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Capillary-gravity waves over a sloping beach

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Abstract. It is shown how the solution for the velocity potential may be determined when the effect of surface tension is included in the linearized theory of surface waves over a sloping beach. In particular, two independent standing wave solutions are found, both of which have finite amplitude at the shoreline. The results agree with those of previous writers when the surface tension force tends to zero.

1. *Introduction.* Solutions in the linearized theory of surface gravity waves over a sloping beach are well known. In particular, Lewy (2) and Stoker (7) have considered the two-dimensional problem in which the slope angle has the special form $p\pi/2q$, p, q relatively prime integers. Their method was later extended to the corresponding problem in three dimensions with arbitrary incidence. Thus, two independent standing wave solutions are found, of which one is regular at the shoreline and behaves like $e^{ay}\cos(bx+c)$ at infinity, whilst the other behaves like $e^{ay}\sin(bx+c)$ at infinity and has a logarithmic singularity at the shoreline. These are combined to produce arbitrary progressive waves. By permitting at most this source-type singularity, together with certain boundedness conditions at infinity, the progressive wave solutions are shown to be uniquely determined by the frequency and amplitude at infinity. Solutions with singularity of the form $r^{-2nq/p}$ (n integer, $\geqslant 1$) in a neighbourhood of the origin $r = 0$, have been obtained by Brillouet (1). For arbitrary angle of slope, α, general solutions in the form of Laplace integrals were subsequently provided by Peters (5) and Roseau (6) but without uniqueness statements. In fact, Roseau obtains solutions which die out at infinity, but each has a singularity of the form $r^{-n\pi/\alpha}$ at the shoreline. More recently, Williams (9) considers a simpler contour integral and obtains solutions by taking the real and imaginary parts of this representation. None of the above writers considers the effect of surface tension.

In a recent paper, however, Packham (4) discusses the special case of incoming waves normally incident on a vertical cliff in the presence of surface tension. By imposing the (physically desirable) condition that the wave amplitude should remain finite at the shoreline, two independent standing wave solutions are found which are uniquely determined by specifying the behaviour at infinity. Furthermore, when the surface tension is equated to zero, the solution for pure gravity waves is recovered. In the present paper we consider the effect of surface tension in the linearized theory of surface

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waves over a sloping beach. By imposing the same condition of boundedness at the shoreline, we obtain two standing wave solutions, behaving at infinity like

$$e^{ay}\cos(bx+c) \quad \text{and} \quad e^{ay}\sin(bx+c)$$

respectively, which may be combined to produce progressive waves. Here too, in the limit as the surface tension force tends to zero the solutions revert to those obtained by previous writers.

2. *Formulation.* We consider the small amplitude oscillations of a heavy ideal fluid with a free surface over a beach of constant slope α. Cylindrical polar coordinates (r, θ, z) are chosen with the z-axis parallel to the shoreline such that the fluid occupies the region Γ: $0 \leqslant r < \infty$, $0 \leqslant \theta \leqslant \alpha$, $-\infty < z < \infty$. Under the assumption of irrotational motion, there exists a velocity potential $\Phi(r, \theta, z, t)$ which is harmonic in Γ and, in the presence of surface tension, may be shown to satisfy the linearized free surface condition (10)

$$\frac{T}{r}\left(\frac{\partial^3\Phi}{\partial^2 r\,\partial\theta}+\frac{\partial^3\Phi}{\partial^2 z\,\partial\theta}-\frac{2}{r}\frac{\partial^2\Phi}{\partial r\,\partial\theta}+\frac{2}{r^2}\frac{\partial\Phi}{\partial\theta}\right)+\rho\left(\frac{\partial^2\Phi}{\partial t^2}-\frac{g}{r}\frac{\partial\Phi}{\partial\theta}\right)=0 \quad \text{on} \quad \theta=0, r>0, \tag{2·1}$$

where T is the surface tension and ρ is the density of the fluid. On the beach, the normal velocity is zero. Thus

$$\partial\Phi/\partial\theta = 0 \quad \text{on} \quad \theta=\alpha, \quad r>0. \tag{2·2}$$

