - \widetilde{f}(x) = \vec{H}_{\vec{\lambda}}(x)\alpha + m(x), \quad m : X \to \mathbb{Z}.$$

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COROLLARY 4.2. *If f is a somewhere continuous eigenfunction of $T_{\pi,\vec{\lambda},\vec{H}}$ with eigenvalue α, then there is a vector $\vec{m} \in \mathbb{Z}^n$ and $\vec{H}_{\vec{\lambda}}(x)\alpha - \vec{m}_{\vec{\lambda}}(x)$ is a $T_{\pi,\vec{\lambda}}$ coboundary. (We refer the reader to Definition 1.1 for the meaning of $\vec{m}_{\vec{\lambda}}$.)*

Definition 4.3. We call a function of the form $\vec{H}_{\vec{\lambda}}(x)\alpha - \vec{m}_{\vec{\lambda}}(x)$ an affine function, and if it is a coboundary, an affine coboundary.

COROLLARY 4.4. *If $T_{\pi,\vec{\lambda}}$ has an affine coboundary $\vec{H}_{\vec{\lambda}}(x)$, then α is in the point spectrum of $T_{\pi,\vec{\lambda},\vec{H}}$.*

Thus if $T_{\pi,\vec{\lambda},\vec{H}}$ has a continuous eigenfunction then $T_{\pi,\vec{\lambda}}$ has an affine coboundary, and if $T_{\pi,\vec{\lambda}}$ has an affine coboundary, then it has a borel eigenfunction. Certainly maps can have borel but not continuous eigenfunctions. We pose the question, can this happen for interval exchange maps?

Assuming $\vec{\pi}$ and $\vec{H}$ are fixed, we want to define a special subsequence $k_i = k_i(\vec{\lambda}) \nearrow \infty$. Fix some generic $\vec{\lambda}_0 \in \Lambda_0$. Let k' be an index with

$$\frac{\sup_i(\lambda_{0,i}^{k'})}{\inf_i(\lambda_{0,i}^{k'})} < 2 \quad \text{and} \quad \frac{\sup_i(N_i^{k'}(\vec{\lambda}_0))}{\inf_i(N_i^{k'}(\vec{\lambda}_0))} \leq \frac{1}{c(\pi)}. \tag{11}$$

The second condition is just that k' is a good recurrence time, and the first is an open property on $\Lambda^{k'}(\vec{\lambda}_0)$ and is forced by requiring an appropriate word $\vec{\pi}' = (\pi_0 = \pi, \pi_1, \ldots, \pi_t)$ in the path of $\vec{\lambda}_0$ in $G(\pi)$ starting at index k.

Consider the induced map on $[0, \lambda_{0,1}^{k'})$. This lies in the sequence of Rauzy inductions following k', say at index $k' + s = k$.

Certainly there is some $\overline{C} < \infty$ with

$$\frac{\sup_i(\lambda_{0,i}^{k})}{\inf_i(\lambda_{0,i}^{k})} < \frac{\overline{C}}{2} < \infty.$$

This inequality is then a non-trivial open property on $\Lambda^k(\lambda_0)$. Thus there is a word $(\pi_s, \pi_{s+1}, \ldots, \pi_w)$ in the path of $\vec{\lambda}_0$ forcing this inequality.

Let $\vec{\pi}$ be the full word $(\pi_0, \pi_1, \ldots, \pi_s, \pi_{s+1}, \ldots, \pi_w)$ and for any $\vec{\lambda}$, if k' is a good recurrence time for $\vec{\lambda}$, and $\pi_{k'+t}(\lambda) = \pi_t$, $i = 0, \ldots, w$, then for $k = k' + s$,

(a) $$\frac{\sup_i(\lambda_i^{k'})}{\inf_i(\lambda_i^{k'})} < 2 \tag{12}$$

(b) $$\frac{\sup_i(N_i^{k'}(\vec{\lambda}))}{\inf_i(N_i^{k'}(\vec{\lambda}))} \leq \frac{1}{c(\pi)} \text{ and}$$

(c) $$\frac{\sup_i(\lambda_i^{k})}{\inf_i(\lambda_i^{k})} < \overline{C}.$$

Since $B^{\pi_{w-1},\pi_w} \circ \cdots \circ B^{\pi_s,\pi_{s+1}}$ is a fixed sequence of matrices, we can also assume that $\overline{C}$ is sufficiently large so that

(d) $$\frac{\sup_i(N_i^{k}(\vec{\lambda}))}{\inf_i(N_i^{k}(\vec{\lambda}))} < \overline{C}.$$

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Definition 4.5. Let $k_i(\vec{\lambda})$ be the sequence of indices of the form $k' + s$ described above. Such sequences exist for a.e. $\vec{\lambda} \in \Lambda_0$.

LEMMA 4.6. *Fix $\vec{M} \in \mathbb{N}^n$ and $\vec{M} \in \mathbb{Z}^n$. For a.e. $\vec{\lambda}$, if $\vec{H}_{\vec{\lambda}}(x)\alpha - \vec{m}_{\vec{\lambda}}(x)$ is an affine coboundary, then for $k_i = k_i(\vec{\lambda})$,*

$$\lim_{i\to\infty} B^{(k_i)}\vec{\lambda} \circ \cdots \circ B^{(1)}(\vec{\lambda})(\vec{H}\alpha + \vec{m}) = 0.$$

Proof. Let $k_i = k_i' + s$ be a point in the sequence. Consider the interval $L_i = [0, \lambda_1^{k_i})$ and the tower of intervals over it after inducing $T_{\pi,\vec{\lambda}}$ for k_i'-times. Call them

$$L_0^i = L^i, L_1^i, \ldots, L^i_{N_i^{k'(\vec{\lambda})-1}}.$$

The union of these intervals occupies at least $[c(\pi)/2n]$ of X.

For $x \in L_0^i, 0 \le t < N_i^{k_i}$, $\widetilde{f}(T_{\pi,\vec{\lambda}}(x)) - \widetilde{f}(x) = k(i,t)$ a constant, as the affine coboundary is constant on each L_t^i, the L_t^i cover $c(\pi)/2n$ of X, independent of i. Thus for any $\varepsilon > 0$, once i is large enough, for some one of them, say $L_{t_0}^i$, there is a subset $\widetilde{L}_{t_0}^i \subseteq L_{t_0}^i$ with $\mu_0(\widetilde{L}_{t_0}^i)/\mu_0(L_{t_0}^i) > 1 - \varepsilon$ and for $x, x' \in \widetilde{L}_{t_0}^i$, $|\widetilde{f}(x) - \widetilde{f}(x')| < \varepsilon$. But $\widetilde{f}|_{L_0^i} = \widetilde{f}|_{L_{t_0}^i} - k(i,t_0)$, hence $\widetilde{L}_t^i$ exists for all t satisfying the above.

