not have property (A), since $\dfrac{e^{-px}}{e^{-x}} = e^{(1-p)x}$, which is decreasing. Another such example, easily verified, is $\dfrac{1}{x+1}$.

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105.10 Right-angled triangles link circular and hyperbolic functions

The circular functions $\sin x$ and $\cos x$ are usually defined and visualised using right-angled triangles in the unit circle. The hyperbolic functions, on the other hand, are normally introduced by means of the exponential function. Students are usually surprised that they are named $\sinh x$ and $\cosh x$, since there seems to be no immediate justification for this, even when the circular and hyperbolic functions are shown to have some similar properties.

This motivates us to propose a geometrical construction, which can be seen in Figure 1, which provides some inter-relationships between circular and hyperbolic functions by means of right-angled triangles.

Figure 1 shows a semicircle, part of $x^2 + y^2 = 1$, centred at the origin and a branch of the rectangular hyperbola $x^2 - y^2 = 1$ through $(1, 0)$. A ray from $(-1, 0)$ intersects the y-axis at $(0, t)$ (where $0 < t < 1$), the semicircle at (x_C, x_C) and the hyperbola at (x_H, y_H). The angle between the ray and the x-axis is α, where $0 < \alpha < \frac{\pi}{4}$. Then the angle between the radius from the origin to (x_C, x_C) and the x-axis is 2α and this is bisected by the ray from the origin to $(1, t)$.

It follows immediately that $t = \tan\alpha, x_C = \cos 2\alpha$ and $y_C = \sin 2\alpha$.

The right-angled triangles with vertices $\{(-1, 0), (0, 0), (0, t)\}$ and $\{(-1, 0), (x_C, 0), (x_C, y_C)\}$ are similar. As a result, we have $\dfrac{t}{1} = \dfrac{y_C}{1 + x_C}$. Since $x_C^2 + y_C^2 = 1$, we obtain the formulae

$$x_C = \frac{1 - t^2}{1 + t^2} = \cos 2\alpha \qquad \text{and} \qquad y_C = \frac{2t}{1 + t^2} = \sin 2\alpha.$$

These are the familiar t-formulae.

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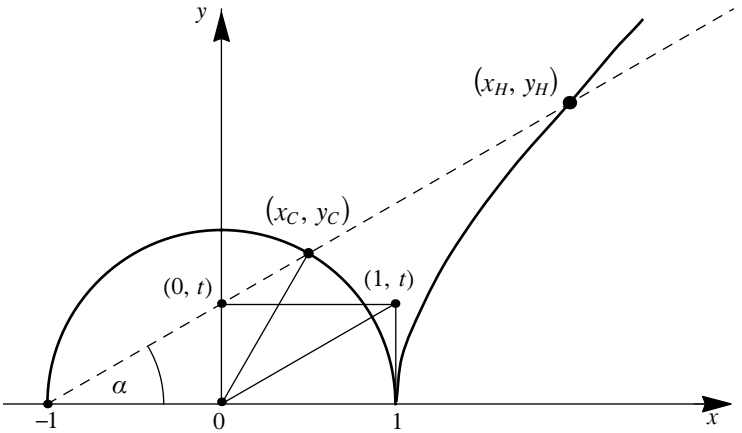


FIGURE 1

Turning now to the hyperbola, the right-angled triangles with vertices $\{(-1, 0), (0, 0), (0, t)\}$ and $\{(-1, 0), (x_H, 0), (x_H, y_H)\}$ are similar, with $\frac{t}{1} = \frac{y_H}{1 + x_H}$. Since $x_H^2 - y_H^2 = 1$, we obtain the formulae $x_H = \frac{1 + t^2}{1 - t^2}$ and $y_H = \frac{2t}{1 - t^2}$, and using familiar trigonometrical identities, these become $x_H = \tan 2\alpha$ and $y_H = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$.

Let $\mu = \ln(x_H + y_H)$, so that $\mu = \ln\left(\frac{1 + t^2}{1 - t^2} + \frac{2t}{1 - t^2}\right) = \ln\left(\frac{1 + t}{1 - t}\right)$. It follows that

$$x_H = \frac{1 + t^2}{1 - t^2} = \frac{e^{\mu} + e^{-\mu}}{2} = \cosh \mu$$

and

$$y_H = \frac{2t}{1 - t^2} = \frac{(1 + t)^2 - (1 - t)^2}{2(1 - t)(1 + t)} = \frac{1}{2}\left(\frac{1 + t}{1 - t} - \frac{1 - t}{1 + t}\right) = \frac{e^{\mu} - e^{-\mu}}{2} = \sinh \mu$$

and so $(x_H, y_H) = (\cosh \mu, \sinh \mu)$.

We also have $\frac{y_C}{x_C} = \tan 2\alpha$ and $\frac{y_H}{x_H} = \tanh \mu$, where α and μ are linked by the equation $t = \tan \alpha = \tanh \frac{\mu}{2}$.

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