The geometry of the problem allows us to extract the z-variation by assuming a solution of the form

$$\Phi(r,\theta,z,t) = \phi(r,\theta)\begin{Bmatrix}\cos kz\\ \sin kz\end{Bmatrix}\cos(\sigma t+\epsilon), \tag{2·3}$$

where σ is the frequency of the motion and k is a positive, real parameter. We have therefore to determine the function $\phi(r, \theta)$. To this end it is convenient to introduce the dimensionless variable $r' = Kr$ together with the parameters $M' = MK^2 = TK^2/\rho g$ and $k' = k/K$, where $K = \sigma^2/g$. The conditions on Φ show that, after dropping the dashes on all quantities, we require a function $\phi(r, \theta)$ satisfying

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}-k^2\phi=0 \quad \text{in} \quad \Gamma, \tag{2·4}$$

together with the boundary conditions

$$\frac{M}{r^3}\left(\frac{\partial^3\phi}{\partial\theta^3}+3r\frac{\partial^2\phi}{\partial r\,\partial\theta}-2\frac{\partial\phi}{\partial\theta}\right)+\frac{1}{r}\frac{\partial\phi}{\partial\theta}+\phi=0 \quad \text{on} \quad \theta=0, r>0 \tag{2·5}$$

and

$$\partial\phi/\partial\theta = 0 \quad \text{on} \quad \theta=\alpha, r>0. \tag{2·6}$$

From (2·4) and (2·5) the behaviour of ϕ at infinity is assumed to be of the form

$$\phi(r,\theta) \sim a\exp\{-\kappa r\sin\theta + i\kappa' r\cos\theta\} \quad \text{as} \quad r\to\infty \text{ in } \Gamma, \tag{2·7}$$

where $\kappa' = (\kappa^2 - k^2)^{\frac{1}{2}}$, the root being such that $\kappa' = \kappa$ when $k = 0$ and κ is the real positive root of

$$M\kappa^3+\kappa-1=0. \tag{2·8}$$

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We restrict our discussion to motions with non-zero amplitude at infinity.

For such surface waves to exist in Γ, we require $\kappa > k$ and thus, since (2·8) implies $0 < \kappa \leqslant 1$, we must have $k < 1$. The restriction on the magnitude of κ shows that, for a fixed frequency σ, the presence of surface tension increases the wavelength at infinity.

Finally, we specify the behaviour of ϕ at the shoreline. In this investigation we assume that

$$\phi \text{ remains finite as } r \to 0 \text{ in } \Gamma. \tag{2·9}$$

Following Williams(9), we represent a solution of (2·4) as the Laplace integral

$$\phi(r,\theta) = \int_C z^{-1} \exp\{\tfrac{1}{2}kr(z+z^{-1})\}\,[g_1(ze^{i\theta}) + g_2(ze^{-i\theta})]\,dz, \tag{2·10}$$

where, apart from possible poles, g_1 and g_2 are continuous, analytic functions of z. The path C lies in the upper half z-plane going to infinity in the negative direction of the real axis and entering the origin tangentially to the real axis and from the left. Continuity of the solution with respect to θ is guaranteed if no singularity of the integrand crosses C for $0 < \theta < \alpha$. If $g_1(z)$, $g_2(z)$ and their first derivatives are suitably bounded near the end points of C, direct differentiation shows that ϕ is a solution of (2·4). The boundary conditions (2·5) and (2·6) become

$$\int_C \exp\{\tfrac{1}{2}kr(z+z^{-1})\}\{[Mik^3z^2(1-z^{-2})^3 - 4ik(1-z^{-2})]\,[g_1(z) - g_2(z)] + 8z^{-1}[g_1(z)+g_2(z)]\}\,dz = 0,$$

$$\int_C (1-z^{-2}) \exp\{\tfrac{1}{2}kr(z+z^{-1})\}\,[g_1(ze^{i\alpha}) - g_2(ze^{-i\alpha})]\,dz = 0.$$

These are satisfied if

$$g_1(z) = g(z),$$

$$g_2(z) = g(ze^{2i\alpha}), \tag{2·11}$$

where $g(z)$ satisfies the functional equation

$$g(z) = \frac{Mk^2z^6 - (3Mk^2+4)\,z^4 + 8iz^3/k + (3Mk^2+4)\,z^2 - Mk^2}{Mk^2z^6 - (3Mk^2+4)\,z^4 - 8iz^3/k + (3Mk^2+4)z^2 - Mk^2}\, g(ze^{2i\alpha}). \tag{2·12}$$

We therefore require a solution $g(z)$ of (2·12).

