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Nonlinear non-stationary Alfvén resonance

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Abstract. For a cold plasma that is inhomogeneous (in the direction across an external homogeneous magnetic field), the nonlinear equation describing the spatial structure and temporal behaviour of a non-stationary disturbance in a resonance layer is obtained. The matching conditions for a disturbance through the resonance layer are obtained, and in the linear limit give a well-known linear matching. It is shown that the spatial and temporal behaviour of the resonance disturbance and the evolution of the resonant absorption in terms of nonlinear theory are determined by the ratio of the nonlinear to linear non-stationary spatial scales. The spatial–temporal profile of the disturbance in the resonance layer and the resonant absorption for different values of this ratio are calculated. A nonlinear decrease in the resonant absorption and a stratification of the resonance disturbance are revealed.

1. Introduction

Alfvén resonance is a key element in the theory of energy transfer from large-scale to small-scale disturbances and then to plasma particles in such diverse areas of research into plasma processes as solar physics (solar coronal heating), magnetospheric physics (geomagnetic pulsations and associated phenomena of plasma heating and particle precipitation) and the physics of plasma heating for thermonuclear fusion. In terms of linear theory for Alfvén resonance, it was established that the spatial structure of a resonance disturbance in the direction across an external magnetic field and the mechanism for extracting energy from a large-scale disturbance are determined through effects characterized by small parameters. Specifically, these parameters are the absorption of energy due to resistivity (Southwood 1974) and viscosity (Davila 1987), and the escape of energy from the resonance layer in the form of small-scale kinetic (Chen and Hasegawa 1974) or inertial (Swanson 1975) Alfvén waves. The importance of the role of small effects (small in the sense that their influence is small everywhere except for the resonance layer) and a relatively large disturbance amplitude in the resonance region inevitably leads to the question: What is the role of a further small effect, namely nonlinearity? Interest in this question has also been stimulated by the disagreement of some predictions of linear theory with observational data (Ofman et al. 1994). Nonlinear effects associated with the presence of Alfvén resonance, were treated by Ofman et al. (1994) and Ofman and Davila (1995) via numerical solution of nonlinearized MHD equations for a slab of low-β resistive plasma with non-uniform density modelling a coronal loop and the time-dependent disturbance exited by a

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driver. Deformation of the resonance layers and a drop in the heating rate were found. Analytical methods were employed by Lundberg and Brodin (1995) to obtain the equations that govern the temporal behaviour of surface waves and the spatial structure of the resonant disturbance in an incompressible plasma. Differences between the linear and nonlinear evolutions of the initial disturbance were revealed.

The objective of this paper is to study nonlinear Alfvén resonance for the case where the effect of weak non-stationarity is dominant among linear factors. A disturbance will be said to be non-stationary if its amplitude outside the resonance region varies slowly with time. Such a disturbance in the resonance layer shows a relatively complicated space–time behaviour.

The work described in this paper proceeds in two stages. The first stage includes the derivation of the equation for the disturbance in the resonance layer, and takes account simultaneously of non-stationarity and nonlinearity, and of the matching conditions for the outer large-scale solution through the nonlinear non-stationary resonance layer. This equation and the matching conditions are suited to any inhomogeneity profile. To obtain the nonlinear equation in the resonance layer, the technique of Dmitrienko (1997), based on the method of different scales, is used. The influence of nonlinearity on the resonance disturbance structure in the case where resistivity is dominant among linear structure-forming factors was studied by Dmitrienko (1997). It was shown that nonlinearity manifests itself in the self-action of the disturbance through mean parameters of the resonance layer that it changes; a nonlinear equation was obtained that describes the spatial structure of the disturbance in the resonance layer and self-consistently takes into account the change in the medium produced by the disturbance. An analogous equation is obtained in this case; however, the mean density, magnetic field and plasma flow velocity are inferred from the effect of disturbance non-stationarity, and differ substantially from those obtained in Dmitrienko (1997). The latter work on predominant small effects may be characterized as a study of nonlinear resistive Alfvén resonance, while the case taken up in this paper may be defined as nonlinear non-stationary Alfvén resonance; it can be treated as a different limiting case from that dealt with by Dmitrienko (1997).

The second stage of this study involves using the resulting nonlinear equation and matching conditions to investigate nonlinear non-stationary Alfvén resonance in a plasma with a linear inhomogeneity profile. On the coordinate along the inhomogeneity, there is a region of disturbance propagation along this coordinate; the amplitude of the incident wave is assumed to be known at every instant of time, and we calculate the resonance disturbance produced by this wave and investigate resonant absorption, both for the energy flux difference in the incident and reflected waves and for the total energy accumulated in the resonance layer. The incident and reflected waves can be classified, according to their characteristics, as FMS-type disturbances, and the resonant disturbance as an Alfvén-type disturbance. Since effects causing the Alfvén waves to propagate across the magnetic field are neglected, energy transferred by the FMS wave is accumulated in the resonance layer in the form of an Alfvén-type disturbance. The nonlinear evolution of the energy accumulation process in the resonance layer is compared with the linear evolution, and conditions are determined that permit either regime, for two types of behaviour of the incident

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wave: a wave with an established constant amplitude, and a wave train of finite duration.

2. Basic equations

We shall model the plasma inhomogeneity by the dependence of its undisturbed density on x; $n_0 = n_0(x)$. The undisturbed magnetic field $\mathbf{B}_0$ is considered to be homogeneous: $\mathbf{B}_0 = \{0, 0, B_0\}$. To describe the disturbance, we employ the quantities

$$\mathbf{b} = \frac{\mathbf{B}-\mathbf{B}_0}{B_0}, \qquad \mathbf{u} = \frac{\omega}{k}\mathbf{v}, \qquad r = \frac{n-n_0}{n_0(x_0)}.$$

To represent these quantities in a uniform fashion, we introduce the notation $\mathbf{\Psi} = \{\mathbf{b}, \mathbf{u}, r\}$. All Ψ_i will be assumed to be functions of the variables

$$\theta = -\omega t + k_z z + k_y y, \qquad \tau = \omega t, \qquad \xi = \frac{x-x_0}{l},$$

where

$$l = \left(\frac{d\ln n_0}{dx}\bigg|_{x=x_0}\right)^{-1}.$$

Where appropriate, we shall also use $\tilde{\mathbf{\Psi}}$ and $\bar{\mathbf{\Psi}}$ for those parts of $\mathbf{\Psi}$ that are respectively dependent on and independent of θ. With this notation, the ideal magnetohydrodynamic equations may be written as

$$\partial_\tau b_x - \partial_\theta(b_x + u_x) = n_1, \tag{2.1}$$

$$\partial_\tau b_y - \partial_\theta(b_y + u_y) = n_2, \tag{2.2}$$

$$\partial_\tau b_z - \partial_\theta b_z + \frac{1}{k_z l}\partial_\xi u_x + \frac{k_y}{k_z}\partial_\theta u_y = n_3, \tag{2.3}$$

$$\partial_\tau u_x - \partial_\theta\left(u_x + \frac{k_z^2}{q^2}b_x\right) + \frac{k_z}{q^2 l}\partial_\xi b_z = n_4, \tag{2.4}$$

$$\partial_\tau u_y - \partial_\theta\left(u_y + \frac{k_z^2}{q^2}b_y\right) + \frac{k_z k_y}{q^2}\partial_\theta b_y = n_5, \tag{2.5}$$

$$\partial_\tau u_z - \partial_\theta u_z = n_6, \tag{2.6}$$

$$\partial_\tau r - \partial_\theta r + r_0\left(\frac{1}{k_z l}\partial_\xi u_x + \frac{k_y}{k_z}\partial_\theta u_y + \partial_\theta u_z\right) + u_x\frac{1}{k_z l}\partial_\xi r_0 = n_6. \tag{2.7}$$

Here

$$q^2 = \frac{4\pi m_i n_0 \omega^2}{B_0^2}, \qquad r_0 = \frac{n_0(x)}{n_0(x_0)},$$

and the nonlinear parts of (2.1)–(2.7) have the form

$$n_1 = \frac{k_y}{k_z}\partial_\theta(u_x b_y - u_y b_x) - \partial_\theta(u_z b_x - u_x b_z), \tag{2.8}$$

$$n_2 = \partial_\theta(u_y b_z - u_z b_y) - \frac{1}{k_z l}\partial_\xi(u_x b_y - u_y b_x), \tag{2.9}$$