Now induce the remaining s times to reach $k_i(\vec{\lambda})$. This is inducing on L_0^i. The intervals $I_1^{k_i}, \ldots, I_n^{k_i}$ partition L_0^i and satisfy

$$\frac{\sup_i[\lambda_j^{k'}]}{\inf_i[\lambda_j^{k'}]} < \overline{C}.$$

If $\varepsilon < 1/2\overline{C}n$, then for all $j = 1, \ldots, n$,

$$\mu_0(I_j^{k_i} \cap \widetilde{L}_0^i) > \tfrac{1}{2}\mu(I_j^{k_i}).$$

Hence there is an $x \in I_j^{k_i} \cap \widetilde{L}^i$ with $T_{\pi,\vec{\lambda}}^{N_j^{k_i}(\vec{\lambda})}(x) \in \widetilde{L}^i$.

$$\varepsilon > |\widetilde{f}(T_{\pi,\vec{\lambda}}^{N_j^{k_i}(\vec{\lambda})}(x)) - \widetilde{f}(x)| = |B^{k_i}(\vec{\lambda}) \circ \cdots \circ B^1(\vec{\lambda})(\vec{H}\alpha + \vec{m})_j|$$

and we are done. □

COROLLARY 4.7. *If $\vec{H} \in \mathbb{N}^n, \vec{m} \in \mathbb{Z}^n$, then for a.e. $\vec{\lambda}$ if*

$$\vec{H}_{\vec{\lambda}}(x)\alpha - \vec{m}_{\vec{\lambda}}(x)$$

is an affine coboundary, then

$$\vec{H}\alpha + \vec{m} \in \lambda^{\perp}.$$

Proof. As

$$\langle \vec{H}\alpha + \vec{m}, \lambda\rangle = \langle B^k(\vec{\lambda}) \circ \cdots \circ B^1(\vec{\lambda}) \cdot (\vec{H}\alpha + \vec{m}), \Lambda^k(\vec{\lambda}) \circ \cdots \circ \Lambda^1(\vec{\lambda}) \cdot \vec{\lambda}\rangle$$

and both terms of the right-hand side go to zero along a subsequence, we are done. □

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LEMMA 4.8. *If $\vec{H} \in \mathbb{N}^n$ and $\vec{m} \in \mathbb{Z}^n$, then for $\vec{\lambda} \in \Lambda_0$ there is a unique continuous $t(\vec{\lambda})$ with $\vec{H}t(\vec{\lambda}) + \vec{m} \in \vec{\lambda}^{\perp}$.*

Proof.

$$t(\vec{\lambda}) = \frac{\langle \vec{\lambda}, \vec{m} \rangle}{\langle \vec{\lambda}, \vec{H} \rangle} \quad \text{as } \langle \vec{\lambda}, \vec{H} \rangle \neq 0. \qquad \square$$

Definition 4.9. We call $(\vec{H}, \vec{m})$ a non-trivial pair if $\vec{H} \in \mathbb{N}^n, \vec{m} \in \mathbb{Z}^n$ and $\vec{m}$ is not a multiple of $\vec{H}$.

LEMMA 4.10. *For a.e. $\vec{\lambda} \in \Lambda_0$ if $\vec{H}$ satisfies $\gcd(H_i) = 1$ then $T_{\pi,\vec{\lambda},\vec{H}}$ has no rational eigenvalues with non-constant somewhere-continuous eigenfunctions. If $T_{\pi,\vec{\lambda},\vec{H}}$ has an irrational eigenvalue with a somewhere continuous eigenfunction, then for some $\vec{m} \in \mathbb{Z}^n$, $(\vec{H}, \vec{m})$ is a non-trivial pair and $\vec{H}_{\vec{\lambda}}(x)t(\vec{\lambda}) + \vec{m}_{\vec{\lambda}}(x)$ is an affine coboundary.*

Proof. If $T_{\pi,\vec{\lambda},\vec{H}}$ has a somewhere continuous eigenfunction, then $\vec{H}_{\vec{\lambda}}(x)t(\vec{\lambda}) + \vec{m}_{\vec{\lambda}}(x)$ is an affine coboundary. If $t(\vec{\lambda})$ is rational, by Lemma 4.4, as all B matrices are $(0, 1)$-matrices of determinant 1,

$$\vec{H}t(\vec{\lambda}) + \vec{m} = \vec{0}.$$

As $\gcd(H_i) = 1, t(\vec{\lambda}) = 0$, and as $T_{\pi,\vec{\lambda},\vec{H}}$ is a.s. ergodic, the eigenfunction is constant.

If $(\vec{H}, \vec{m})$ is a trivial pair, then $\vec{H}t(\vec{\lambda}) + \vec{m} \in \vec{\lambda}^{\perp}$ says $\vec{H}t(\vec{\lambda}) + \vec{m} = \vec{0}$ and $t(\vec{\lambda})$ is rational. $\square$

If we fix a non-trivial pair $(\vec{H}, \vec{m})$, the function $\vec{H}_{\vec{\lambda}}(x)t(\vec{\lambda}) + \vec{m}_{\vec{\lambda}}(x)$ is measurable on $\overline{\Lambda}_0$. We want to consider the set of $\vec{\lambda}$ for which this is a coboundary and show it has μ_0-measure zero. As a first step we must show it measurable.

LEMMA 4.11. *Suppose $(x, \mathcal{S}, \mu, T)$ is a dynamical system, with*

$$\int (X_z, \mathcal{S}_z, \mu_z, T_z)\, d\mu(z)$$

as its ergodic decomposition. Suppose $g : X \to \mathbb{R}$ is measurable. $S = \{z$; there is a μ_z-measurable f_z with $g(x) = f_z(T_z(x)) - f_z(x)$, μ_z−a.e.$\}$ is a measurable set. Further, there is a μ-measurable

$$f : S \to \mathbb{R} \quad \text{with } g(x) = f(T(x)) - f(x) \text{ for } x \in S.$$

Proof. For $j \in \mathbb{N}, x \in X$ set $a_j(x) = \sum_{k=0}^{j-1} g(T^k(x))$. For $M, N \in \mathbb{N}$, $(M - N)$ even, set

$$A_{M,N}(x) = \{a_j(x); N \leq j < M\}$$

counted with multiplicity. Now $A_{M,N}$ has $(M - N)$ elements; set $\overline{A}_{M,N}$ to be that half of $A_{M,N}$ consisting of the smaller values $a_j(x)$.