3. *Solution of the functional equation.* We first observe the following factorization:

$$Mk^2z^6 - (3Mk^2+4)\,z^4 + 8iz^3/k + (3Mk^2+4)\,z^2 - Mk^2 = Mk^2(z^2 - 2i\kappa z/k - 1)\,(z^2 + 2iA^+z - 1)\,(z^2 + 2iA^-z - 1), \tag{3·1}$$

where

$$A^\pm = \frac{\kappa}{2k}\left\{1 \pm i\sqrt{\left(\frac{4}{M\kappa^3} - 1\right)}\right\}$$

and κ satisfies (2·8). It is convenient to write $A^\pm = ce^{\pm i\omega}$, where

$$c = \frac{1}{k}\sqrt{\frac{1}{M\kappa}} \quad \text{and} \quad 0 < \omega = \tan^{-1}\sqrt{\left(\frac{4}{M\kappa^3} - 1\right)} < \tfrac{1}{2}\pi \quad \text{for} \quad M > 0. \tag{3·2}$$

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It may now be shown that the right-hand side of (3·1) reduces to

$$Mk^2(z-ir_1)(z-ir_2)(z+ic_1e^{i\Omega})(z+ic_2e^{-i\Omega})(z+ic_2e^{i\Omega})(z+ic_1e^{-i\Omega}), \tag{3·3}$$

where

$$r_{1,2}=\frac{\kappa\pm\kappa'}{k}, \quad c_{1,2}=b\pm\sqrt{(b^2-1)},$$

$$b=\frac{1}{\sqrt{2}}[c^2+1+(c^4-2c^2\cos 2\omega+1)^{\frac{1}{2}}]^{\frac{1}{2}} \quad \text{and} \quad 0<\Omega=\tan^{-1}\left\{\frac{b\tan\omega}{\sqrt{(b^2-1)}}\right\}<\tfrac{1}{2}\pi. \tag{3·4}$$

The roots in the expression for b are chosen so that b behaves like c as $c\to\infty$.

Furthermore, under condition (2·8), $c_1>r_1$ and hence $c_2<r_2$. It is of interest to note that (2·8) has just one real root for $M\geqslant 0$. Finally, we have that

$$c_1\to\infty \quad \text{and} \quad \Omega\to\tfrac{1}{2}\pi \quad \text{as} \quad M\to 0. \tag{3·5}$$

The functional equation (2·12) for $g(z)$ can now be written:

$$g(z)=\frac{(z-ir_1)(z-ir_2)(z+ic_1e^{i\Omega})(z+ic_1e^{-i\Omega})(z+ic_2e^{i\Omega})(z+ic_2e^{-i\Omega})}{(z+ir_1)(z+ir_2)(z-ic_1e^{i\Omega})(z-ic_1e^{-i\Omega})(z-ic_2e^{i\Omega})(z-ic_2e^{-i\Omega})}g(ze^{2i\alpha}). \tag{3·6}$$

We obtain a solution of (3·6) in terms of $h(z)$, a solution of

$$h(z)=\frac{z-i}{z+i}h(ze^{2i\alpha}). \tag{3·7}$$

This function has been studied by Williams(9) and Peters(5) in connexion with the corresponding wave problem in the absence of surface tension, and by Williams(8) in an electromagnetic diffraction problem. We require the following properties:

$h(z)$ is defined on a Riemann surface with rth sheet $-\pi-2r\pi<\arg z<\pi+2r\pi$ and is regular apart from simple poles on the unit circle at

$$\arg z=\tfrac{1}{2}\pi+2(r+1)\alpha+2n\pi \quad \text{and} \quad -\tfrac{1}{2}\pi-2r\alpha-2n\pi,$$

where n, $r\geqslant 0$. Zeros of $h(z)$ are at $z=\exp\{i[3\pi/2+2(r+1)\alpha+2n\pi]\}$ and at

$$z=\exp\{-i[3\pi/2+2r\alpha+2n\pi]\}.$$

$h(z)$ satisfies the functional relations

$$h(z)=-iz^{\frac{1}{2}\tau}h(z^{-1}e^{2i\alpha}) \quad \text{and} \quad h(z)\,h(ze^{-i\pi})=z^{\tau}/(z^{\tau}-e^{\frac{1}{2}i\pi\tau}), \tag{3·8}$$

where $\tau=\pi/\alpha$. Furthermore, $\overline{h(z)}=h(\bar{z}e^{2i\alpha})$, where the bar denotes a complex conjugate. We also have the asymptotic behaviour

$$h(z)\to 1 \text{ as } |z|\to\infty \quad \text{and} \quad h(z)\sim -iz^{\frac{1}{2}\tau} \quad \text{for small } z. \tag{3·9}$$