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$$n_3 = \frac{1}{k_z l}\partial_\xi(u_z b_x - u_x b_z) - \frac{k_y}{k_z}\partial_\theta(u_y b_z - u_z b_y), \tag{2.10}$$

$$n_4 = \frac{r}{r_0}(\partial_\theta u_x - \partial_\tau u_x) - \left(1+\frac{r}{r_0}\right)\left(\mathbf{u}\cdot\frac{\boldsymbol{\nabla}}{k_z}\right)u_x + \frac{k_z^2}{q^2}\left[b_z\left(\partial_\theta b_x - \frac{1}{k_z l}\partial_\xi b_z\right) - b_y\left(\frac{1}{k_z l}\partial_\xi b_y - \frac{k_y}{k_z}\partial_\theta b_x\right)\right], \tag{2.11}$$

$$n_5 = \frac{r}{r_0}(\partial_\theta u_y - \partial_\tau u_y) - \left(1+\frac{r}{r_0}\right)\left(\mathbf{u}\cdot\frac{\boldsymbol{\nabla}}{k_z}\right)u_y + \frac{k_z^2}{q^2}\left[b_x\left(\frac{1}{k_z l}\partial_\xi b_y - \frac{k_y}{k_z}\partial_\theta b_x\right) - b_z\left(\frac{k_y}{k_z}\partial_\theta b_z - \partial_\theta b_y\right)\right], \tag{2.12}$$

$$n_6 = \frac{r}{r_0}(\partial_\theta u_z - \partial_\tau u_z) - \left(1+\frac{r}{r_0}\right)\left(\mathbf{u}\cdot\frac{\boldsymbol{\nabla}}{k_z}\right)u_z + \frac{k_z^2}{q^2}\left[b_y\left(\frac{k_y}{k_z}\partial_\theta b_z - \partial_\theta b_y\right) - b_x\left(\partial_\theta b_x - \frac{1}{k_z l}\partial_\xi b_z\right)\right], \tag{2.13}$$

$$n_7 = -\frac{1}{k_z l}\partial_\xi(r u_x) - \frac{k_y}{k_z}\partial_\theta(r u_y) - \partial_\theta(r u_z). \tag{2.14}$$

Along the coordinate ξ, two regions can be identified: an inner region where nonlinearity and non-stationarity can determine the disturbance structure (the resonance layer), and an outer region where these effects play the role of small corrections in the formation of the transverse structure of the disturbance.

3. Outer region

3.1. Small parameters and corresponding expansions

In the outer region, it is assumed that

$$\boldsymbol{\Psi}(\xi,\theta,\tau) = \sum_{n=1}^{\infty} \epsilon^n \boldsymbol{\Psi}^{(n)}(\xi,\theta,\tau),$$

where ϵ is a small parameter characterizing the disturbance amplitude, and

$$\boldsymbol{\Psi}^{(n)}(\xi,\theta,\tau) = \boldsymbol{\Psi}_0^{(n)}(\xi,\tau) + \sum_{l=1}^{\infty}[\boldsymbol{\Psi}_l^{(n)}(\xi,\tau)\mathbf{e}^{il\theta} + \text{c.c.}].$$

We introduce a parameter $\Gamma \sim |\partial_\tau \ln \Psi|$ as a measure of non-stationarity (it will be assumed that $\Gamma \ll 1$). Further, we introduce a 'slow time'

$$T = \Gamma\tau,$$

and we shall formally assume that

$$\boldsymbol{\Psi}_l^{(n)}(\xi,\tau) = \sum_{m=0}^{\infty} \Gamma^m \boldsymbol{\Psi}_l^{(n,m)}(\xi,\tau), \qquad l = 0, 1, \ldots \quad .$$

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3.2. Fundamental harmonic

At order $\epsilon^1\Gamma^0$, we obtain from (2.1)–(2.2) the familiar equation describing a linear stationary disturbance (Southwood 1974):

$$\partial^2_{\xi\xi} b_{z1}^{(1,0)} - \partial_\xi \ln\left(\frac{q^2}{k_z^2}-1\right)\partial_\xi b_{z1}^{(1,0)} + (k_z l)^2\left(\frac{q^2}{k_z^2}-1-\frac{k_y^2}{k_z^2}\right) b_{z1}^{(1,0)} = 0. \tag{3.1}$$

The other $\Psi_i^{(1,0)}$ are related to $b_{z1}^{(1,0)}$ by

$$b_{x1}^{(1,0)} = \frac{i}{k_z l}\frac{\partial_\xi b_{z1}^{(1,0)}}{q^2/k_z^2-1}, \quad b_{y1}^{(1,0)} = -\frac{k_y}{k_z}\frac{b_{z1}^{(1,0)}}{q^2/k_z^2-1}, \tag{3.2a,b}$$

$$\mathbf{u}_{\perp 1}^{(1,0)} = -\mathbf{b}_{\perp 1}^{(1,0)}, \qquad u_{z1}^{(1,0)} = 0, \tag{3.2c,d}$$

$$r_1^{(1,0)} = r_0 b_{z1}^{(1,0)} - \frac{1}{(k_z l)^2}\frac{\partial_\xi b_{z1}^{(1,0)}}{q^2/k_z^2-1}. \tag{3.2e}$$

In the opacity region, when $\xi = 0$, Alfvén resonance occurs; that is, a disturbance of FMS type transforms to a disturbance of Alfvén type. Mathematically, the resonance point is a regular singular point of (3.1); therefore its solution may be represented as

$$b_{z1}^{(1,0)} = A_\pm F_A(\xi) + B_\pm F_B(\xi), \tag{3.3}$$

where $F_A(\xi)$ is regular in 0 and $F_B(\xi)$ has a logarithmic singularity. The '+' and '−' signs refer to the regions $\xi > 0$ and $\xi < 0$ respectively. We choose F_A and F_B such that the series expansions of the solutions of (3.1) have the form

$$b_{z1}^{(1,0)} = A_\pm\left[\frac{k_z k_y l^2}{2}\xi^2 + O(\xi^4)\right] + B_\pm\left[\frac{k_z}{k_y} + \frac{k_z k_y l^2}{2}\xi^2 \ln|\xi| + O(\xi^4)\right]. \tag{3.4}$$

Note that to obtain the first terms of the series in (3.4), it is sufficient to limit attention to the linear term in the expansion $r_0(\xi)$ as a series near $\xi = 0$:

$$r_0 = 1 + \xi, \tag{3.5}$$

or, instead of (3.1), to the equation

$$\partial^2_{\xi\xi} b_{z1}^{(1,0)} - \frac{1}{\xi}\partial_\xi b_{z1}^{(1,0)} + (k_z l)^2\left(\xi - \frac{k_y^2}{k_z^2}\right) b_{z1}^{(1,0)} = 0. \tag{3.6}$$

The other $\Psi_i^{(1,0)}$ as $\xi \to 0$ are determined from (3.2) through (3.4) and (3.5):

$$b_{x1}^{(1,0)} \sim u_{x1}^{(1,0)} \sim r_1^{(1,0)} \sim \ln\xi, \qquad b_{y1}^{(1,0)} \sim u_{y1}^{(1,0)} \sim \xi^{-1}. \tag{3.7a,b}$$

Subsequently, to match the solution through the resonance region, we shall use the expansion $b_{z1}^{(1,0)}$ in the form (3.4) and the expansion $b_{x1}^{(1,0)}$, the explicit form of which is

$$b_{x1}^{(1,0)} = ik_y l\{A_\pm[1 + O(\xi^2)] + B_\pm[\ln|\xi| + O(\xi^2)]\}. \tag{3.8}$$

The matching of A_+ and A_- and of B_+ and B_- is done by specializing from the general model of a stationary, ideal magnetohydrodynamic disturbance. A linear initial disturbance evolution corresponds most closely to the case of a non-stationary disturbance considered in this paper; in terms of this problem, one obtains the choice rule for the algorithm branch in (3.4), and, as a

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consequence, the matching $B_+ = B_-$ and $A_+ - A_- = -\pi i B$. From linear theory, it is also known that the jump $A_+ - A_-$ is associated with energy absorption of a large-scale disturbance in the resonance layer (Southwood and Kivelson 1986), and does not depend on which particular physical mechanism is responsible for the regularization of the singularity in the solution. Further, it will be shown that a linear matching of A_+ and A_- is the linear limit of a nonlinear matching dependent on the disturbance amplitude.