Let $b_{M,N}(x) = \max(\overline{A}_{M,N}(x))$. These are measurable functions of x. Set

$$\overline{f}(x) = -\lim_{M\to\infty} \lim_{N\to\infty} \sup b_{M,N}(x)$$

taking values in $\mathbb{R} \cup \{\pm\infty\}$.

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If g is a coboundary on X_z, i.e. $g(x) = f_z(T_z(x)) - f_z(x)$, then for μ_z-a.e. x,

$$A_{M,N}(x) = \{f_z(T_z^j(x)) - f_z(x);\, M \le j < N\}.$$

As f is bounded when restricted to some subset of μ_z measure $<\frac{1}{2}$, by the pointwise ergodic theorem

$$\overline{f}(x) \text{ is finite}, \quad \mu_z\text{-a.s, } z \in S.$$

On the other hand suppose $\overline{f}(x)$ is finite at x. As

$$A_{M,N}(T(x)) = \{a_j - g(x);\, a_j \in A_{M-1,N-1}\},$$
$$b_{M,N}(T(x)) = b_{M-1,N-1}(x) - g(x)$$

and $\overline{f}(T(x)) - \overline{f}(x) = g(x)$, then $\overline{f}$ is finite on the full orbit of x. Hence the set of x where $\overline{f}$ is finite is T invariant, and on this set

$$\overline{f}(T(x)) - \overline{f}(x) = g(x). \qquad \square$$

COROLLARY 4.12. *For $(\vec{H}, \vec{m})$ a non-trivial pair,*

$$S_{\vec{H},\vec{m}} = \{\vec{\lambda} \in \Lambda_0;\, \vec{H}_{\vec{\lambda}}(x)t(\vec{\lambda}) + \vec{m}_{\vec{\lambda}}(x) \text{ is a coboundary for } T_{\pi,\vec{\lambda}}\}$$

is measurable.

We are now ready to construct our main result, that if π is not of rotation class, then all $S_{\vec{H},\vec{m}}$ have measure zero.

LEMMA 4.13. *If $(\vec{H}, \vec{m})$ is a non-trivial pair, and $\mu_0(S_{\vec{H},\vec{m}}) > 0$, then for any $\varepsilon > 0$, there is a non-trivial $(\vec{H}_1, \vec{m}_1)$ with $\mu_0(S_{\vec{H}_1,\vec{m}_1}) > 1 - \varepsilon$.*

Proof. As $S_{\vec{H},\vec{m}}$ is measurable, for a.e. $\vec{\lambda}_0 \in S_{\vec{H},\vec{m}}$,

$$\lim_{k\to\infty} \frac{\mu_0(\Lambda^k(\vec{\lambda}_0) \cap S_{\vec{H},\vec{m}})}{\mu_0(\Lambda^k(\vec{\lambda}_0))} = 1.$$

Choose k to be a good recurrence time for $\vec{\lambda}_0$ with

$$\frac{\mu_0(\Lambda^k(\vec{\lambda}_0) \cap S_{\vec{H},\vec{m}})}{\mu_0(\Lambda^k(\vec{\lambda}_0))} > 1 - \frac{\varepsilon}{c(\pi)^n}.$$

For

$$\Lambda_k(\vec{\lambda}_0) = \Lambda^k(\vec{\lambda}_0) \circ \cdots \circ A^1(\vec{\lambda}_0)(\Lambda_k(\vec{\lambda}_0))$$

and

$$\vec{H}_1 t(\vec{\lambda}_0) + \vec{m}_1 = B^k(\vec{\lambda}_0) \circ \cdots \circ B^1(\vec{\lambda}_0)(\vec{H}t(\vec{\lambda}_0) + \vec{m})$$

we get, for any

$$\vec{\lambda} \in p(\Lambda_k(\vec{\lambda}_0)) \cap (A^k(\vec{\lambda}) \circ \cdots \circ A^1(\vec{\lambda}_0) S_{\vec{H},\vec{m}}), \quad \vec{\lambda} \in S_{\vec{H}_1,\vec{m}_1}.$$

Thus $\mu_0(S_{\vec{H}_1,\vec{m}_1}) > 1 - \varepsilon$, as the $R - N$ derivative of $p(A^k \circ \cdots \circ A^1)$ has ratios bounded by $c(\pi)^n$. $\square$

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Consider the $3n$ finite words in $G(\pi)$, π standard:

$$W(\pi) = \{P(i,1), P(i,2), P(i,3), i = (1,\dots,n)\}.$$

For each $\vec{\pi} \in W(\pi)$ let $\Lambda(\vec{t}_{\vec{\pi}}) = A^{\vec{\pi}}(\Lambda^{\vec{\pi}})$. As there are only finitely many words here, there is a k so that

$$\sup_{\vec{\pi}\in W(\pi)} \left[\frac{\sup[dp(\mu_{\vec{t}_{\vec{\pi}}})/d\mu_0]}{\inf[dp(\mu_{\vec{t}_{\vec{\pi}}})/d\mu_0]} \right] \le k.$$

COROLLARY 4.14. *For π standard, if for some non-trivial $(\vec{H},\vec{m})$, $\mu_0(S_{\vec{H},\vec{m}}) > 0$, then for some non-trivial $(\vec{H}_1,\vec{m}_1)$,*

$$\mu_0\left(\bigcap_{\vec{\pi}\in W(\pi)} S_{B^{\vec{\pi}}(\vec{H}_1),B^{\vec{\pi}}(\vec{m}_1)} \right) \ge \tfrac{1}{2}.$$

Proof. We know

$$P(A^{\pi}(\Lambda^{\vec{\pi}} \cap S_{\vec{H},\vec{m}})) \subseteq S_{B^{\vec{\pi}}(\vec{H}_1),B^{\vec{\pi}}(\vec{m}_1)}.$$