As can be seen from the above mentioned studies by Williams and Peters, the solution for general α is difficult to compute. However, when the slope angle has the special form $\alpha=p\pi/2q$ (p, q integers; p, $2q$ relatively prime) a particular solution of (3·7) is defined by (2)

$$h(z)=\frac{z^{q/p}\prod_{k=1}^{\frac{1}{2}(p-1)}z^{2q/p}-\exp\{i\pi(4k-1)\,q/p\}}{\prod_{j=1}^{q}z-\exp\{i\pi(jp/q+\frac{1}{2})\}}. \tag{3·10}$$

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When considering permissible forms for $g(z)$, it is useful to note that any poles introduced into the integrand of (2·10) only when M is non-zero, must lie outside C; that is, we do not expect the surface tension effect *per se* to produce terms of the surface-wave type.

As a candidate for our solution we consider the function

$$g(z) = h_0(z)\, H_1(z)\, H_2(z e^{-i\pi}), \tag{3·11}$$

where $H_1(z) = h(z/r_1)\, h(z/r_2)$, $H_2(z) = h(z/c_1 e^{i\Omega})\, h(z/c_1 e^{-i\Omega})\, h(z/c_2 e^{i\Omega})\, h(z/c_2 e^{-i\Omega})$ and $h_0(z)$ satisfies

$$h_0(z) = h_0(z e^{2i\alpha}). \tag{3·12}$$

Clearly $g(z)$ satisfies (3·6). With $g_1(z)$, $g_2(z)$ given by (2·11) and (2·12), it may be verified that $H_1(z)$ contributes no poles to the integrand in (2·10) in the region $-\frac{1}{2}\pi < \arg z < \frac{1}{2}\pi$. The poles due to $H_2(z e^{-i\pi})$, however, are at corresponding points on circles of radius c_1 and c_2 with

$$\arg z = \tfrac{1}{2}\pi + 2(r+1)\alpha + (2n+1)\pi \pm \Omega - \theta, \quad -\tfrac{1}{2}\pi - 2r\alpha - (2n-1)\pi \pm \Omega - \theta,$$
$$\tfrac{1}{2}\pi + 2r\alpha + (2n+1)\pi \pm \Omega + \theta, \quad -\tfrac{1}{2}\pi - 2(r+1)\alpha - (2n-1)\pi \pm \Omega + \theta,$$

but, for $0 \leqslant \theta \leqslant \alpha$, these do not lie in $\frac{1}{2}\pi + \Omega < \arg z \leqslant \pi$. The above results lead to our choice of contour C as illustrated in Fig. 1. We see that for M, κ satisfying (2·8), the poles of the integrand introduced by surface tension effects lie outside C.

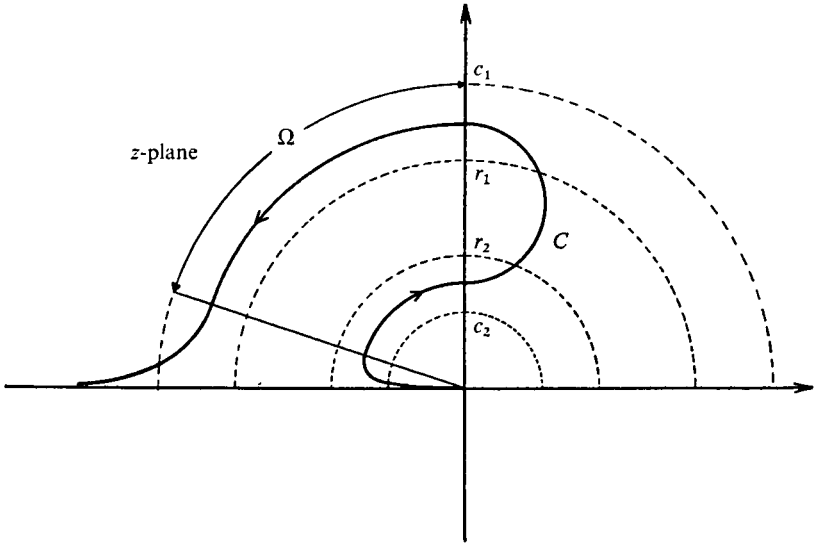


Fig. 1. Path of integration in the z-plane.

