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Classification of blow-up for Hele-Shaw flow solutions driven by suction

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In this paper, we consider rational or multi-cut solutions to Hele-Shaw flow driven by suction. A lower bound of the distance between the moving boundary and the suction point is given and this lower bound enables us to find necessary conditions for rational or multi-cut solutions to blow up due to the boundary touching the sink.

Key words: Hele-Shaw; Polubarinova–Galin equation; quadrature domain; Löwner–Kufarev Equation

1 Introduction

The classical Hele-Shaw problem models an incompressible viscous fluid which occupies parts of the space between two parallel plates which have a small gap. This problem is described by Darcy's law and is considered to be a two-dimensional problem since the gap is small. We consider the case that the incompressible viscous fluid occupies a bounded domain and the same fluid is continuously injected or sucked through a hole on the top plate. Denote the initial viscous fluid domain by $\Omega(0)$. The domain starts moving due to the injection/suction and surface tension which acts on the boundary of the viscous fluid domain. Therefore, we can obtain a moving domain $\Omega(t)$ which changes with time. Let $p(x, y, t)$ be the pressure at $(x, y) \in \Omega(t)$, γ to be the surface tension and Q to be the injection or suction rate ($Q > 0$ injection and $Q < 0$ suction). This problem can be formulated explicitly as follows:

$$\begin{cases} \Delta p = -Q\delta_0 & \text{in } \Omega(t), \\ p = \gamma\kappa & \text{on } \partial\Omega(t), \\ v_n = -\frac{\partial p}{\partial n} & \text{on } \partial\Omega(t), \end{cases}$$

where n is the unit outward normal direction and v_n is the outward normal velocity. In this paper, we ignore the surface tension and only consider the effect of suction through a single point.

We first list all notations which are used in this paper:

(1) $\mathbb{D} = \{\zeta \in \mathbb{C} || \zeta | < 1\}$.
(2) $\dot{f}(\zeta, t) = \frac{\partial}{\partial t} f(\zeta, t)$, $f'(\zeta, t) = \frac{\partial}{\partial \zeta} f(\zeta, t)$.

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(3) $h^*(\zeta) = \sum_{j=1}^{\infty} \overline{b_j}\zeta^{-j}$, if $h(\zeta) = \sum_{j=1}^{\infty} b_j\zeta^j$.

(4) Let $\mathbb{E}$ be a set which contains the origin.

$$\mathcal{O}(\mathbb{E}) = \{f | f \text{ is analytic, 1–1 and onto in a neighbourhood of } \mathbb{E},\ f(0) = 0, f'(0) > 0\}. \tag{1.1}$$

(5) For a set $A \subset \mathbb{R}^n$ and a point $p \in \mathbb{R}^n$,

$$\operatorname{dist}(A, p) = \inf_{z \in A} |z - p|. \tag{1.2}$$

This work is mainly devoted to the so-called Polubarinova–Galin equation driven by suction at the origin arising from the reformulation of zero-surface-tension Hele-Shaw flow. A solution $f(\zeta, t) \in \mathcal{O}(\overline{\mathbb{D}})$ is a *strong solution* to the Polubarinova–Galin equation driven by suction at the rate 2π if it satisfies

$$Re[\dot{f}(\zeta, t)\overline{f'(\zeta, t)\zeta}] = -1, \quad \zeta \in \partial\mathbb{D} \tag{1.3}$$

and $f(0, t) = 0, f'(0, t) > 0$ and $\dot{f}(\zeta, t)$ is continuous for $(\xi, t) \in \overline{\mathbb{D}} \times [0, \epsilon)$ for some $\epsilon > 0$. The moving domain $\Omega(t)$ is exactly $f(\mathbb{D}, t)$ and it has been shown that $\Omega(t_1) \subset \Omega(t_2)$ if $t_1 > t_2$. The time reversal of the suction case is the case of injection and in this case $\Omega(t_1) \subset \Omega(t_2)$ if $t_1 < t_2$. We say that a solution blows up at a specific time if this solution ceases to exist at that time. In Reissig and von Wolfersdorf [9], the authors show that given $f(\zeta, 0) \in \mathcal{O}(\mathbb{D})$, a unique solution $f(\zeta, t)$ to (1.3) exists at least for short time. For these solutions, it has been known that they all blow up before all the fluid is sucked out except for the solution $f(\zeta, t) = \sqrt{a_1^2(0) - 2t}\zeta$ if $f(\zeta, 0) = a_1(0)\zeta$. Therefore, it is very important to classify types of blow-up in the suction problem.