Choose ε in the previous lemma to be at most $\min_{\vec{\pi}\in W(\pi)} \mu_0(\Gamma^{\vec{\pi}})/6nK$. We compute

$$\mu_0(S_{B^{\vec{\pi}}(\vec{H}_1),B^{\vec{\pi}}(\vec{m}_1)}) \ge 1 - \frac{1}{6n}. \qquad \square$$

COROLLARY 4.15. *For π standard, if $\mu(S_{\vec{H}_1,\vec{m}}) > 0$ for some non-trivial $(\vec{H},\vec{m})$, then for some non-trivial $(\vec{H}_1,\vec{m}_1)$, for a set of $\vec{\lambda} \in \Lambda_0$ of measure at least $1/2$, for $k_i = k_i(\vec{\lambda})$,*

$$\lim_{i\to\infty} B^{ki}(\lambda) \circ \cdots \circ B^{1}(\vec{\lambda}) \circ (B^{\vec{\pi}}(\vec{H}_1)t_{\vec{\pi}}(\vec{\lambda}) + B^{\vec{\pi}}(\vec{m}_1)) = 0$$

($t_{\vec{\pi}}(\vec{\lambda})$ is that value putting $(B^{\vec{\pi}}(\vec{H}_1)t + B^{\vec{\pi}}(\vec{m}_1))$ in $\vec{\lambda}^{\perp}$).

We now have to do a bit of computing, and so list the set

$$\begin{aligned}
&L_0(\vec{\lambda}) \\
&= \{B^{\pi}(\vec{H}_1)t_{\pi}(\vec{\lambda}) + B^{\vec{\pi}}(\vec{m}_1); \vec{\pi} \in W(\pi)\} \\
&= \{\vec{H}_1 t_{1,i}(\vec{\lambda}) + \vec{m}_1 + (H_{1,i}t_{1,i}(\vec{\lambda}) + m_{1,i})(\vec{v}_1 - \vec{v}_n) + (H_{1,n}t_{1,i}(\vec{\lambda}) + m_{1,n})(\vec{v}_n), \\
&\quad \vec{H}_1 t_{1,n}(\vec{\lambda}) + \vec{m}_1 + (H_{1,n}t_{1,n}(\vec{\lambda}) + \vec{m}_{1,n})(-\vec{v}_n)\} \\
&\quad \cup \{\vec{H}_1 t_{2,i}(\vec{\lambda}) + \vec{m}_1 + 2(H_{1,i}t_{2,i}(\vec{\lambda}) + m_{1,i})(\vec{v}_i - \vec{v}_n) + (H_{1,n}t_{2,n}(\vec{\lambda}) + m_{1,n})(-\vec{v}_n), \\
&\quad \vec{H}_1 t_{2,n}(\vec{\lambda}) + \vec{m}_1 + 2(H_{1,n}t_{2,n}(\vec{\lambda}) + \vec{m}_{1,n})(-\vec{v}_n)\} \\
&\quad \cup \{\vec{H}_1 t_{3,i}(\vec{\lambda}) + \vec{m}_1 + 3(H_{1,i}t_{3,i}(\vec{\lambda}) + m_{1,i})(\vec{v}_i - \vec{v}_n) + (H_{1,n}t_{3,n}(\vec{\lambda}) + m_{1,n})(-\vec{v}_n), \\
&\quad \vec{H}_1 t_{3,n}(\vec{\lambda}) + \vec{m}_1 + 3(H_{1,n}t_{3,n}(\vec{\lambda}) + \vec{m}_{1,n})(-\vec{v}_n)\}. \qquad (13)
\end{aligned}$$

The values $t_{1,i}(\vec{\lambda})$, $t_{2,i}(\vec{\lambda})$ and $t_{3,i}(\vec{\lambda})$ are the unique values putting these vectors in $\vec{\lambda}^{\perp}$.

LEMMA 4.16. *$t_{1,i}(\vec{\lambda}) \ne t_{2,i}(\vec{\lambda}) \ne t_{3,i}(\vec{\lambda})$, μ_0-a.s.*

Proof. Suppose $t_{1,i}(\vec{\lambda}) = t_{2,i}(\vec{\lambda})$. Subtracting the ith elements of the first set from the second, we obtain

$$(\vec{H}_{1,i}t_{1,i}(\vec{\lambda}) + \vec{m}_{1,i})(\vec{v}_i - \vec{v}_n), \qquad \text{for } i \ne n$$

or

$$(\vec{H}_{1,n}t_{1,n}(\vec{\lambda}) - \vec{m}_{1,i})(-\vec{v}_n) \in \vec{\lambda}^{\perp}.$$

As $t_{1,i}(\vec{\lambda})$ is $\vec{\lambda}$-a.s. irrational, these are non-zero vectors and as $(\vec{v}_i - \vec{v}_n)$ and $(-\vec{v}_n)$ are in the positive cone, as is $\vec{\lambda}$, then we have a conflict. The other cases are identical. $\square$

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Definition 4.17. Let $L(\vec{\lambda}) = \{\vec{v}; \lim_{i\to\infty} B^{k_i}(\lambda) \circ \cdots \circ B^1(\vec{\lambda})\vec{v} = \vec{0}\}$, a subspace of $\mathbb{R}^n$ containing $L_0(\vec{\lambda})$ for at least half the λ's.

Take the differences between the ith elements of the first and second and the second and third subsets of $L_0(\vec{\lambda})$ to get

$$\begin{aligned} L_1(\vec{\lambda}) \quad = \quad & \{\vec{H}_1(t_{2,i}(\vec{\lambda}) - t_{1,i}(\vec{\lambda})) + (2H_{1,i}t_{2,i}(\vec{\lambda}) - H_{1,1}t_{1,i}(\vec{\lambda}) + m_i)(\vec{v}_i - \vec{v}_n) \\ & + H_{1,n}(t_{2,i}(\vec{\lambda}) - t_{1,i}(\vec{\lambda}))(-\vec{v}_n), \\ & \vec{H}_1(t_{2,n}(\vec{\lambda}) - t_{1,n}(\vec{\lambda})) + (2H_{1,n}t_{2,n}(\vec{\lambda}) - H_{1,n}t_{1,n}(\vec{\lambda}) + m_n)(-\vec{v}_n)\} \\ & \cup\{\vec{H}_1(t_{3,i}(\vec{\lambda}) - t_{2,i}(\vec{\lambda})) + (3H_{1,i}t_{3,i}(\vec{\lambda}) - 2H_{1,i}t_{2,i}(\vec{\lambda}) + m_i)(\vec{v}_i - \vec{v}_n) \\ & + H_{1,n}(t_{3,i}(\vec{\lambda}) - t_{2,i}(\vec{\lambda}))(-\vec{v}_n), \\ & \vec{H}_1(t_{3,n}(\vec{\lambda}) - t_{2,n}(\vec{\lambda})) + (3H_{1,n}t_{3,n}(\vec{\lambda}) - 2H_{1,n}t_{2,n}(\vec{\lambda}) + m_n)(-\vec{v}_n)\}. \end{aligned} \tag{14}$$