Thus far, there are no poles of the integrand in $0 \leqslant \arg z < \frac{1}{2}\pi$ which lie within C. To ensure boundedness of ϕ for large r, the appropriate choice of $h_0(z)$ is a polynomial with constant coefficients. Thus

$$h_0(z) = \sum_{k=-m}^{n} C_k z^{k\tau}. \tag{3·13}$$

The form of h_0 is further specified by our requirement that $\phi(r, \theta)$ remains finite as $r \to 0$. The following result is appropriate:

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LEMMA. *Suppose that* $g(z) \sim Az^{-\lambda}$ *as* $|z| \to \infty$ *and* $g(z) \sim Bz^{\mu}$ *as* $z \to 0$. *Then*

$$I = \int_0^{\infty e^{i\pi}} z^{-1} g(z \exp\{i\theta\}) \exp\{\tfrac{1}{2}kr(z+z^{-1})\}\, dz$$

has the following behaviour for small r:

(i) *if* $\lambda, \mu > 0$, *then* $I \sim C$, *where* C *is a constant, possibly zero*;

(ii) *if* $\lambda = \mu = 0$, *then* $I \sim (A+B)\log r$, A, B *constant*;

(iii) *if either* $\lambda < 0$ *or* $\mu < 0$ (*or both*), *then* $I \sim O(r^{\nu})$, *where* $\nu = \min(\lambda, \mu)$.

It follows therefore that ϕ will remain finite at the origin if

$$h_0(z) = C_{-1} z^{-\tau} + C_{-2} z^{-2\tau}$$

and we take

$$g(z) = Ac_1^{\tau} z^{-\tau}(1 + Bz^{-\tau})\, H_1(z)\, H_2(z e^{-i\pi}) \tag{3·14}$$

as our solution to the functional equation, where A and B are arbitrary complex constants. The factor c_1^{τ} is inserted so that, from (3·5) and (3·9), $H_2(z)$ remains non-zero in the limit as $M \to 0$.

4. *Standing wave solutions*. The constants A, B in (3·14) are now determined from the condition at infinity that

$$\phi(r, \theta) \sim a \exp\{-\kappa r \sin\theta + i\kappa' r \cos\theta\}.$$

When $\theta = 0$, we have

$$\phi(r, 0) = \int_C Ac_1^{\tau}(1 + Bz^{-\tau})\, z^{-\tau-1}[H_1(z)\, H_2(z e^{-i\pi}) + H_1(z e^{2i\alpha})\, H_2(z\, e^{2i\alpha - i\pi})] \exp\{\tfrac{1}{2}kr(z+z^{-1})\}\, dz. \tag{4·1}$$

To describe the behaviour of $\phi(r, 0)$ as $r \to \infty$, we deform C into a contour C' enclosing only the poles of the integrand in $\frac{1}{2}\pi < \arg z \leqslant \pi$. Taking into account the contributions from the poles of $H_1(z e^{2i\alpha})$ at ir_1 and ir_2, we find that

$$\begin{aligned}\phi(r, 0) = {} & \frac{4\pi i h(i)\, c_1^{\tau} A}{e^{i\tau\pi/2}} \left[\frac{1 + Br_1^{-\tau} e^{-i\tau\pi/2}}{r_1^{\tau}} h\left(i \frac{r_1}{r_2} e^{2i\alpha}\right) H_2(ir_1 e^{2i\alpha - i\pi})\, e^{i\kappa' r} \right. \\ & \left. + \frac{1 + Br_2^{-\tau} e^{-i\tau\pi/2}}{r_2^{\tau}} h\left(i \frac{r_2}{r_1} e^{2i\alpha}\right) H_2(ir_2 e^{2i\alpha - i\pi})\, e^{-i\kappa' r} \right] \\ & + \int_{C'} Ac_1^{\tau}(1 + Bz^{-\tau})\, z^{-\tau-1}[H_1(z)\, H_2(z e^{-i\pi}) \\ & + H_1(z e^{2i\alpha})\, H_2(z e^{2i\alpha - i\pi})] \exp\{\tfrac{1}{2}kr(z+z^{-1})\}\, dz. \end{aligned} \tag{4·2}$$