In Gustafsson [2], the author gives a reformulation of the Polubarinova–Galin equation (1.3) to be the Löwner–Kufarev type equation

$$\dot{f}(\zeta, t) = -\zeta f'(\zeta, t) P\left[\frac{1}{|f'|^2}\right](\zeta, t), \quad |\zeta| < 1 \tag{1.4}$$

where

$$P\left[\frac{1}{|f'|^2}\right](\zeta, t) = \frac{1}{2\pi i}\int_{|z|=1} \frac{1}{|f'(z, t)|^2}\frac{z + \zeta}{z - \zeta}\frac{dz}{z}.$$

An equation similar to equation (1.4) is also derived in Shraiman and Bensimon [11]. In Gustafsson [2], the author also shows that if initial functions $f(\zeta, 0)$ are rational functions of ζ, then the solutions $f(\zeta, t)$ are rational functions of ζ also, and though the poles of $f(\zeta, t)$ move around in the complex plane, they never collide or produce new poles. Furthermore, if the initial functions are polynomials of ζ, then the solutions are polynomials of the same degree. In Abanov, Mineev-Weinstein and Zabrodin [1], the authors show a more general set of special type of solutions and the form of them is

$$f(\zeta, t) = \sum_{j=1}^{l} b_{j0}\ln(\zeta - \zeta_j(t)) + \sum_{j=1}^{l}\sum_{k=1}^{n_j} \frac{b_{jk}(t)}{(\zeta - \zeta_j(t))^k} + \sum_{k=0}^{s} b_k(t)\zeta^k = \sum_{j=1}^{\infty} a_j(t)\zeta^j, \tag{1.5}$$

where $\zeta_i(t) \neq \zeta_j(t)$ if $i \neq j$ and b_{j0} are independent of time. ζ_j must be located outside the unit circle centred at the origin since $f(\zeta, t)$ is defined for $\zeta \in \overline{\mathbb{D}}$. The right-hand

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side of equation (1.5) is the Taylor expansion of $f(\zeta,t)$ and it holds for $\zeta \in \overline{\mathbb{D}}$. If b_{j0} does not vanish for some $1 \leqslant j \leqslant l$, these solutions are called multi-cut solutions since the domains of the definition of these solutions have branch cuts. If b_{j0} vanishes for all $1 \leqslant j \leqslant l$, these solutions are rational solutions including polynomial solutions. We would like to point out that, if $s \geqslant 2$, $b_s(t)$ never vanishes as long as this solution exists. However, if $s = 1$, it is possible that $b_s(t)$ vanishes at one single moment during the time span of existence. When $b_s(t) \not\equiv 0$, we denote zeros and poles of $f'(\zeta,t)$ to be

$$\omega_1(t),\ldots,\omega_m(t) \quad \text{and} \quad \theta_1(t),\ldots,\theta_n(t), \tag{1.6}$$

respectively, counting multiplicity. The two integers m, n are both constant and $m-n+1 = s$. Then, if $b_s(t) \not\equiv 0$, we can represent $f'(\zeta,t)$ by

$$f'(\zeta,t) = (m-n+1)b_{m-n+1}(t)\frac{\prod_{j=1}^{m}(\zeta-\omega_j(t))}{\prod_{j=1}^{n}(\zeta-\theta_j(t))}. \tag{1.7}$$

In this paper, we consider rational or multi-cut solutions driven by suction rate 2π at the origin with general form (1.5) where $s = m-n+1$ and classify their blow-up.