Taking differences between the ith elements of the two subsets of $L_1(\vec{\lambda})$, we get

$$\begin{aligned} & L_2(\vec{\lambda}) \\ & = \{\vec{H}_1(t_{3,i}(\vec{\lambda}) - 2t_{2,i}(\vec{\lambda}) + t_{1,i}(\vec{\lambda})) + H_{1,i}(3t_{3,i}(\vec{\lambda}) - 4t_{2,i}(\vec{\lambda}) + t_{1,i}(\vec{\lambda}))(\vec{v}_i - \vec{v}_n) \\ & \quad + H_{1,n}(t_{3,i}(\vec{\lambda}) - 2t_{2,i}(\vec{\lambda}) + t_{1,i}(\vec{\lambda}))(-\vec{v}_n), \\ & \quad \vec{H}_1(t_{3,n}(\vec{\lambda}) - 2t_{2,n}(\vec{\lambda}) + t_{1,n}(\vec{\lambda})) + H_{1,n}(3t_{3,n}(\vec{\lambda}) - 4t_{2,n}(\vec{\lambda}) + t_{1,n}(\vec{\lambda}))(-\vec{v}_n)\} \\ & \subseteq \quad L(\vec{\lambda}). \end{aligned} \tag{15}$$

LEMMA 4.18. *For $i = 1, \ldots, n$, $t_{3,i}(\vec{\lambda}) - 2t_{2,i}(\vec{\lambda}) + t_{1,i}(\vec{\lambda}) \neq 0$, μ_0-a.s.*

Proof. If the expression was equal to zero, then in $L_2(\vec{\lambda})$ we would have

$$H_{1,i}(2t_{3,i}(\vec{\lambda}) - 2t_{2,i}(\vec{\lambda}))(\vec{v}_i - \vec{v}_n) \quad \text{or} \quad H_{1,n}(2t_{3,n}(\vec{\lambda}) - 2t_{2,n}(\vec{\lambda}))(-\vec{v}_n) \in \vec{\lambda}^{\perp}.$$

But as these are non-zero vectors and $\vec{v}_i - \vec{v}_n$ and $-\vec{v}_n$ are in the positive cone this is a.s. impossible. □

COROLLARY 4.19. *For $i = 1, \ldots, n$, $3t_{3,i}(\vec{\lambda}) - 4t_{2,i}(\vec{\lambda}) + t_{1,i}(\vec{\lambda}) \neq 0$, μ_0-a.s.*

Proof. If the expression was equal to zero, for $i \neq n$ then $H_1 + H_{1,i}(\vec{v}_n) \in \vec{\lambda}^{\perp}$, and for $i = n$ then $H_1 \in \vec{\lambda}^{\perp}$, both impossible as they are in the positive cone. □

COROLLARY 4.20. *For π a standard permutation, if for some non-trivial $(\vec{H}, \vec{m})$, $\mu_0(S_{\vec{H},\vec{m}}) > 0$, then for a set of $\vec{\lambda}$ of measure $\geq 1/2$, $L(\vec{\lambda})$ contains vectors of the form*

$$\vec{H}_1 a_i + b_i(\vec{v}_i - \vec{v}_n) \quad \textit{and} \quad \vec{H}_1 a_n + b_n(-\vec{v}_n),$$

where $a_i, \ldots, a_n$ and $b_i, \ldots, b_n \neq 0$.

COROLLARY 4.21. *In this situation $L(\vec{\lambda})$ contains vectors of the form $\vec{v}_i - c_i\vec{v}_n$, $i = 1, \ldots, n-1$.*

COROLLARY 4.22. *In this situation $L(\vec{\lambda}) \supseteq H(\pi) \cap \lambda^{\perp}$.*

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Proof. As $-\vec{v}_n \neq \lambda^\perp$, μ_0-a.s. there is a unique c_i with $\vec{v}_i - c_i\vec{v}_n \in \lambda^\perp$. These span $H(\pi) \cap \lambda^\perp$. □

THEOREM 4.23. *If π is irreducible, and for some non-trivial $(\vec{H}, \vec{m})$, $\mu_0(S_{\vec{H},\vec{m}}) > 0$, then* $\dim(H(\pi)) = 2$ *and π is of rotation class.*

Proof. First, we can assume π is standard, as we can always induce from π to a standard π. By Corollary 4.18 for a set of $\vec{\lambda} \in \Lambda_0$ of measure $\geq 1/2$, for any $\vec{v} \in H(\pi) \cap \lambda^\perp$, $\lim_{i\to\infty}(B^{ki(\vec{\lambda})}(\vec{\lambda}) \circ \cdots \circ B^{(1)}(\vec{\lambda})(\vec{v})) = 0$.

The $\vec{v}_i$ are integer vectors spanning $H(\pi)$, and for any word $\vec{\pi}$ beginning and ending with π,

$$B^{\vec{\pi}} : H(\pi) \to H(\pi)$$

and is of determinant one, hence takes $\mathbb{Z}^n \cap H(\pi) \longleftrightarrow \mathbb{Z}^n \cap H(\pi)$. As this is a cocompact lattice, $B^{\vec{\pi}}$ is measure preserving on $H(\pi)$.

Consider the two vectors $\tau(\vec{\lambda})$ and $\vec{v}_i - \vec{v}_n$, both in $H(\pi)$. We will show they span $H(\pi)$, and as one is in the positive cone and the other is not, we will conclude $\dim(H(\pi)) = 2$.

Extend this pair with $\vec{b}_1, \ldots, \vec{b}_\ell \in H(\pi) \cap \lambda^\perp$ so that $\{\vec{\tau}(\lambda), \vec{v}_i - \vec{v}_n, \vec{b}_\ell\}$ are a basis for $H(\pi)$.