The integral in (4·2) approaches zero as $r \to \infty$. A short calculation then shows that with

$$B = -r_2^{\tau} e^{i\tau\pi/2}, \quad A = \frac{ar_1^{\tau} h(ir_2/r_1)\, e^{i\tau\pi/4}}{4\pi h(i)\, c_1^{\tau} r_2^{\tau} H_2(ir_1 e^{2i\alpha - i\pi})}, \tag{4·3}$$

$\phi(r, \theta)$ satisfies (2·7). The real and imaginary parts, ϕ_1 and ϕ_2 respectively, of ϕ lead, for example, to two independent standing wave solutions

$$\Phi_{1,2} = \phi_{1,2}(r, \theta) \begin{Bmatrix} \cos kz \\ \sin kz \end{Bmatrix} \cos(\sigma\tau + \epsilon) \tag{4·4}$$

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which are bounded at the shoreline. By taking appropriate linear combinations of ϕ_1 and ϕ_2 we may construct a solution Φ which corresponds to an incoming wave at infinity of prescribed frequency and amplitude. Thus, for example,

$$\Phi(r,\theta,z,t) = \phi_1(r,\theta)\cos(kz+\sigma t) - \phi_2(r,\theta)\sin(kz+\sigma\tau)$$

behaves at infinity like $\bar{a}\,e^{\kappa y}\cos(\kappa' x + kz + \sigma t)$, where $x+iy = re^{i\theta}$.

In the limiting case as $M \to 0$, we find from (3·5), (3·9), (3·14) and (4·3) that

$$g_0(z) \equiv \lim_{M\to 0} g(z) = \frac{ah(ir_2/r_1)\,(r_1^\tau - e^{i\tau\pi/2}z^{-\tau})}{4\pi h(i)\,e^{i\tau\pi/4}}\,H_1(z). \tag{4·5}$$

Furthermore, with $g(z)$ in (2·11) replaced by $g_0(z)$, the corresponding function $\phi_0(r,\theta)$ of (2·10) is a solution of (2·4)–(2·8) with $M = 0$. From our lemma in section 3, the behaviour of ϕ_0 at the shoreline is given by

$$\phi_0(r,\theta) \sim \frac{ah(ir_2/r_1)}{2\pi h(i)\,e^{i\tau\pi/4}}\,(r_1^\tau + e^{i\tau\pi/2})\log r. \tag{4·6}$$

Thus both the real and imaginary parts of ϕ_0 have a logarithmic singularity unless $r_1^\tau + e^{i\tau\pi/2} = 0$; that is, unless $k = 1$ and $\alpha = \pi/2m$, with m an odd integer. To bring our solutions into line with other writers we seek a function $\chi(r,\theta)$ whose real and imaginary parts lead to standing wave solutions, 90° out of phase at infinity and such that the real and imaginary parts of the corresponding function χ_0 when $M = 0$ are respectively regular and logarithmically singular at the shoreline. This may be achieved by multiplying ϕ by $e^{i\beta}$ for an appropriate choice of β. The behaviour of χ as $r \to \infty$ is then

$$\chi(r,\theta) \sim a\exp\{-\kappa r\sin\theta + i(\kappa' r\cos\theta + \beta)\}.$$

A particular value is

$$\beta = (\tau-2)\,\pi/4 + \arg\left[\frac{h(i)}{h(ir_2/r_1)\,\{r_1^\tau + e^{i\tau\pi/2}\}}\right].$$

Alternatively, if $\chi = \chi_1 + i\chi_2$, then χ_1 and χ_2 can each be obtained as the integral expression in (2·10), by replacing $g(z)$ in (2·11) with, respectively,

$$\tilde{g}_1(z) = A_1\frac{z^{-\tau}+1}{z^\tau}\,H_1(z)\,H_2(ze^{-i\pi}) \quad \text{and} \quad \tilde{g}_2(z) = iA_2\frac{z^{-\tau}-1}{z^\tau}\,H_1(z)\,H_2(ze^{-i\pi}), \tag{4·7}$$

where (real) A_1, A_2 are chosen to give the appropriate amplitude at infinity.

5. *Solutions in the two-dimensional case* ($k = 0$). Solutions to the two-dimensional problem of normally incident waves on a sloping beach in the presence of surface tension, which are bounded at the shoreline, are easily obtained from a formal limiting procedure on the results of the previous sections. Eliminating time dependence as before, we require a function $\hat{\phi}(r,\theta)$ which satisfies Laplace's equation in Γ together with the boundary conditions (2·5) and (2·6).