In Howison and Hohlov [6], the authors show that

(H_1) In the suction case, polynomial solutions must blow up with the formation of cusps before their boundaries touch the sink, except for the solutions $f(\zeta,t) = a_1(t)\zeta$. Continuation beyond the blow-up time occurs only for $(4N+1)/2$-power cusps (Howison [7]). A time-reversal argument shows that such behaviour must necessarily be observable in the injection problem.

(H_2) In the suction case, rational solutions might blow up when their boundaries touch the sink with other forms of blow-up occurring (for example, the formation of cusps at the same time).

The authors leave lower bound of distance between the boundary and the sink in the case of suction to be an open problem. As far as we know, there is still no result regarding the lower bound of the distance between the boundary and the suction point. In this work, we consider rational or multi-cut solutions driven by suction at the origin with strength 2π and obtain a lower bound for the distance between the sink and the moving boundary. Furthermore, we give a more detailed description for these rational or multi-cut solutions which blow up due to touching the sink, i.e., $\mathrm{dist}(\partial\Omega(t),0) \to 0$ as t approaches the blow-up time.

We now list our main results in this paper. Let $f(\zeta,t)$ be a rational or multi-cut solution driven by suction with the form (1.5) and the corresponding moving domain be $\Omega(t) = f(\mathbb{D},t)$. This paper proves the following.

(L_1) We obtain a lower bound of the distance between the boundary and the suction point:

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$$\operatorname{dist}(\partial\Omega(t),0)\geqslant\begin{cases}\dfrac{1}{4}\left[\dfrac{|(m-n+1)b_{m-n+1}(0)a_1^{m-n+1}(0)|}{|\prod_{j=1}^{n}\theta_j(0)|}\right]^{\frac{1}{m-n+2}} & \text{if } m>n,\\ \dfrac{1}{4}\left[\dfrac{|a_1(0)b_1(0)-2t|}{|\prod_{j=1}^{n}\theta_j(0)|}\right]^{\frac{1}{2}} & \text{if } m=n.\end{cases}\tag{1.8}$$

(L_2) If $f(\zeta,t)$ blows up due to touching the sink, then

(a) $m=n$ and $0<b_1(0)<a_1(0)$;

(b) this solution must touch the sink as $t=T^*=\frac{a_1(0)b_1(0)}{2}$;

(c) as $t=T^*-\epsilon$, then $\operatorname{dist}(\partial\Omega(t),0)=O(\epsilon^{1/2})$. This rate is attained if and only if $\prod_{j=1}^{m}|\omega_j(t)|$ is uniformly bounded and we can show that this case happens;

(d) as $t=T^*$, all poles and branch points touch the unit circle centred at the origin.

Note that equation (1.8) holds for the case of polynomials if we replace $|\prod_{j=1}^{n}\theta_j(0)|$ by 1.

The proof for the former result (H_1), when $n=0$, is based on this quantity $a_1^{m-n+1}(t)\overline{a_{m-n+1}}(t)$. From Richardson [10], this quantity is the $m-n$th Richardson's complex moment which is invariant if $m>n=0$ and $a_1^2(t)=a_1^2(0)-2t$ if $m=n=0$. In this work, we derive similar identities and the proofs of $(L_1)-(L_2)$ are based on them. They can be expressed:

$$\begin{cases}a_1^{m-n+1}(t)b_{m-n+1}(t)=a_1^{m-n+1}(0)b_{m-n+1}(0) & \text{if } m>n,\\ a_1(t)b_1(t)\qquad\qquad\quad =a_1(0)b_1(0)-2t & \text{if } m=n.\end{cases}\tag{1.9}$$

Parts of our results can be observed from the point of view of quadrature domains if $f(\zeta,t)$ is a rational solution. The organization of this work is as follows. In Section 2, we derive equation (1.9) and hence prove the main results $(L_1)-(L_2)$. In Section 3, by restricting consideration to the case of rational solutions, we can explain parts of our results from the point of view of quadrature domains.

2 Main results

In this section, we prove our main results. In Section 2.1, we derive the two identities equation (1.9). In Section 2.2, we prove these main results $(L_1)-(L_2)$ as stated in Section 1.