Let A be the $\dim(H(\pi))$-dimensional volume of the parallelepiped Q whose edges are the following vectors:

$$\begin{aligned} 0 < A &= \operatorname{vol}(Q) = \operatorname{vol}(B^{k_i}(\vec{\lambda}) \circ \cdots \circ B^1(\lambda)(Q)) \\ &\leq \|B^k(\vec{\lambda}) \circ \cdots \circ B^1(\vec{\lambda})\tau(\vec{\lambda})\|_2 \times \|B^k(\vec{\lambda}) \circ \cdots \circ B^1(\vec{\lambda})(\vec{v}_i - \vec{v}_n)\|_2 \\ &\quad \times \prod_{i=1}^{\ell} \|B^k(\vec{\lambda}) \circ \cdots \circ B^1(\vec{\lambda})(\vec{b}_i)\|_2. \end{aligned}$$

Now $\vec{v}_i - \vec{v}_n = (1, 2, \ldots, 1)$ so

$$N_i^k(\vec{\lambda}) \leq B^k(\vec{\lambda}) \circ \cdots \circ B^1(\vec{\lambda})(\vec{v}_i - \vec{v}_n) \leq 2N_i^k(\vec{\lambda}) \tag{16}$$

and so

$$\|B^k(\vec{\lambda}) \circ \cdots \circ B^1(\vec{\lambda})(\vec{v}_i - \vec{v}_n)\|_2 \leq 2\|N_i^k(\vec{\lambda})\|_2$$

as

$$B^k(\vec{\lambda}) \circ \cdots \circ B^1(\vec{\lambda})\vec{\tau}(\vec{\lambda}) = \vec{\tau}(\vec{\lambda}^k) = V(\pi_k(\vec{\lambda}))\vec{\lambda}^k,$$
$$\|B^k(\vec{\lambda}) \circ \cdots \circ B^1(\vec{\lambda})\vec{\tau}(\vec{\lambda})0\|_2 \leq n\|\vec{\lambda}^k\|_2.$$

For any of the values $k_i = k_i(\vec{\lambda})$ remember

$$\frac{\sup_i(\lambda_j^{k_i})}{\inf_i(\lambda_j^{k_i})} \leq \overline{C} \quad \text{and} \quad \frac{\sup_i(\lambda_j^{k_i})}{\inf_i(\lambda_j^{k_i})} \leq \overline{C}.$$

This says

$$1 = \vec{\lambda}^{k_i} \ldots N^{k_i} \geq \|\vec{\lambda}^{k_i}\| \|N^{k_i}\| / \overline{C}^2.$$

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Thus

$$0 < A \leq 2n\overline{C}^2 \prod_{j=1}^{\ell} \|B^{k_i}(\vec{\lambda}) \circ \cdots \circ B^1(\vec{\lambda})(\vec{b}_i)\|_2. \tag{17}$$

If any $\vec{b}_j$ exists, this is a conflict. Hence $\dim H(\pi) = 2$ and π is of rotation class. □

THEOREM 4.24. *If π is irreducible and not of rotation class and $\vec{H}$ has $\gcd(H_i) = 1$, then for a.e. $\vec{\lambda} \in \Lambda_0$, $T_{\pi,\vec{\lambda},\vec{H}}$ has no somewhere continuous eigenfunction.*

Proof. We know $\mu_0(S_{\vec{H},\vec{m}}) = 0$ and there are only countably many choices for $\vec{m}$. □

COROLLARY 4.25. *If π is irreducible and not of rotation class, and $\vec{H}$ is such that $T_{\pi,\vec{\lambda},\vec{H}}$ has a somewhere continuous eigenfunction for a set of $\vec{\lambda}$ of positive measure, then the eigenvalue is k/p, where $p = \gcd(H_i)$.*

Proof. If $p = \gcd(H_i)$, then setting $\vec{H}_1 = \vec{H}/p$, the ergodic components of $T_{\pi,\vec{\lambda},\vec{H}}$ are a.s. p copies of $T_{\pi,\vec{\lambda},\vec{H}_1}$. These have only constants as somewhere continuous eigenfunctions, by Theorem 4.20. □

5. *Rotation class*

For completeness we also want to understand the somewhere continuous eigenfunctions when π is of rotation class.

To begin, let $\pi = (1, 2)$, the interchange on two symbols. $T_{\pi,\vec{\lambda}}$ has a continuous eigenfunction $f(x) = x$ with eigenvalue λ_1. If $\vec{H} = (H, H)$ is a tower of constant height, we still have an irrational eigenvalue λ_1/H. If we induce such a system we, of course, keep the eigenfunction.

LEMMA 5.1. *If $\vec{H} \in \mathbb{N}^n$, then $T_{\pi,\vec{\lambda},\vec{H}_1}, \pi = (1, 2)$, is induced from a tower of constant height over some $T_{\pi,\vec{\lambda}'}$.*

Proof. For $H_1 = H_2$ we are done.

If $H_1 > H_2$, then $T_{\pi,\vec{\lambda},\vec{H}}$ is induced from $T_{\pi,\vec{\lambda}',\vec{H}'}$, where $\lambda'_2 > \lambda'_1$ and $H'_1 = H_2$, $H'_2 = H_1 - H_2$.

If $H_1 < H_2$, then $T_{\pi,\vec{\lambda},\vec{H}}$ is induced from $T_{\pi,\vec{\lambda}',\vec{H}'}$, where $\lambda'_2 < \lambda'_1$ and $H'_1 = H_1$, $H'_2 = H_2 - H_1$.

This process continues until $H'_1 = H'_2$ (perhaps $= 1$). □

THEOREM 5.2. *If $\pi = (1, 2)$ and $\vec{H} \in \mathbb{N}^2$, for a.e. $\vec{\lambda} \in \Lambda_0$, if $T_{\pi,\vec{\lambda},\vec{H}}$ has somewhere continuous eigenfunction with eigenvalue α then:*

(a) *if α is rational, then $\alpha = k/p$, $p = \gcd(H_i)$;*

(b) *if α is rational, then $T_{\pi,\vec{\lambda},\vec{H}}$ is induced from some $T_{\pi,\vec{\lambda}',\vec{H}'}$, $\vec{H}' = (H, H)$ and*

$$\alpha = \frac{\lambda'_1}{\|\vec{\lambda}\|_1 H}.$$

Now let us examine $\pi = (1, 3)(2)$, the simplest non-rotation of rotation class.