To take account of the effects of non-stationarity and nonlinearity in the outer region, we can apply perturbation theory and find the corresponding corrections (see Appendices A and B). In view of these corrections, for the fundamental harmonic in the outer region we obtain

$$b_{z1} = \epsilon\left\{A_\pm\left[\frac{k_z k_y l^2}{2}\xi^2 + O(\xi^4)\right] + B_\pm\left[\frac{k_z}{k_y} + \frac{k_z k_y l^2}{2}\xi^2 \ln|\xi| + O(\xi^4)\right] + \Gamma O(\xi)\right\} + \epsilon^3 O(\xi^{-3}) + \dots, \tag{3.9a}$$

$$b_{x1} = \epsilon\{ik_y l\{A_\pm[1 + O(\xi^2)] + \mathrm{B}_\pm[\ln|\xi| + O(\xi)]\} + \Gamma O(\xi^{-1})\} + \epsilon^3 O(\xi^{-3}) + \dots \quad . \tag{3.9b}$$

One can see that the non-stationary and nonlinear corrections are comparable to the main-order solution when $\xi \sim \Gamma$ and $\xi \sim \epsilon^{2/3}$ respectively. What this means is, we believe, that non-stationarity and nonlinearity play a structure-forming role in the resonance layer, and the spatial scales Δ_Γ and Δ_N on which their influence is important are Γ and $\epsilon^{2/3}$.

4. Resonance layer

We now introduce the variable $\zeta = \epsilon^{-2/3}\xi$. In accordance with (3.4) and (3.7), and (A1), (A2) and (B1)–(B3) from the appendices, the solution can be represented in the resonance layer as

$$\Psi_i = \begin{cases} \epsilon\Psi_i^{(1)}(\zeta,\theta,\tau) + \epsilon^{4/3}\Psi_i^{(4/3)}(\zeta,\theta,\tau) + \dots, & i = 1,4, \\ \epsilon^{1/3}\Psi_i^{(1/3)}(\zeta,\theta,\tau) + \epsilon^{2/3}\Psi_i^{(2/3)}(\zeta,\theta,\tau) + \dots, & i = 2,5, \\ \epsilon^{2/3}\Psi_i^{(2/3)}(\zeta,\theta,\tau) + \epsilon\Psi_i^{(1)}(\zeta,\theta,\tau) + \dots, & i = 3,6,7. \end{cases}$$

It follows from (2.4) that

$$\tilde{b}_z^{(1)} = \overbrace{-b_y^{(2/3)}\, b_y^{(1/3)}} + \Phi(\theta, T), \tag{4.1}$$

where $\Phi(\theta, T)$ is a certain function. From (2.1) and (2.2), we obtain

$$\tilde{b}_x^{(1)} + \tilde{u}_x^{(1)} = 0, \qquad \tilde{b}_y^{(1/3)} + \tilde{u}_y^{(1/3)} = 0, \tag{4.2a,b}$$

$$\tilde{b}_x^{(4/3)} + \tilde{u}_x^{(4/3)} = 0, \qquad \tilde{b}_y^{(2/3)} + \tilde{u}_y^{(2/3)} = 0. \tag{4.3a,b}$$

Similarly, we obtain from (2.4), (2.6) and (2.7),

$$\tilde{b}_z^{(2/3)} + \tilde{u}_z^{(2/3)} = 0, \qquad \tilde{r}^{(2/3)} = 0, \tag{4.4a,b}$$

and from (2.1) and (2.3), in view of the corresponding solutions in the outer region, we get

$$\bar{b}_x^{(4/3)} = 0, \qquad \bar{u}_x^{(4/3)} = 0. \tag{4.5a,b}$$

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The relationships (4.1)–(4.5) permit us to obtain from (2.2) and (2.5)

$$-2\frac{\Gamma}{\epsilon^{2/3}}\partial_T \tilde{b}_y^{(1/3)} + \zeta\partial_\theta \tilde{b}_y^{(1/3)}$$

$$+\left[\bar{r}^{(2/3)} - 2(\bar{b}_z^{(2/3)} + \bar{u}_z^{(2/3)}) - 2\frac{k_y}{k_z}(\bar{b}_y^{(2/3)} + \bar{u}_y^{(2/3)})\right]\partial_\theta \tilde{b}_y^{(1/3)}$$

$$= -\frac{k_y}{k_z}\partial_\theta \Phi. \tag{4.6}$$

According to (3.9), when $\zeta \to \infty$, $\tilde{b}_z^{(1)}$ must have the form

$$\tilde{b}_z^{(1)} = \left(B_\pm \frac{k_z}{k_y} e^{i\theta} + \text{c.c.}\right) + O(\zeta^{-3}).$$

Since $\widetilde{b_y^{(2/3)}} b_y^{(1/3)} = O(\zeta^{-3})$ according to (3.7), (B1) and (B2) when $\zeta \to \infty$, for this condition to be satisfied, one should put in (4.1)

$$\Phi = \frac{k_z}{k_y} B e^{i\theta} + \text{c.c.}, \tag{4.7a}$$

$$B_+ = B_- = B. \tag{4.7b}$$

Hence the matching condition for one of the two coefficients in (3.4) is obtained.

Using (2.3) and (2.7), and the expressions for $\bar{n}_3^{(4/3)}$ and $\bar{n}_7^{(4/3)}$ from (2.2), we obtain

$$\partial_T(\bar{r}^{(2/3)} - \bar{b}_z^{(2/3)}) = 0,$$

whence, in view of the initial condition $\mathbf{\Psi} = 0$,

$$\bar{r}^{(2/3)} = \bar{b}_z^{(2/3)}. \tag{4.8}$$

For $\bar{b}_z^{(2/3)}$, it follows from (2.4) that

$$\partial_\zeta \bar{b}_z^{(2/3)} = -\tfrac{1}{2}\overline{\partial_\zeta (\tilde{b}_y^{(1/3)})^2}.$$

By representing $\tilde{b}_y^{(1/3)}$ as

$$\tilde{b}_y^{(1/3)} = H_y e^{i\theta} + \text{c.c.}, \tag{4.9}$$

we obtain

$$\bar{b}_z^{(2/3)} = -|H_y|^2. \tag{4.10}$$

From (2.6), we find that

$$\bar{u}_z^{(2/3)} = 2|H_y|^2, \tag{4.11}$$

and, from (2.2) and (2.5),

$$\bar{b}_y^{(2/3)} + \bar{u}_y^{(2/3)} = 2\frac{k_y}{k_z}|H_y|^2. \tag{4.12}$$

We use (4.8)–(4.12) to bring (4.6) to the form

$$2i\frac{\Gamma}{\epsilon^{2/3}}\partial_T H_y + \left[\zeta - \left(3 + 4\frac{k_y^2}{k_z^2}\right)|H_y|^2\right]H_y = -B. \tag{4.13}$$

We already have the equation for a disturbance in the resonance layer; it remains to match the coefficients A_+ and A$_-$. By representing $\tilde{b}_x^{(1)}$ as

$$\tilde{b}_x^{(1)} = H_x e^{i\theta} + \text{c.c.}, \tag{4.14}$$

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we find from (3.9) that, when $\zeta \to \infty$, we have

$$H_x(\zeta, T) - H_x(-\zeta, T) = ik_y l(A_+ - A_-) + O\left(\frac{\Gamma}{\epsilon^{2/3}}\zeta^{-1}\right) + O(\zeta^{-3}), \tag{4.15}$$

while it follows from $(\nabla \cdot \mathbf{b})^{(1/3)}$ that

$$\partial_\zeta b_x^{(1)} + k_y l \partial_\theta b_y^{(1/3)} = 0,$$

or

$$H_x(\zeta, T) - H_x(-\zeta, T) = -ik_y l \int_{-\zeta}^{\zeta} H_y \, d\zeta'.$$