2.1 Formulation of equations

Now, assume $f(\zeta,t)$ is a rational or multi-cut solution with the form (1.5). We formulate equations of $a_1(t)$ and $b_{m-n+1}(t)$ in order to obtain equation (1.9). The main tool for this proof is the Löwner–Kufarev type equation (1.4).

Theorem 2.1 *Assume $f(\zeta,t)$ has the form (1.5). If $m>n$, then*

$$\frac{d}{dt}b_{m-n+1}=-(m-n+1)b_{m-n+1}\left[\frac{a_1}{a_1}\right].\tag{2.1}$$

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If $m = n$, then

$$\frac{d}{dt}b_1 = -\frac{b_1}{a_1}\dot{a}_1 - \frac{2}{a_1}. \tag{2.2}$$

Proof Without loss of generality, we assume $b_{m-n+1} \neq 0$.

In equation (1.4), the Poisson integral satisfies

$$\frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{|f'|^2}\frac{z+\zeta}{z-\zeta}\frac{dz}{z} = A_0 + F(f'), \quad |\zeta| < 1 \tag{2.3}$$

where $F(f')$ is a rational function with possible poles $\omega_1, \ldots, \omega_m$, $\lim_{\zeta\to\infty} F(f') = 0$ and A_0 satisfies

$$A_0 = \lim_{\zeta\to\infty}\frac{2}{f'(f')^*} - \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|f'(e^{i\alpha},t)|^2}\,d\alpha. \tag{2.4}$$

For the first term in equation (2.4),

$$\lim_{\zeta\to\infty}\frac{1}{f'(f')^*} = \begin{cases}\frac{1}{a_1 b_1} & \text{if } m = n\\ 0 & \text{if } m > n.\end{cases} \tag{2.5}$$

For the second term in equation (2.4), from Howison and Hohlov [6],

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|f'(e^{i\alpha},t)|^2}\,d\alpha = \frac{-\dot{a}_1}{a_1}, \tag{2.6}$$

which holds for any solution f to equation (1.3). Finally, applying equations (2.5) and (2.6) to equation (2.4),

$$A_0 = \begin{cases}\frac{\dot{a}_1}{a_1} & \text{if } m > n\\ \frac{2}{a_1 b_1} + \frac{\dot{a}_1}{a_1} & \text{if } m = n.\end{cases} \tag{2.7}$$

From equation (1.4),

$$\frac{d}{dt}b_{m-n+1} = -(m-n+1)b_{m-n+1} \times A_0 \tag{2.8}$$

and, hence, these two equations in this theorem are derived. □

Theorem 2.2 *Assume $f(\zeta,t)$ has the form (1.5). If $m > n$, then $a_1^{m-n+1}(t)b_{m-n+1}(t)$ is conserved. If $m = n$, then $a_1(t)b_1(t) = a_1(0)b_1(0) - 2t$.*

Proof Theorem 2.2 follows directly from Theorem 2.1 by solving the two equations in Theorem 2.1. □

2.2 Proof of main results

In this subsection, we assume $f(\zeta,t)$ is with the form (1.5) and prove our main results $(L_1) - (L_2)$ in Section 1.

The following two lemmas are very important for our estimate of lower bound.

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Lemma 2.3 *All the poles and branch points of a rational or multi-cut solution of the form (1.5), called $\zeta_k(t)$, must satisfy $|\zeta_k(t)| \leqslant |\zeta_k(0)|$.*

Proof From Gustafsson and Lin [3],

$$\frac{-\frac{d}{dt}|\zeta_j(t)|}{|\zeta_j(t)|} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|f'(e^{i\alpha},t)|^2}\frac{|\zeta_j(t)|^2-1}{|\zeta_j(t)-e^{i\alpha}|^2}d\alpha > 0. \tag{2.9}$$

Therefore, all poles and branch points move towards the origin. □

Remark 2.1 By restricting to rational solutions, Tanveer [12] obtains an equation similar to equation (2.9). The method of derivation of this equation in Tanveer [12] can also be applied to the case of multi-cut solutions since the derivation only requires f' to be rational.