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THEOREM 5.3. *For $\pi = (1,3)(2)$ and $\vec{H} \in \mathbb{N}^3$ for a.e. $\vec{\lambda} \in \Lambda_0$, if $T_{\pi,\vec{\lambda},\vec{H}}$ has a somewhere continuous eigenfunction with eigenvalue α then:*

(a) *if α is rational, then $\alpha = k/p$, $p = \gcd(H_i)$;*

(b) *if α is irrational, then $T_{\pi,\vec{\lambda},\vec{H}}$ is induced from*

$$T_{\pi,\vec{\lambda}',\vec{H}'}, \quad \text{where } \pi' = \begin{cases} (1,2,3) \\ \text{or} \\ (1,3,2) \end{cases}$$

and $\vec{H}' = (H, H, H)$. As π' is a rotation,

$$\alpha = \begin{cases} \lambda_1'/\|\vec{\lambda}'\|_1 H \\ \text{or} \\ \lambda_1' + \lambda_2'/\|\vec{\lambda}'_1\| H. \end{cases}$$

Proof. Part (a) is as usual. If f is a somewhere continuous eigenfunction, then f is continuous on the base of the tower by Theorem 4.1. Consider the six numbers

$$\begin{aligned} a_1 &= \lim_{x\to 0} f(x), \\ b_1 &= \lim_{x\to\lambda_1^-} f(x) = a_2 = \lim_{x\to\lambda_1^+} f(x), \\ b_2 &= \lim_{x\to(\lambda_1+\lambda_2)^-} f(x) = a_3 = \lim_{x\to(\lambda_1+\lambda_2)^+} f(x) = b_3 = \lim_{x\to 1^-} f(x) \end{aligned}$$

and the six numbers

$$\begin{aligned} c_3 &= \lim_{x\to(\lambda_1+\lambda_2)^+} f(T_{\pi,\vec{\lambda}}(x)), \\ d_3 &= \lim_{x\to\lambda_1^-} f(T_{\pi,\vec{\lambda}}(x)) = c_2 = \lim_{x\to\lambda_1^+} f(T_{\pi,\vec{\lambda}}(x)), \\ d_2 &= \lim_{x\to(\lambda_1+\lambda_2)^-} f(T_{\pi,\vec{\lambda}}(x)) = \lim_{x\to 0^+} f(T_{\pi,\vec{\lambda}}(x)) = c_1, \\ d_1 &= \lim_{x+\lambda_2^-} f(T_{\pi,\vec{\lambda}}(x)). \end{aligned}$$

Now

$$\begin{aligned} &c_1 = a_1 + H_1\alpha, \quad c_2 = a_2 + H_2\alpha, \quad c_3 = a_3 + H_3\alpha, \\ &d_1 = b_1 + H_1\alpha, \quad d_2 = b_2 + H_2\alpha, \quad d_3 = b_3 + H_3\alpha, \end{aligned}$$

and

$$a_1 = c_3, \quad b_3 = d_1.$$

Starting at a_1,

$$\begin{aligned} a_1, c_1 &= a_1 + H_1\alpha = d_2 = b_2 + H_2\alpha = H_2\alpha \\ &= c_3 - H_3\alpha + H_2\alpha = a_1 - H_3\alpha + H_2\alpha \pmod 1. \end{aligned}$$

Thus $(H_1\alpha + H_3\alpha - H_2\alpha) \pmod 1 = 0$. As α is irrational, $H_1 + H_3 - H_2 = 0$, or $H_2 = H_1 + H_3$. Thus $T_{\pi,\vec{\lambda},\vec{H}}$ is induced from $T_{\pi',\vec{\lambda}',\vec{H}'}$, where $\pi' = (1,3,2)$, $\vec{\lambda}' = (\lambda_1 + \lambda_2, \lambda_3, \lambda_2)$ and $\vec{H}^1 = (H_1, H_3, H_3)$. This is isomorphic to a tower over a rotation on two intervals which we have seen is induced from a tower of constant height. □

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THEOREM 5.4. *If $T_{\pi,\vec{\lambda},\vec{H}}$ is of rotation class and has a somewhere continuous eigenfunction with eigenvalue α then a.s.:*

(a) *if α is rational, $\alpha = k/p$, $p = \gcd(H_i)$; and*

(b) *if α is irrational then a discontinuity of $\vec{H}_{\vec{\lambda}}(x)$ can occur only at a discontinuity of $T_{\pi,\vec{\lambda}}(x)$ and $T_{\pi,\vec{\lambda},\vec{H}}$ is induced from a tower of constant height over a rotation.*

Proof. Part (a) is standard. For (b), consider the affine coboundary $\widetilde{f}(T_{\pi,\vec{\lambda}}(x)) - \widetilde{f}(x) = \vec{H}_{\vec{\lambda}}(x)\alpha(\vec{\lambda}) - \vec{m}_{\vec{\lambda}}(x)$, where $\widetilde{f}$ is a continuous lift of f. Clearly $\vec{H}_{\vec{\lambda}}$ is continuous wherever $T_{\pi,\vec{\lambda}}$ is continuous. Thus $T_{\pi,\vec{\lambda},\vec{H}}$ is isomorphic to a tower over a rotation on two or three intervals and we apply our previous Theorems 5.2 and 5.3. □

We must add that the condition that the eigenfunction be somewhere continuous can be removed in both Theorems 5.2 and 5.3. For rotations this is classical. For (1, 3, 2) see Veech [**9**]. We do not know if this is true of Theorem 4.20.

6. *Topological weakly mixing*

We show in this section that when $T_{\pi,\vec{\lambda},\vec{H}}$ has no somewhere continuous eigenfunction then in the cartesian square $T_{\pi,\vec{\lambda},\vec{H}} \times T_{\pi,\vec{\lambda},\vec{H}}$, a.e. orbit is dense. This is little more than an application of the following.