We consider the behaviour of $g(z)$ given by (3·9) in the limit as $k \to 0$. Thus, as $k \to 0$, we find that

$$r_1 = \frac{2\hat{\kappa}}{k} + O(k), \quad r_2 = \frac{k}{2\hat{\kappa}} + O(k^3), \tag{5·1}$$

$$c_1 = \frac{2a}{k} + O(k), \quad c_2 = \frac{k}{2a} + O(k^3),$$

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where $a = (M\hat{\kappa})^{-\frac{1}{2}}$ and $\hat{\kappa}$ is the real positive root of (2·8); that is, $\hat{\kappa}$ satisfies

$$M\hat{\kappa}^3 + \hat{\kappa} - 1 = 0. \tag{5·2}$$

It follows immediately from (5·2) that $a > \hat{\kappa}$. Using the asymptotic behaviour of $h(z)$ in (3·9), we obtain the result

$$\hat{g}(u) \equiv \lim_{k\to 0}\{g(2u/k)\} = Aa^{\tau}\frac{h(u/\hat{\kappa})}{u^{\tau}}\hat{H}(ue^{-i\pi}), \tag{5·3}$$

where $\hat{H}(u) = h\left(\frac{u}{a}e^{i\hat{\omega}}\right)h\left(\frac{u}{a}e^{-i\hat{\omega}}\right)$ and $0 < \hat{\omega} = \tan^{-1}\{(4/M\hat{\kappa}^3 - 1)^{\frac{1}{2}}\} < \frac{1}{2}\pi$.

It is now straightforward to verify that

$$\hat{\phi}(r,\theta) = \int_C u^{-1}[\hat{g}(ue^{i\theta}) + \hat{g}(ue^{2i\alpha - i\theta})]\,e^{ru}du, \tag{5·4}$$

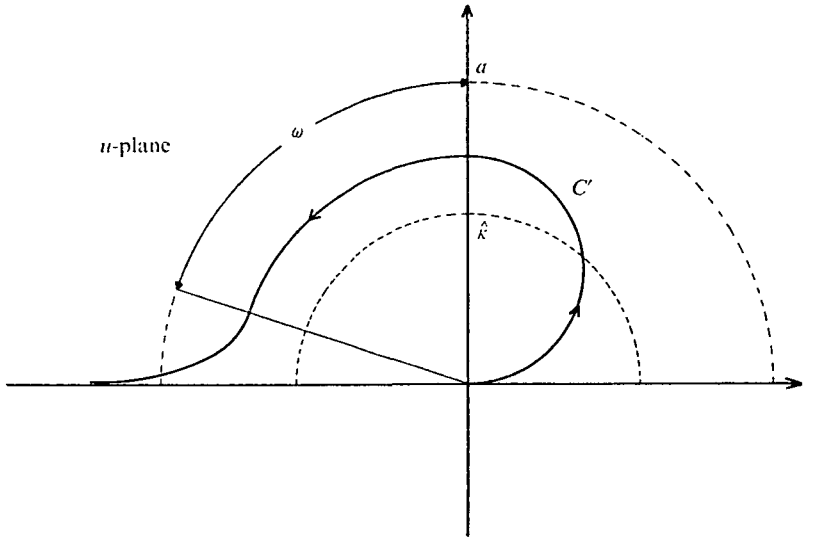


Fig. 2. Path of integration in the u-plane.

where $\hat{C}$ is illustrated in Fig. 2, satisfies all the conditions of the problem. Furthermore, if we choose

$$A = \frac{\hat{a}\,\hat{\kappa}^{\tau}\exp\{\frac{1}{2}i\pi(2\tau+1)\}}{4\pi a^{\tau}\,|h(i)|\,|H(i\hat{\kappa}\exp\{2i\alpha - i\pi\})|},$$

then $\hat{\phi}(r,0) \sim \hat{a}\,e^{i\hat{\kappa}r + i\hat{\beta}}$ at infinity and the real and imaginary parts of $\hat{\phi}$ lead to solutions which are respectively regular and logarithmically singular at the shoreline in the absence of surface tension, thus in agreement with the work of previous authors.

After the present paper had been prepared for publication, the author's attention was called to the work of Mondal (3) who has investigated this latter, specialized case in some detail.

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