Lemma 2.4 *If $f(\zeta,t)$ is a solution of the form (1.5) and if $b_{m-n+1}(t) \neq 0$ at time t, then*

$$a_1(t) = (m-n+1)b_{m-n+1}(t)\left(\frac{\omega_1(t)\cdots\omega_m(t)}{\theta_1(t)\cdots\theta_n(t)}\right)(-1)^{m+n}. \tag{2.10}$$

Proof It follows from equation (1.7) by letting $\zeta = 0$. □

We first consider $m > n$ and obtain a lower bound of the distance between the boundary and the sink as follows.

Theorem 2.5 *Assume $f(\zeta,t)$ has the form (1.5) and $m > n$. Then we show that the solution never blows up due to the boundary $\partial\Omega(t)$ touching the sink and the distance to the sink at the moment t satisfies*

$$dist(\partial\Omega(t),0) \geqslant \frac{1}{4}\left[\frac{|(m-n+1)a_1^{m-n+1}(0)b_{m-n+1}(0)|}{|\prod_{j=1}^n \theta_j(0)|}\right]^{\frac{1}{m-n+2}}, \quad 0 \leqslant t < T^*$$

where T^* is the blow-up time.

Proof From Theorem 2.2 and Lemma 2.4, we have

$$a_1^{m-n+2}(t) = (m-n+1)(-1)^{n+m}a_1^{m-n+1}(0)b_{m-n+1}(0)\frac{\omega_1(t)\cdots\omega_m(t)}{\theta_1(t)\cdots\theta_n(t)}. \tag{2.11}$$

With the fact that $|\omega_j(t)| > 1$ and $|\theta_j(t)| \leqslant |\theta_j(0)|$ from Lemma 2.3, equation (2.11) implies that

$$a_1(t) \geqslant \left[\frac{|(m-n+1)b_{m-n+1}(0)a_1^{m-n+1}(0)|}{|\prod_{j=1}^n \theta_j(0)|}\right]^{\frac{1}{m-n+2}}. \tag{2.12}$$

Applying the Koebe one-quarter theorem as stated in Corollary 1.4 in Pommerenke [8],

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we have

$$\operatorname{dist}(\partial\Omega(t),0) \geqslant \frac{1}{4}|a_1(t)| \geqslant \frac{1}{4}\left[\frac{|(m-n+1)b_{m-n+1}(0)a_1^{m-n+1}(0)|}{|\prod_{j=1}^{n}\theta_j(0)|}\right]^{\frac{1}{m-n+2}}. \tag{2.13}$$

□

In the following, we want to give a lower bound of the distance between the boundary and the suction point in the case of $m = n$.

Theorem 2.6 *Assume $f(\zeta,t)$ has the form (1.5) and $m = n$. Then for $0 \leqslant t < T^*$,*

$$dist(\partial\Omega(t),0) \geqslant \frac{1}{4}\sqrt{\frac{|a_1(0)b_1(0)-2t|}{|\prod_{j=1}^{n}\theta_j(0)|}}, \tag{2.14}$$

where T^* is the blow-up time.

Proof First assume $b_1(t) \neq 0$. From Theorem 2.2 and Lemma 2.4, we have

$$a_1^2(t) = a_1(t)b_1(t)\frac{\omega_1(t)\cdots\omega_m(t)}{\theta_1(t)\cdots\theta_n(t)} = (a_1(0)b_1(0)-2t)\frac{\omega_1(t)\cdots\omega_m(t)}{\theta_1(t)\cdots\theta_n(t)}. \tag{2.15}$$

From equation (2.15), with the fact that $1 < |\omega_j(t)|$ and $1 < |\theta_j(t)| < |\theta_j(0)|$ from Lemma 2.3, we have

$$a_1^2(t) \geqslant \frac{|a_1(0)b_1(0)-2t|}{|\theta_1(0)\cdots\theta_n(0)|}. \tag{2.16}$$