THEOREM 6.1. (Keynes and Robertson [**5**], Furstenberg [**2**]) *Suppose T and S are minimal homeomorphisms of compact metric spaces X and Y with invariant measures μ and ν. If T has no continuous non-constant eigenfunctions then $(\mu \times \nu)$-a.e. orbit of $T \times S$ is dense in $X \times Y$.*

Proof. (McMahon [**7**]) Suppose there is a $\mu \times \nu$ set of positive measure without dense orbits. We can then find a closed $(T \times S)$-invariant set $F \subsetneqq X \times Y$ with $0 < \mu \times \nu(F) < 1$. Let $F_x = \{Y : (x, y) \in F\}$. If $x_i \to {}_i x$ then $F_x \supseteq \limsup(F_{x_i})$, hence as T is minimal $\mu(F_x) = \sup_{x \in X} \mu(F_x) = c$, a constant everywhere. Thus if $x_i \to {}_i x$,

$$\mu(F_{x_i} \triangle F_x) \longrightarrow 0.$$

Define $m(x_1, x_2) = \mu(F_{x_i} \triangle F_{x_2})$, and now if $x_i^1 \to {}_i x_1$, $x_i^2 \to {}_i x_2$ then $m(x_i^1, x_i^2) \longrightarrow {}_i m(x_1, x_2)$.

Thus the relation $x_1 \sim x_2$ if $m(x_1, x_2) = 0$ is a closed equivalence relation, and the map $x \to [x]_\sim$ is continuous. The space of equivalence classes is non-trivial, since if $x \sim T(x)$, then $G = \bigcap_{i=-\infty}^{\infty} F_{T^i(x)}$ would be a closed S-invariant set with $\nu(G) = c \neq 0, 1$.

By compactness, m is equivalent to the induced metric on the factor space of equivalence classes, and hence T has a continuous isometric factor. Minimal isometries are monethetic groups and always have continuous characters (eigenfunctions). □

To apply this to interval exchanges all we need to do is to find appropriate minimal, uniquely ergodic models. This is standard and easily described. Here is one way.

Let $T = T_{\pi,\vec{\lambda}}$. The intervals I_j are half-open half-closed. Let $P = \{I_1, \ldots, I_n\}$ be the partition into these intervals, and let $P_n = V_{i=-n}^{n} T^{-i}(P)$; again a partition into left-closed, right-open intervals. The elements of P_n correspond to T, P-names from $-n$ to n.

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A chain $\mathcal{C}$ is a sequence $\{C_n\}$, $C_n \in P_n$ and $C_{n+1} \subseteq C_n$. If $\vec{\lambda}$ has Keane's i.d.o.c. then for any chain $\mathcal{C}$, $\bigcap_n C_n$ is either empty, or is a single point $\{x\}$. It is empty iff for all n sufficiently large, C_{n+1} is the rightmost subinterval of C_n. No matter what, $\bigcap \overline{C}_n = \{x\}$, and we say $x = \varphi(\mathcal{C})$.

Now if $\mathcal{C}_1 \neq \mathcal{C}_2$ but $\varphi(\mathcal{C}_1) = \varphi(\mathcal{C}_2)$, then one of the two is an empty chain and $x = \varphi(\mathcal{C}_1)$ is the endpoint of an interval in some P_n and these are the only two chains with $\varphi(\mathcal{C}) = x$.

The space of chains $\mathbb{C}$ carries a natural compact metric topology:

$$m(\mathcal{C}_1, \mathcal{C}_2) = (\min(k : C_{1,k} \neq C_{2,k}))^{-1}.$$

For this topology, $\varphi : \mathbb{C} \to x$ is continuous.

The action of T lifts to $\mathbb{C}$, as $P_{n+1} = T(P_n)VT^{-1}(P_n)$. Set $\widetilde{T}_{\pi,\vec{\lambda}}(\mathcal{C}) = \mathcal{C}'$, where

$$T(C_{n+1}) \subseteq C'_n.$$

One easily checks that this is a homeomorphism and

$$\phi(\widetilde{T}(\mathcal{C})) = T(\phi(\mathcal{C})).$$

The system $(\widetilde{T}, \mathbb{C})$ is an almost one-to-one lift of $(T_{\pi,\vec{\lambda}}, x)$.

A tower construction $T_{\pi,\vec{\lambda},\vec{H}}$ lifts to $(\widetilde{T}, \mathbb{C})$ but notice the tower heights can be made functions of C_0 above, hence continuous on $\mathbb{C}$. This will not project everywhere to $T_{\pi,\vec{\lambda},\vec{H}}$ as we may give distinct heights for the two preimages of an endpoint of an I_j, but μ_0-a.s. it does.

LEMMA 6.2. *For π irreducible and a.e. $\vec{\lambda} \in \Lambda_0$, $(\widetilde{T}_{\pi,\vec{\lambda}}, \mathbb{C})$ is minimal and uniquely ergodic.*

Proof. The Veech [**8**] and Kerckhoff [**4**] pictures apply equally well here. □

LEMMA 6.3. *If $(\widetilde{T}_{\pi,\vec{\lambda}}, \mathbb{C})$ is minimal and uniquely ergodic so is $\widetilde{T}_{\pi,\vec{\lambda},\vec{H}}$.*

LEMMA 6.4. *If π is irreducible, then μ_0-a.s. if $\widetilde{T}_{\pi,\vec{\lambda},\vec{H}}$ has a continuous eigenfunction then $T_{\pi,\vec{\lambda},\vec{H}}$ has a somewhere continuous eigenfunction.*

THEOREM 6.5. *If $T_{\pi,\vec{\lambda},\vec{H}}$ is minimal and has no non-constant somewhere continuous eigenfunction, and $T_{\pi',\vec{\lambda}',\vec{H}'}$ is minimal then the $(\mu \times \mu)$-a.e. orbit of $T_{\pi,\vec{\lambda},\vec{H}} \times T_{\pi',\vec{\lambda}',\vec{H}'}$ is dense.*

Maps of the form $T_{\pi,\vec{\lambda},\vec{H}} \times T_{\pi',\vec{\lambda}',\vec{H}'}$ are a special class of 'rectangle exchanges' and our conclusion tells us exactly when they have many dense orbits. In this situation there seems to be no good two-dimensional analogue of Rauzy induction.

We are convinced much of our efforts here will be obviated as surely a.e. $T_{\pi,\vec{\lambda},\vec{H}}$, which is topologically weakly mixing, is actually weakly mixing. The proof of this could take one of two forms. Perhaps the dynamics of Rauzy induction can be well-enough understood to push our argument for eigenfunctions only assumed measurable. On the other hand, perhaps some other line of reasoning can show eigenfunctions must be at least somewhere continuous. As no examples are known for an interval exchange with an irrational eigenvalue with discontinuous eigenfunction, perhaps this is not totally out of the question.

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