In the limit $\zeta \to \infty$, we obtain

$$A_+ - A_- = -\int\!\!\!\!\!\!- H_y \, d\zeta' \tag{4.16}$$

where the symbol $\int\!\!\!\!\!\!-$ means $\lim_{\zeta\to\infty} \int_{-\zeta}^{\zeta}$. For ease of comparing the nonlinear and linear cases, we rearrange (4.13). We introduce the variable

$$\eta = \frac{\epsilon^{2/3}}{2\Gamma}\zeta = \frac{\xi}{2\Gamma},$$

and

$$G(\eta, T) = \frac{2\Gamma}{\epsilon^{2/3}} H(\zeta, T).$$

We then obtain

$$i\partial_T G + (\eta - \lambda_{\text{in}} |G|^2) G = -B, \tag{4.17}$$

$$A_+ - A_- = -\int\!\!\!\!\!\!- G \, d\eta'. \tag{4.18}$$

Here the notation λ_{in} is introduced for the parameter characterizing the relative role of the non-stationarity and nonlinearity:

$$\lambda_{\text{in}} = \frac{\Delta_N^3}{\Delta_\tau^3}, \qquad \Delta_N = \left(3 + 4\frac{k_y^2}{k_z^2}\right)^{1/3} \epsilon^{2/3}, \qquad \Delta_\tau = 2\Gamma.$$

When $\lambda_{\text{in}} = 0$, the equation takes the form

$$i\partial_T G + \eta G = -B. \tag{4.19}$$

This equation describes a linear non-stationary disturbance in the resonance layer. Its solution under the initial condition $G(\eta, T_0) = 0$ is

$$G(\eta, T) = i \int_{T_0}^{T} B(T') \exp[i\eta(T - T')] \, dT'. \tag{4.20}$$

Substitution of (4.20) into (4.18) gives the usual condition of linear matching:

$$A_+ - A_- = -\pi i B. \tag{4.21}$$

5. Complete solution

We now return to (3.1). The dependence $n_0(x)$ will be considered such that $\xi > \xi_r$ is the propagation region of a disturbance of FMS type. The solution in this region is then representable as

$$b_{z1}^{(1,0)} = C_I(T) F_I(\xi) + C_R(T) F_R(\xi), \tag{5.1}$$

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where $F_I(\xi)$ is the solution of (3.1) corresponding to the incident wave and $F_R(\xi)$ is that corresponding to the reflected wave. $C_I(T)$ will be considered a specified function. The relationship (5.1) implies that in the outer region we shall be using a quasistationary approximation to describe the variation of $\Psi_i(\xi, \tau)$ with time. Such an approach is possible if the size of the system considered is not too large and the source amplitude varies slowly. Also, the relation of the coefficients C_I and C_R to A_+ and B may be represented as

$$\begin{pmatrix} C_I \\ C_R \end{pmatrix} = \hat{\mathbf{D}} \begin{pmatrix} A_+ \\ B \end{pmatrix}.$$

The boundary condition $b_{z1}^{(1,0)} \to 0$ when $\xi \to \infty$ gives the relation between A_- and B in the form

$$A_- = \delta B.$$

The matrix elements of $\hat{\mathbf{D}}$ and the coefficients δ can be determined by solving (3.1) numerically. We denote

$$p = \frac{d_{11}}{d_{12} + \delta d_{11}}, \qquad s = d_{12} + (\delta - i\pi)\, d_{11}.$$

Let $C_I(T)$ be specified in the form

$$C_I(T) = sf(T). \tag{5.2}$$

Equation (4.17) then takes the form

$$i\partial_T G + (\eta - \lambda_{\text{in}} |G|^2)\, G = -f(T)(1 - \pi i p) - p \fint G(\eta', T)\, d\eta'. \tag{5.3}$$

In the linear case,

$$i\partial_T G + \eta G = -f(T), \tag{5.4}$$

and hence ϵ is equal to the amplitude of the linear FMS wave in the resonance layer. Its relation to the incident-wave amplitude is determined by (5.1) and (5.2).

For the linear plasma density profile specified by (3.5) for all ξ, the asymptotic expansions $F_I(\xi)$ and $F_R(\xi)$ when $\xi \to \infty$ may be chosen in the form

$$F_I(\xi) = (k_z l)^{1/2}\, \xi^{1/4} \exp\{-i\tfrac{2}{3}[(k_z l)^{2/3}\, \xi - \lambda_{\text{ex}}^2]^{3/2} - \tfrac{1}{4} i\pi\} + O(\xi^{-3/4}), \tag{5.5a}$$

$$F_R(\xi) = (k_z l)^{1/2}\, \xi^{1/4} \exp\{i\tfrac{2}{3}[(k_z l)^{2/3}\, \xi - \lambda_{\text{ex}}^2]^{3/2} + \tfrac{1}{4} i\pi\} + O(\xi^{-3/4}), \tag{5.5b}$$

where

$$\lambda_{\text{ex}} = k_y l (k_z l)^{-2/3}.$$

With such a choice, p and s depend on λ_{ex} only. The energy lost by the incident wave can be characterized by the coefficient

$$\kappa = \frac{|C_I|^2 - |C_R|^2}{|C_I|^2}, \tag{5.6}$$

which we shall refer to (as is conventional) as the absorption coefficient. A large-scale disturbance loses energy by transformation to a small-scale disturbance, which in our case does not escape the layer since effects causing the Alfvén waves to propagate across the magnetic field are neglected. Hence κ can also be

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referred to as the transformation coefficient. The energy storage in the resonance layer may be described by the function

$$\mathscr{E}(T) = \int_{-\infty}^{\infty} |G|^2 \, d\eta'. \tag{5.7}$$

From (4.17), we obtain

$$i\partial_T \mathscr{E} = B(A_+^* - A_-^*) - B^*(A_+ - A_-).$$

Using the properties of the matrix $\hat{\mathbf{D}}$ that follow from (3.6) ($d_{21}^* = d_{11}$, $d_{22}^* = d_{12}$, $\det \hat{\mathbf{D}} = \frac{1}{2}i$) and (5.6), we have

$$\tfrac{1}{2}\partial_T \mathscr{E} = \kappa |C_I|^2. \tag{5.8}$$

Note that (5.8) implies the coincidence of the zeros of resonant absorption and jump $\mathrm{Im}[(A_+ - A_-)/B]$.

To clarify the relation of (5.8) to the law of conservation of energy, we return to the basic MHD equations. They give

$$\frac{\Gamma}{\epsilon^{2/3}} \partial_T \int_{-\infty}^{\infty} [\overline{(b_y^{(1/3)})^2} + \overline{(u_y^{(1/3)})^2} + 2\bar{b}_z^{(2/3)}] \, d\zeta' = -\frac{1}{k_z l} (2\overline{b_x^{(1)} b_z^{(1)}} + \bar{u}_x^{(2)}) \Big|_{-\infty}^{\infty}. \tag{5.9}$$

It is easy to see from (5.9) that (5.8) implies that the energy of the resonant disturbance in a given volume is equal to half of the flux of energy lost by the incident FMS wave when it passes through the surface of this volume. The other half of the energy left by the FMS wave in the resonance layer according to (5.9) is taken out of the resonant layer by a flow of plasma with a small velocity $\bar{u}_x \sim \epsilon^2$. This flow 'carries away' the undisturbed magnetic field B_0, causing a decrease in the mean magnetic field in the resonance layer. The total energy stored in the resonance layer is equal to $\epsilon^{4/3} \mathscr{E} B_0^2 l / 8\pi\Gamma$. The energy of the oscillations of the magnetic field, the energy of the oscillations of the plasma and the decrease in the mean magnetic field energy are also equal to this quantity. Hence the total energy accumulated in magnetic field and plasma oscillations is just equal to the loss of energy from the FMS wave.

We shall consider an incident wave with the temporal behaviour of one of two types: case (i), with

$$f(T) = \begin{cases} \frac{1}{2}[\cos(T-\pi)+1] & (0 \leqslant T \leqslant \pi), \\ 1 & (T > \pi), \end{cases}$$

reaching a constant amplitude in a time $T \sim 1$ (or $\tau \sim \Gamma^{-1}$); and case (ii), with

$$f(T) = \begin{cases} \frac{1}{2}[\cos(T-\pi)+1] & (0 \leqslant T \leqslant 2\pi), \\ 0 & (T > 2\pi), \end{cases}$$

a time-bounded $T \sim 1$ (or $\tau \sim \Gamma^{-1}$) wave train. The results of numerical solution of (5.3) and (4.18) for the two cases are summarized in graphical form later. The values of p and s were obtained by numerical solution of (3.6).