If $b_1(t) = 0$ at a specific time, then $a_1(0)b_1(0) - 2t = a_1(t)b_1(t) = 0$. Therefore, in this case, equation (2.16) still holds at this specific time since the right-hand side of equation (2.16) vanishes. Finally, we obtain a lower bound by using the Koebe one-quarter theorem, as stated in Corollary 1.4 in Pommerenke [8], and equation (2.16). □

It has been shown in Howison and Hohlov [6] that some rational solutions with poles blow up due to the boundary touching the sink. In Theorem 2.5, it is clear that this kind of blow-up only happens as $m = n$. In Theorem 2.7 and Theorem 2.8, we show some properties of this kind of blow-up.

Theorem 2.7 *Assume $f(\zeta,t)$ has the form (1.5) and blows up due to the boundary touching the sink, then:*

(1) $m = n$, $0 < b_1(0) < a_1(0)$.

(2) *Its time of blow-up T^* equals to $\frac{a_1(0)b_1(0)}{2}$.*

(3) *As $t = T^* - \epsilon$, then $dist(\partial\Omega(t),0) = O(\epsilon^{1/2})$. This rate is attained if and only if $\prod_{j=1}^{m}|\omega_j(t)|$ is uniformly bounded.*

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Proof If the solution touches the sink at $t = T^*$, then the lower bound in Theorem 2.5 or Theorem 2.6 has to vanish at $t = T^*$. Therefore, $T^* = \frac{a_1(0)b_1(0)}{2}$ and $m = n$. The value T^* has to be positive and therefore $b_1(0) > 0$.

From Howison and Hohlov [6], $a_1\dot{a}_1 + 1 < 0$ for $0 < t < T^*$, which holds for any solution f to equation (1.3). Therefore,

$$2T^* = \lim_{t \to T^*} a_1^2(t) + 2T^* < a_1^2(0) \tag{2.17}$$

and it follows that $b_1(0) < a_1(0)$.

From equation (2.14), one can conclude that as $t = T^* - \epsilon$, then $\text{dist}(\partial\Omega(t), 0) = O(\epsilon^{1/2})$. Since

$$\frac{1}{4}a_1(t) \leqslant \text{dist}(\partial\Omega(t), 0) \leqslant a_1(t), \tag{2.18}$$

as shown in Corollary 1.4 in Pommerenke [8] and by equation (2.15), this rate is attained if and only if $\prod_{j=1}^{m} |\omega_j(t)|$ is uniformly bounded. □

Remark 2.2 We cannot prove that, under the assumption of Theorem 2.6, the value $\prod_{j=1}^{m} |\omega_j(t)|$ is uniformly bounded. However, Example 2.3 shows that this case happens. Finally, these initial functions which give rise to rational or multi-cut solutions that blow up due to the boundary touching the sink must fall within this set as shown in Theorem 2.7. Obviously, this is a very limited set compared with the whole rational or multi-cut solution set.

Next, we would like to prove if the boundary touches the sink, then all poles and branch points touch the unit circle centred at the origin.

Theorem 2.8 *If a rational or multi-cut solution blows up due to its boundary touching the sink, then all poles and branch points must touch the unit circle centred at the origin at the moment of blow-up.*

Proof Assume $\zeta_j(t)$ is one of poles or branch points of this solution $f(\zeta, t)$. From Gustafsson and Lin [3], we have

$$\frac{-|\dot{\zeta_j}(t)|}{|\zeta_j(t)|} \geqslant \frac{|\zeta_j(t)| - 1}{|\zeta_j(t)| + 1}\left(\frac{-\dot{a}_1(t)}{a_1(t)}\right). \tag{2.19}$$

Equation (2.19) implies

$$(|\zeta_j(t)| - 1)^2 \leqslant a_1(t)\frac{|\zeta_j(t)|}{|\zeta_j(0)|}\frac{(|\zeta_j(0)| - 1)^2}{a_1(0)}. \tag{2.20}$$

Since $a_1(t) \to 0$ as the boundary approaches the sink, by the Koebe one-quarter theorem, we have $|\zeta_j(t)|$ approaching 1 as t approaches the blow-up time. □

The following example is from Howison and Hohlov [6] (with some revision such that $a_1(t) > 0$). The solution blows up due to the boundary touching the sink. We use this example to verify that our theorems shown in this paper hold.