5.1. Case (i)

Figure 1 shows the temporal behaviour of the resonance absorption coefficient. The straight line (1) for a linear non-stationary resonance layer was calculated from (5.3) and (4.18) for $\lambda_{\mathrm{in}} = 0$. Although the amplitude of the incident FMS wave varies with time, the absorption coefficient is constant, and its value

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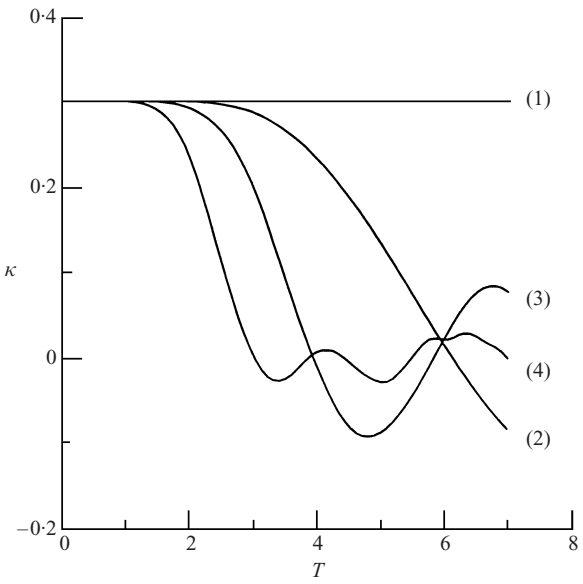


Figure 1. Temporal behaviour of the resonant absorption coefficient κ for case (i) of the time behaviour of the incident-wave amplitude when $\lambda_{ex} = 1$ and $\lambda_{in} = 0$, 0.125, 1, 8 (curves (1)–(4) respectively).

depends only on the parameter λ_{ex} ($\lambda_{ex} = 1$ was chosen), which determines the extent to which the FMS disturbance amplitude on the resonance surface is smaller than that in the propagation region. The value of the resonant absorption coefficient, corresponding to curve (1), coincides, as it must, with the resonant absorption coefficient calculated (for a stationary disturbance) by Forslund et al. (1975) for the problem of an electromagnetic wave incident on an inhomogeneous plasma layer; the curve κ (λ_{ex}^2) was also given by Southwood and Kivelson (1986) for the case of Alfvén resonance. Curves (2)–(4) were calculated from (5.3) and (4.18) for $\lambda_{in} = 0.25$, 1 and 8 respectively (at $\lambda_{ex} = 1$, as in the linear case). One can see that in the initial build-up stage of the amplitude of the incident FMS wave, the coefficients of nonlinear resonant absorption coincide with the linear coefficient; however, as the disturbance amplitude increases, the nonlinearity causes the absorption coefficient to decrease. The larger λ_{in} is (i.e. the larger the nonlinear scale is compared with the linear one), the faster is the rate of this decrease. At a certain instant of time T_*, the absorption coefficient equals zero (total reflection occurs); after that, the absorption has a quasiperiodic character: the period of energy absorption alternates with the period of energy emission, and the larger λ_{in} is, the smaller is the oscillation amplitude of the absorption coefficient compared with the linear absorption coefficient. The time taken to reach zero absorption (T_*) depends on λ_{in}: when $\lambda_{in} = 0$, we have $T_* = \infty$, and we obtain the linear absorption regime.

Figure 2 clearly demonstrates the association (following from (5.8)) of the resonant absorption with a jump of $\mathrm{Im}[(A_+ - A_-)/B]$: specifically, the coincidence of the zeros of the resonant absorption coefficient and $\mathrm{Im}[(A_+ - A_-)/B]$. It can also be seen that, in the initial build-up stage of the disturbance amplitude, the quantity $\mathrm{Im}[(A_+ - A_-)/B]$ is equal to $-\pi$; when

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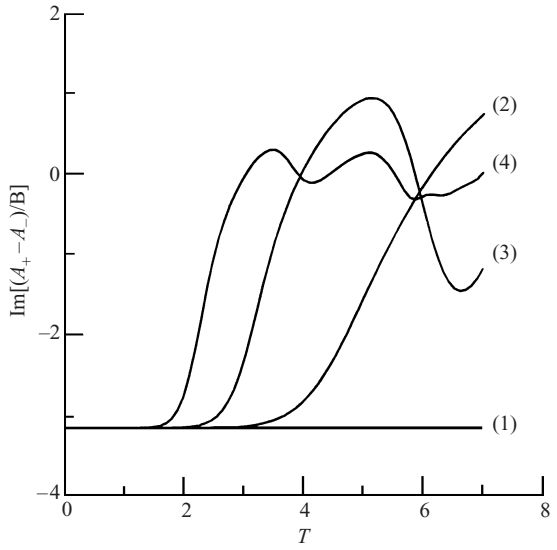


Figure 2. $\mathrm{Im}[(A_+ - A_-)/B]$ as a function of T for case (i). Curves (1)–(4) correspond to $\lambda_{in} =$ 0, 0.125, 1, 8; $\lambda_{ex} = 1$.

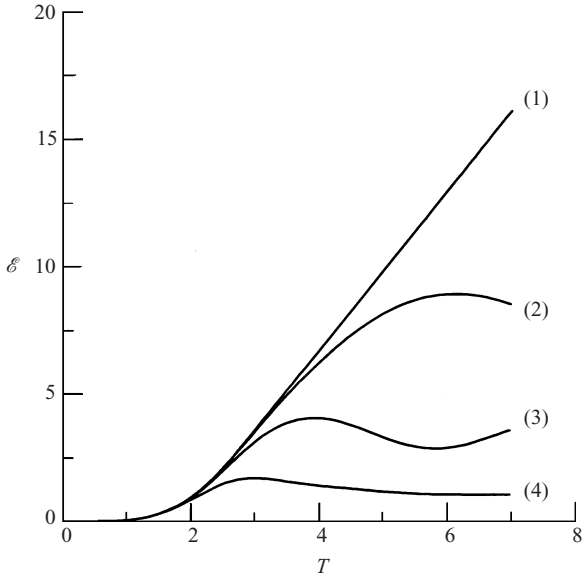


Figure 3. Temporal behaviour of $\mathscr{E}$ for case (i) when $\lambda_{ex} = 1$ and $\lambda_{in} = 0$, 0.125, 1, 8 (curves (1)–(4) respectively).

nonlinear effects are neglected (curve 1), it retains this value, no matter what the disturbance amplitude.

Figure 3 presents plots of $\mathscr{E}$ against T. If nonlinearity is neglected (curve 1) then the energy accumulated in the resonance layer will grow steadily; furthermore, as soon as the amplitude of the incident wave reaches a constant value at $T > \pi$, the accumulated energy will increase by the linear law. If the nonlinearity is taken into account (curves 2–4) then the rate of energy

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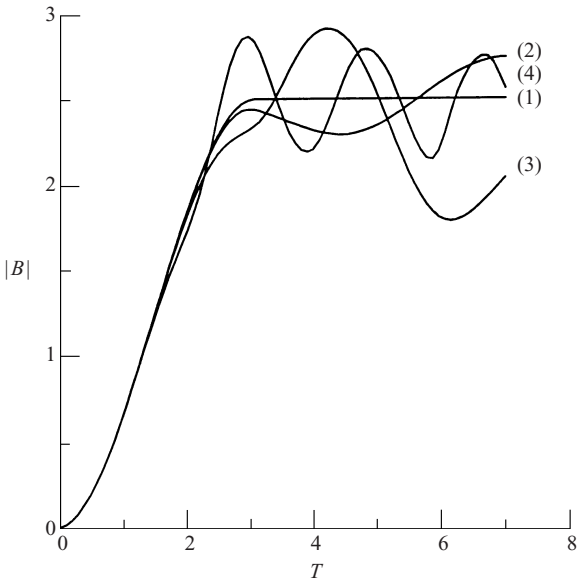


Figure 4. $|B|$ as a function of T for case (i). Curves (1)–(4) correspond to $\lambda_{in} = 0, 0.125, 1, 8$; $\lambda_{ex} = 1$.