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Example 2.3

$$f(\zeta,0)=\zeta\frac{(4+\sqrt{5/2}\zeta)}{2(\zeta+\sqrt{5/2})}. \tag{2.21}$$

In this example, as shown in Howison and Hohlov [6], $f(\zeta,0)$ gives rise to a solution which blows up due to touching the sink. Here, $a_1(0)=\frac{4}{\sqrt{10}}, b_1(0)=\frac{\sqrt{10}}{4}$ and the blow-up time $T^*=1/2$, as shown in Howison and Hohlov [6]. Therefore, this example satisfies $a_1(0)>b_1(0)>0$, $T^*=\frac{1}{2}a_1(0)b_1(0)$, $m=n$ and Theorem 2.7 holds here. Furthermore, Howison and Hohlov [6] also show that poles all touch the unit circle centred at the origin as $t=T^*$ and hence Theorem 2.8 holds here.

In this example, we can obtain that the zeros of $f'(\zeta,t)$ approach 1 and -3, respectively, as t approaches T^*. Therefore, in this case, as $t=T^*-\epsilon$, the rate $\mathrm{dist}(\partial\Omega(t),0)=O(\epsilon^{1/2})$ is attained according to Theorem 2.7.

3 From the point of view of quadrature domains

A domain Ω is called a quadrature domain if there exist $c_k\in\Omega, c_{kj}\in\mathbb{C}$ ($1\leqslant k\leqslant l$ and $0\leqslant j\leqslant l_k-1$ for some $l,l_k\in\mathbb{N}$) such that

$$\frac{1}{\pi}\int_\Omega \phi dA=\sum_{k=1}^{l}\sum_{j=0}^{l_k-1}c_{kj}\phi^{(j)}(c_k)\frac{1}{(j!)} \tag{3.1}$$

for all integrable analytic functions ϕ in Ω (dA represents the area measure). We refer the reader to Gustafsson and Shapiro [4] for more details on quadrature domains.

In this section, we assume that f is a rational solution of the form (1.5) and explain parts of our results from the point of view of quadrature domains and the corresponding Schwarz functions.

Assume that Ω is a quadrature domain and it satisfies (3.1). A Schwarz function $S(z)$ of $\partial\Omega$ is an analytic function in the neighbourhood of $\partial\Omega$ which satisfies

$$\overline{z}=S(z),\quad z\in\partial\Omega. \tag{3.2}$$

From Gustafsson and Vasil'ev [5], this $S(z)$ must satisfy

$$S(z)=\left[\sum_{k=1}^{l}\sum_{j=0}^{l_k-1}\frac{c_{kj}}{(z-c_k)^{j+1}}\right]+\text{regular part in }\mathbb{C}. \tag{3.3}$$

When $f(\zeta,t)$ is a rational solution of the form (1.5), the corresponding moving domains $\Omega(t)$ are quadrature domains. From Gustafsson and Vasil'ev [5], there exist $e_{jk}(0), 0\leqslant k\leqslant n_j-1, 1\leqslant j\leqslant l$ and $e_k(0), 1\leqslant k\leqslant m-n$ such that

$$\frac{1}{\pi}\int_{\Omega(0)}\phi dA=\sum_{j=1}^{l}\sum_{k=0}^{n_j-1}e_{jk}(0)\phi^{(k)}(z_j(0))\frac{1}{(k!)}+\sum_{k=0}^{m-n}e_k(0)\phi^{(k)}(0)\frac{1}{(k!)}, \tag{3.4}$$

where $z_j(0)=f(\frac{1}{\overline{\zeta_j(0)}},0)$. In Richardson [10], it is shown that these quadrature domains