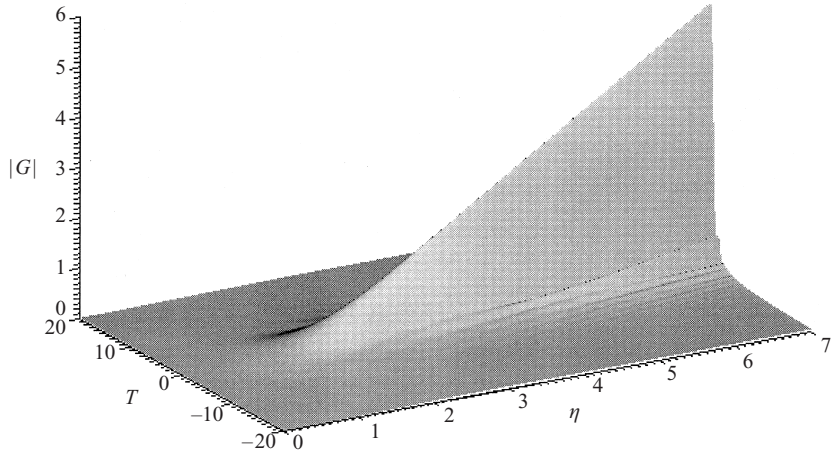


Figure 5. Spatial–temporal profile of $|G|$ obtained from linear theory for case (i).

accumulation will decrease with time; on the interval of time when $\kappa < 0$, there is even a loss of energy.

Figure 4 shows the modulus of the right-hand side of (5.3), which plays the role of the 'internal source'. One can see that the temporal behaviour of the internal source in the nonlinear case (curves 2–4) does not coincide, as in the linear case (curve 1), with the temporal behaviour of the 'external source', the incident wave, but has a quasiperiodic character; the period tends ∞ in the linear limit, and decreases when λ_{in} increases.

Figures 5 and 6 show the spatial–temporal profiles of the modulus of the disturbance amplitude in the resonance layer for a linear non-stationary disturbance ($\lambda_{in} = 0$) and a nonlinear ($\lambda_{in} = 8$) non-stationary disturbance.

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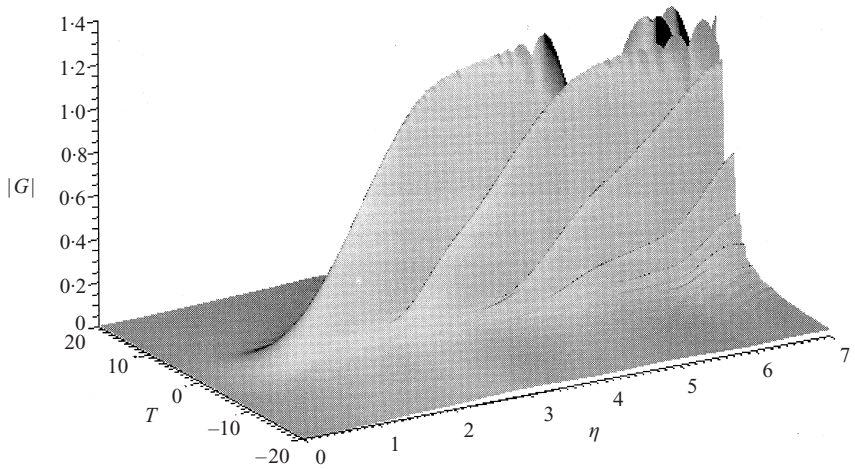


Figure 6. Spatial–temporal profile of $|G|$ obtained from nonlinear theory for case (i) when $\lambda_{in} = 8$.

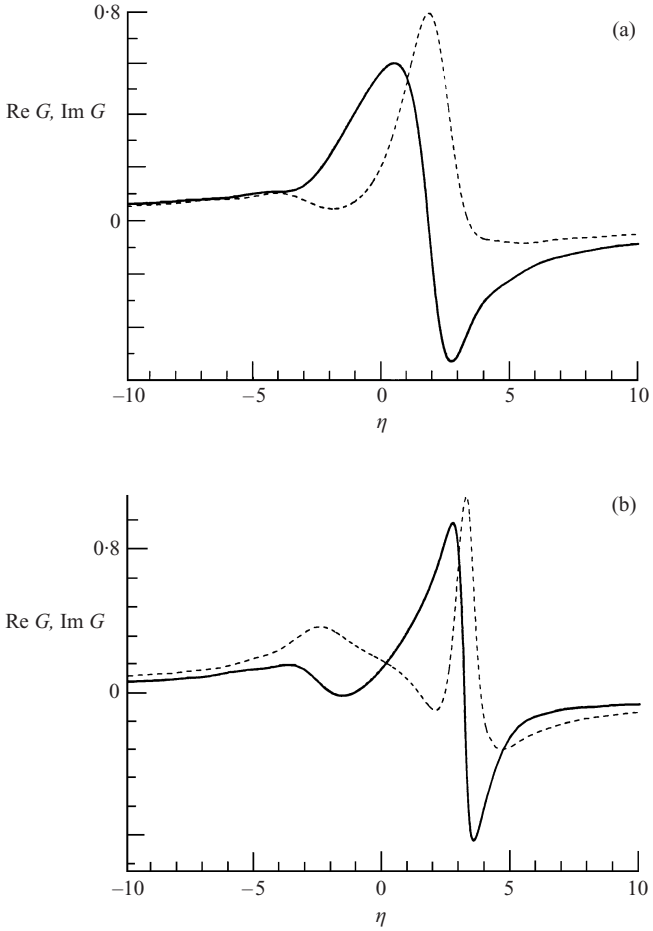


Figure 7. Real and imaginary parts of $G(\eta, T)$ for case (i) ($\lambda_{in} = 8$, $\lambda_{ex} = 1$): ——, Re G, – – –, Im G; (a) $T \approx 2.45$; (b) $T \approx 3.11$.

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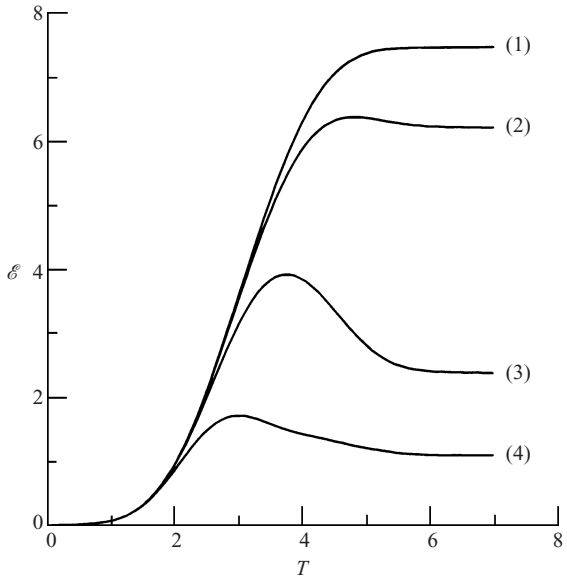


Figure 8. Energy storage in the resonance layer for case (ii) of the time behaviour of the incident-wave amplitude when $\lambda_{ex} = 1$ and $\lambda_{in} = 0, 0.125, 1, 8$ (curves (1)–(4) respectively).

Figure 5 is given for illustrative purposes only, because (5.4) permits a linear disturbance to be investigated analytically. Figure 7 shows the real and imaginary parts of $G(\eta, T)$. The spatial–temporal behaviour of a nonlinear disturbance is more complicated than that of a linear disturbance. It has the following characteristic properties:

(1) a displacement of the disturbance maximum toward the reflection point by an amount Δ_N;
(2) a decrease of the spatial scale with time and a stratification of the resonance layer;
(3) the establishment of a quasistationary amplitude.

That the spatial scale of a disturbance becomes smaller with the passage of time suggests that the resonance layer may become turbulent. Comparison of Figs 1 and 6 leads one to conclude that the time for transition to turbulence is of the order of T_*.

5.2. Case (ii)

Figure 8 shows plots of $\mathscr{E}(T)$ for a linear disturbance (curve 1) and for a nonlinear disturbance (curves 2–4). One can see that when λ_{in} is small, the fraction of energy left by a FMS disturbance in the resonance layer is close in magnitude to that given by linear theory; with increasing λ_{in}, the amount of energy trapped decreases compared with what would be obtained in linear theory.

The spatial–temporal profile calculated from (4.18) and (5.3) when $\lambda_{in} = 8$ and $\lambda_{ex} = 1$ (Fig. 9) demonstrates a displacement of the disturbance maximum and a decrease of the spatial scale with the time. It differs significantly from the linear spatial–temporal profile (Fig. 10). The linear case is also described by (5.4), which permits $G(\eta, T)$ to be calculated analytically.

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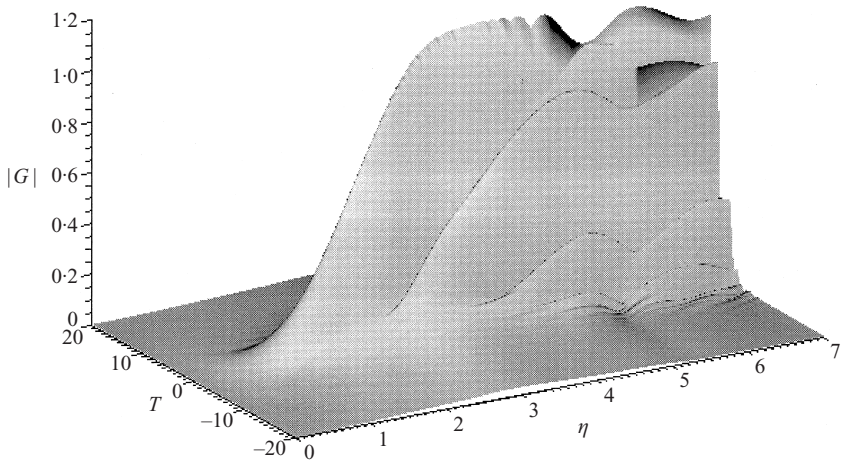


Figure 9. Spatial–temporal profile of $|G|$ obtained from nonlinear theory for case (ii) when $\lambda_{in} = 8$.

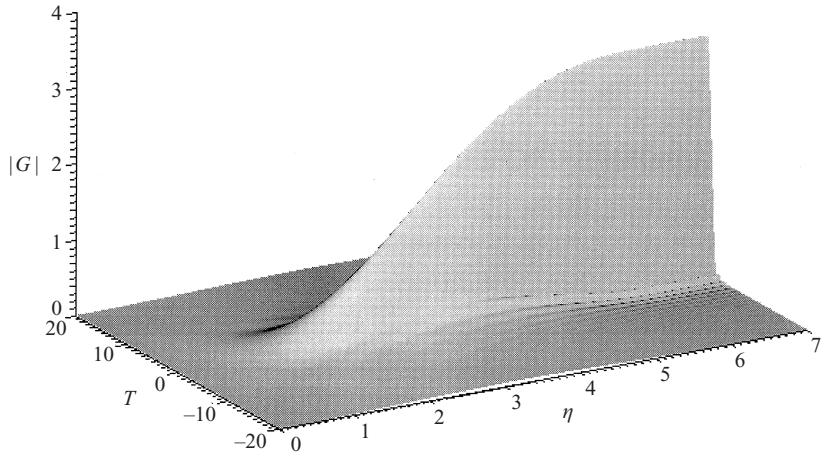


Figure 10. Spatial–temporal profile of $|G|$ obtained from linear theory for case (ii).

In closing, we now discuss qualitatively the influence of other linear effects on the results obtained. It is clear that taking account of small resistivity (small compared with non-stationarity) in the initial stage of establishment of a constant-amplitude source must not have a marked effect on the disturbance structure and the resonant absorption. In the case of the constant-amplitude source, such a situation will exist until a decrease in the spatial scale causes resistivity to be involved in the formation of the spatial structure of a disturbance in the resonance layer. It is likely that the subsequent evolution can lead to the establishment, with a stationary source, of a stationary disturbance where resistivity will smooth out the transverse structure of the disturbance by establishing its own scale as a minimum one, and will eliminate weak oscillations with time. Such a disturbance was described by Dmitrienko (1997).

Such a state will be reached on a resistive timescale (large in the case of small resistivity), and the nonlinear evolution of the resonant absorption during this

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time will proceed under the action of the effect of non-stationarity and corresponding nonlinearity. In this case, the parameter λ_{in} characterizes the relative role of nonlinearity and non-stationarity, while resistivity is essential only for controlling the resonance heating. Hence the spatial structure and the resonant absorption must first conform to linear theory by increasing proportionally to an increase in the source amplitude. In this case, the fundamental harmonic generates the zeroth and higher harmonics, whose amplitudes also increase with increasing source amplitude. When the disturbance amplitude becomes sufficiently large ($|G|^2 \sim 1$), the back-influence of the zeroth harmonic upon the fundamental causes its structure and the resonant absorption to depart from linear theory. A nonlinear non-stationary resonance layer must form, the transverse spatial structure of which becomes smaller with time, and the rate of energy storage in it decreases. At a certain level, this decrease in the disturbance scale is stopped by resistivity, and a stationary disturbance with a constant rate of energy storage is established. The presence of resistivity leads to plasma heating, which, on reaching the nonlinear regime, becomes smaller compared with linear theory because of a decrease in energy supplied to the resonance layer. A decrease in heating in the nonlinear regime was detected by Ofman and Davila (1995) in a numerical simulation of the nonlinear heating of solar coronal loops. This phenomenon was interpreted in terms of the nonlinear shift of the local Alfvén frequency, which is in agreement with the theoretical description suggested in this paper. If the time of source operation is small (a wave train of finite duration) then energy must be accumulated in the resonance layer in the regime governed by a predominance of the effects of non-stationarity and nonlinearity, while the rate with which it is transferred to plasma particles will be determined by small resistivity, both during the source operation and after the source is 'turned off'.

In a similar manner, it is possible to assess the role of other small linear effects on the background of the originally dominant (among them) non-stationarity and the nonlinearity determined by it, in the presence of an external disturbance with an established constant amplitude. The role played by these small effects in energy storage in the resonance layer is unimportant until the reduction in scale occurring in the resonance layer is such that the scale reaches a value corresponding to them, whereupon the transverse structure and energy storage are determined by the combined action of the new linear effect and by the nonlinearity corresponding to this effect.

6. Conclusions

The main results of this work may be summarized as follows.

1. We have obtained a nonlinear equation, describing a disturbance in the resonance layer, with the nonlinearity and non-stationarity simultaneously taken into account: (4.17) or (5.3). The nonlinear equation for the disturbance in the resonance layer differs from the linear one in that it takes into account changes in mean density, magnetic field and plasma flow velocity (zeroth harmonic) under the action of a disturbance that, in turn, depends on the disturbance of the medium that it produces. Higher harmonics are present in the disturbance; the second and zeroth harmonics are of the same order of magnitude, but the back-influence of the second

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harmonic upon the fundamental is not important because of the mutual compensation of the back-influences of its different components.

2. We have obtained the matching conditions for a disturbance through the resonance layer: (4.18). The matching conditions have been obtained in a universal form that makes them applicable in both nonlinear and linear theory; linear theory gives, for the jump in the coefficient of the regular part of the solution through the resonance layer, the familiar relationship (4.21), while nonlinear theory changes this jump substantially (Fig. 2).

3. We have found the parameter that determines the relative role of the nonlinearity and non-stationarity in the formation of the spatial structure and the character of energy accumulation in the resonance layer. The extent to which nonlinearity affects the spatial structure of the disturbance and the resonant absorption are determined by the parameter $\lambda_{\text{in}} = \Delta_N^3/\Delta_\tau^3$, where Δ_N is the nonlinear spatial scale and Δ_τ is the linear non-stationary spatial scale.

4. We have drawn conclusions about the nonlinearity-induced decrease in energy absorption by the resonance layer in comparison with what is given by the linear theory of resonant absorption. The larger the parameter λ_{in}, the more important is the decrease. The temporal behaviour of the coefficient of nonlinear resonant absorption differs from that of the linear resonant absorption coefficient. In the case of a sufficiently long operation of the source, a quasiperiodic regime of resonant absorption can be established where the resonant absorption coefficient fluctuates around zero. Comparison of the results of linear and nonlinear theory has also shown that nonlinearity changes substantially the spatial structure of a disturbance in the resonance layer, leading to the appearance of a stratification of the resonance layer.