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$\Omega(t)$, which are solutions to Hele-Shaw flow driven by suction 2π, must satisfy

$$\frac{d}{dt}\int_{\Omega(t)}\phi dA = -2\pi\phi(0) \tag{3.5}$$

for all integrable analytic functions ϕ in $\Omega(t)$. Therefore, from equations (3.5) and (3.4), we can conclude that there exist e_{jk}, z_j and $e_k(t)$ such that

$$\frac{1}{\pi}\int_{\Omega(t)}\phi dA = \sum_{j=1}^{l}\sum_{k=0}^{n_j-1} e_{jk}\phi^{(k)}(z_j)\frac{1}{(k!)} + \sum_{k=0}^{m-n} e_k\phi^{(k)}(0)\frac{1}{(k!)}, \tag{3.6}$$

where $z_j(t) = f(\frac{1}{\overline{\zeta_j(t)}}, t)$, $\frac{d}{dt}z_j = 0$, $\frac{d}{dt}e_{jk} = 0$ and $\frac{d}{dt}e_k = -2\delta_{k,0}$. It follows that the corresponding Schwarz function of the domain $\Omega(t)$ satisfies

$$\overline{z} = S(z,t) = \left[\sum_{j=1}^{l}\sum_{k=0}^{n_j-1}\frac{e_{jk}}{(z-z_j)^{k+1}} + \sum_{k=0}^{m-n}\frac{e_k}{z^{k+1}}\right] + \text{regular part in } \mathbb{C}, \quad z \in \partial\Omega(t). \tag{3.7}$$

The relation between the conformal mapping $f(\zeta,t)$ satisfies

$$f^*(\zeta,t) = S(f(\zeta,t),t) \tag{3.8}$$

in the neighbourhood of $|\zeta| = 1$. Hence, equation (3.8) holds everywhere except for the points $\zeta = 1/\overline{\zeta_k}$ or $\zeta = 0$. The residue of $\zeta^{m-n} \times f^*(\zeta,t)$ at 0 and the residue of $\zeta^{m-n} \times S(f(\zeta,t),t)$ at 0 are $\overline{b_{m-n+1}}(t)$ and $e_{m-n}/a_1^{m-n+1}(t)$, respectively. Hence, one can obtain $e_{m-n} = a_1^{m-n+1}(t)\overline{b_{m-n+1}}(t)$ since these two residues are equal. Since $\frac{d}{dt}e_k = -2\delta_{k,0}$, this identity $e_{m-n} = a_1^{m-n+1}(t)\overline{b_{m-n+1}}(t)$ explains the two identities in equation (1.9) (or Theorem 2.2).

It is shown in Theorem 3 of Howison and Hohlov [6] that if a solution $f(\zeta,t)$ blows up at $t = T^*$ due to the boundary touching the sink, then $\Omega(t)$ converges to $\{0\}$ in the sense of kernel convergence as defined in Pommerenke [8]. This implies that $f(D_r,t) \to 0$ as $t \to T^*$ for all discs D_r with radius $r < 1$ which are centred at the origin. For rational $f(\zeta,t)$, these constants $z_j = f(\frac{1}{\overline{\zeta_j(t)}}, t) \neq 0$ for $0 \leqslant t < T^*$. If there exists $r_0 < 1$ such that $|\zeta_j| > \frac{1}{r_0}$ for some j, then $z_j \in f(D_{r_0},t)$ for $0 \leqslant t < T^*$ and hence $z_j = 0$, which is impossible. Therefore, in this case, all poles have to touch the unit circle centred at the origin at the time of blow-up, $t = T^*$.

4 Conclusions

For multi-cut solutions to (1.3), we give a lower bound between the sink and the boundary. For these multi-cut solutions to (1.3), which blow up due to the boundary touching the sink, we determine the blow-up time from their initial functions and show that these initial functions are very specific. In the latter case, all poles or branch points of solutions touch the boundary of the unit disc centred at the origin as solutions blow up. Some examples in Howison and Hohlov [6], which illustrate rational solutions blowing up due to the boundary touching the sink, have been used to verify that our theorems hold. Finally, we have explained many of our results from the point of view of quadrature domains.

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