Appendix A. Non-stationary correction for the fundamental harmonic

Equation (3.1) describes a stationary disturbance; in order for a non-stationary disturbance to be included, we may put

$$b_{z1}^{(1)}(\xi,\tau) = \int_{-\infty}^{\infty} b_{z1}^{(1)}(\xi,\nu) e^{-i\nu\tau}\, d\nu.$$

It is easy to see that $b_{z1}(\xi,\tau)$ will be described by (3.1) if the substitution $q^2 \to q^2 + 2\nu k_z^2$ is made, and, when $\xi \to 0$, it will be described by (2.8) with the substitution $\xi \to \xi + 2\nu$. Hence, when $\xi \to 0$, we have

$$b_{z1}^{(1)}(\xi,\tau) = \int_{-\infty}^{\infty} [A_{\pm}(\nu) F_A(\xi+2\nu) + B_{\pm}(\nu) F_B(\xi+2\nu)]\, e^{-i\nu\tau}\, d\nu.$$

Using for $F_{A,B}$ the Taylor series expansion

$$F_{A,B}(\xi+2\nu) = F_{A,B}(\xi) + 2\frac{\partial F_{A,B}(\xi)}{\partial \xi}\nu + \ldots$$

and the expansion (3.4), it is possible to obtain the first terms of the expansion $b_{z1}(\xi,\tau)$ as a series in terms of Γ when $\xi \to 0$. It will suffice to have $B_{\pm} = B_{\pm}(T)$, $A_{\pm} = A_{\pm}(T)$ and

$$b_{z1}^{(1,1)} = O(\xi^{-1}). \tag{A1}$$

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From the linearized equations (2.1)–(2.7) and (A1), we can obtain

$$b_{x1}^{(1,1)} \sim u_{x1}^{(1,1)} \sim \xi^{-1}, \quad b_{y1}^{(1,1)} \sim u_{y1}^{(1,1)} \sim \xi^{-2}, \quad r_1^{(1,1)} \sim \ln \xi. \tag{A2}$$

Note that $\mathbf{\Psi}_1^{(1,1)} \sim \mathbf{\Psi}_1^{(1,0)}$ when $\xi \sim \Gamma$. We believe that this implies that the non-stationarity characterized by the parameter Γ plays a structure-forming role in the resonance layer, and the spatial scale Δ_τ on which its influence is important is given by $\Delta_\tau \sim \Gamma$. Introducing a new variable $\eta = \xi/\Gamma$ and a corresponding expansion $\Psi_i = \Psi_i(\eta, T)$, it is easy to obtain inner 'non-stationary' equations and a matching of the inner and outer solutions for a linear non-stationary layer. We shall obtain these equations and matchings as a linear limit for the nonlinear layer.

Appendix B. Nonlinear correction for the fundamental harmonic

The first (in magnitude) nonlinear correction to the fundamental harmonic is a result of the interaction of the second and zeroth harmonics with it. Since the second harmonic at its main order ($\epsilon^2\Gamma^0$) is determined by the main order of the fundamental harmonic, it does not depend on which linear structure-forming factor is dominant in a disturbance. According to Dmitrienko (1997), we have

$$b_{x2}^{(2,0)} \sim u_{x2}^{(2,0)} \sim r_2^{(2,0)} \sim \xi^{-1}, \quad b_{y2}^{(2,0)} \sim b_{z2}^{(2,0)} \sim u_{y2}^{(2,0)} \sim u_{z2}^{(2,0)} \sim \xi^{-2}. \tag{B1}$$

For the zeroth harmonic at order $\epsilon^2\Gamma^0$, we obtain from (2.1), (2.3) and (2.4)

$$\partial_\xi b_{x0}^{(2,0)} = 0, \quad \partial_\xi u_{x0}^{(2,0)} = \bar{n}_3^{(2,0)}, \quad \frac{k_z}{lq^2}\partial_\xi b_{z0}^{(2,0)} = \bar{n}_4^{(2,0)},$$

and, at order $\epsilon^2\Gamma^1$, we get

$$\partial_T b_{x0}^{(2,0)} = \bar{n}_1^{(2,1)}, \quad \partial_T b_{y0}^{(2,0)} = \bar{n}_2^{(2,1)},$$

$$\partial_T u_{y0}^{(2,0)} = \bar{n}_5^{(2,1)}, \quad \partial_T u_{z0}^{(2,0)} = \bar{n}_6^{(2,1)},$$

$$2\partial_T b_{z0}^{(2,0)} + \partial_{\xi T}^2 (b_{z0}^{(2,0)} - r_0^{(2,0)}) + \xi\partial_{\xi T}^2 b_{z0}^{(2,0)} = 2\bar{n}_3^{(2,1)} + \partial_\xi(\bar{n}_3^{(2,1)} - \bar{n}_7^{(2,1)}) + \xi\partial_\xi \bar{n}_3^{(2,1)}.$$

We denote by $\bar{n}_i$ those parts of n_i that are independent of θ. On rearrangement of $\bar{n}_i$ using the linearized equations (2.1)–(2.7) and the relationships (3.2), we find that $\bar{n}_1 = 0$, $\bar{n}_3^{(2,0)} = 0$, $\bar{n}_4^{(2,0)} = O(\xi^{-3})$, $\bar{n}_2^{(2,1)} \sim \bar{n}_5^{(2,1)} \sim \bar{n}_6^{(2,1)} \sim \xi^{-2}$ and $\bar{n}_3^{(2,1)} \sim \bar{n}_7^{(2,1)} \sim \xi^{-1}$ when $\xi \to 0$. Hence, when $\xi \to 0$, we have

$$b_{x0}^{(2,0)} = 0, \quad u_{x0}^{(2,0)} = O(1),$$

$$b_{y0}^{(2,0)} \sim b_{z0}^{(2,0)} \sim u_{y0}^{(2,0)} \sim u_{z0}^{(2,0)} \sim r_0^{(2,0)} \sim \xi^{-2}. \tag{B2}$$

It is noteworthy that the components $b_{y0}^{(2,0)}$, $u_{y0}^{(2,0)}$, $u_{z0}^{(2,0)}$ and $r_0^{(2,0)}$ of the zeroth harmonic are determined through the effect of non-stationarity. This makes itself evident in the fact that these components are not derivable from equations of order $\epsilon^2\Gamma^0$, which neglect the non-stationarity of the disturbance, but are calculated at order $\epsilon^2\Gamma^1$, and the appearance at this order of non-zero $\bar{n}_2^{(2,1)}$, $\bar{n}_3^{(2,1)}$, $\bar{n}_5^{(2,1)}$, $\bar{n}_6^{(2,1)}$ and $\bar{n}_7^{(2,1)}$ is caused by taking into account at this order the non-stationarity-generated phase shifts between disturbance components of the magnetic field and between magnetic field disturbances and the plasma flow velocity. Note also that, by considering the plasma to be cold, we (unlike Dmitrienko 1997) have neglected gas-kinetic pressure in the input equations

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because its role in establishing the zeroth harmonics, with the non-stationarity being dominant among linear effects, is unimportant in the case of sufficiently small β.

Calculations according to the scheme initiated here can be continued; in particular, at order $\epsilon^3\Gamma^0$, it is possible to determine $\mathbf{\Psi}_1^{(3,0)}$, a nonlinear correction for the fundamental harmonic. It is found in this case that the contributions of the most rapidly growing (when $\xi \to 0$) expansion terms of the components of the second harmonic to the source for $\mathbf{\Psi}_1^{(3,0)}$ compensate each other and that the most rapidly growing (when $\xi \to 0$) terms of the expansion $\mathbf{\Psi}_1^{(3,0)}$ are a consequence of the interaction of $\mathbf{\Psi}_1^{(1,0)}$ and $\mathbf{\Psi}_0^{(2,0)}$. Specifically,

$$b_{z1}^{(3,0)} = O(\xi^{-3}), \qquad b_{x1}^{(3,0)} = O(\xi^{-3}) \tag{B3}$$

when $\xi \to 0$. Hence, according to (3.4), (3.8), (A1), (A2) and (B3), for the fundamental harmonic of a disturbance, we get (3.9).

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