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# Plane elastostatic transmission through an interface layer

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*Abstract*

A theorem is established for generating by differentiation and integration the Airy stress function due to the presence of any arbitrary two-dimensional singularity near a thick layer separating two other dissimilar semi-infinite solids directly from the corresponding Airy stress function for the unbounded homogeneous solid. A systematic application of the theorem yields the dominant distant effect associated with any arbitrary influencing line singularity in a well-structured physically interpretable form. As an illustration, the problem of a concentrated line force of arbitrary orientation near the interface layer is examined from this standpoint. It is found that a force aligned normal to the interface layer produces a far-field effect of an edge dislocation with Burgers vector parallel to the interface layer, together with a semi-infinite plane of centres of dilatation, whereas a force aligned parallel to the interface layer produces a far-field effect of an edge dislocation with Burgers vector perpendicular to the interface layer together with a semi-infinite plane of concentrated couples.

## 1. *Introduction*

Aderogba [**1**] has expressed the Airy stress function due to the presence of any arbitrary two-dimensional singularity in the interior of one of two perfectly bonded dissimilar semi-infinite solids explicitly in terms of the corresponding Airy stress function for the unbounded homogeneous solid. The advantage is that the direct calculation of the desired stresses is thus reduced to routine differentiation and integration, without going through the rigour of a fully-fledged boundary-value problem. Honein and Herrmann [**2**] have also independently recently related the complex potentials for elastic composites with circular or straight interfaces to the corresponding complex potentials for the unbounded homogeneous solid. The presentation of similar results in antiplane elastostatics also seems to have been reserved for Honein *et al.* [**3**].

If the two semi-infinite solids in Aderogba's paper are separated by a dissimilar thick layer and an arbitrary two-dimensional singularity is operative inside or outside the layer, the satisfaction of the new continuity conditions at the two parallel plane interfaces is a formidable boundary-value problem in elasticity theory. However, Maxwell [**4**] has shown that if we have a complete theory of images in the particular case of two bonded semi-infinite solids without an interface layer and the governing

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equation in each phase is harmonic, then a systematic repeated application of such a theory to the general case when an interface layer separates the two semi-infinite solids should result in a well-structured solution, whatever be the physical nature of the influencing singularity in or near the interface layer.

In this paper, it is shown that for the present three-phase plane layered composite governed by elastostatic biharmonic equations, the use of Aderogba's [**1**] theory for the special case of two bonded semi-infinite solids without an interface layer also significantly facilitates the routine calculation of stress, reducing it to the application of the operations of differentiation and integration. The influencing singularity is assumed to be located outside the inhomogeneous interface layer, because the interaction between singularities and inhomogeneities is important in the theory of plastic deformation.

Apart from some interest in generality (we admit the possibility of biharmonic influencing singularities such as concentrated line forces and edge dislocations, as well as harmonic influencing singularities such as concentrated couples and lines of centres of dilatation) our treatment is perhaps rather simple and direct. We just select suitable Airy stress functions that ensure the automatic satisfaction of the continuity conditions at the first interface, then match displacement and traction at the second interface and finally convert the resulting integral solution to a series solution. From the series, far-field asymptotic expressions for the stress functions are extracted and physically interpreted in terms of fundamental singular solutions.

## 2. *Formulation*

We consider the plane strain problem of obtaining complementary solutions for the elastic field due to the presence of any arbitrary line singularity passing through the points $(0, h)$ inside the isotropic elastic half-space $|x| < \infty$, $y > 0, |z| < \infty$ of shear modulus $\mu_1$ and Poisson's ratio $\nu_1$, which is perfectly bonded to the isotropic elastic layer $|x| < \infty, -H < y < 0, |z| < \infty$ of shear modulus $\mu_2$ and Poisson's ratio $\nu_2$. The plane surface $y = -H$ is perfectly bonded to a second dissimilar isotropic elastic half-space $|x| < \infty, y < -H, |z| < \infty$ of shear modulus $\mu_3$ and Poisson's ratio $\nu_3$ (Fig. 1).

In each phase, the elastic field is representable in terms of an Airy stress function $\Psi$ by means of the relations

$$\left.\begin{aligned} 2\mu u_x &= -\frac{\partial \Psi}{\partial x} + (1-\nu)\int \nabla^2 \Psi \, dx, \\ 2\mu u_y &= -\frac{\partial \Psi}{\partial y} + (1-\nu)\int \nabla^2 \Psi \, dy, \\ \sigma_{xx} &= \frac{\partial^2 \Psi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \Psi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \Psi}{\partial x \, \partial y}, \end{aligned}\right\} \tag{1}$$

provided that $(u_x, u_y)$ stands for the displacement vector $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$ for the stress tensor and

$$\nabla^4 \Psi = 0, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{2}$$

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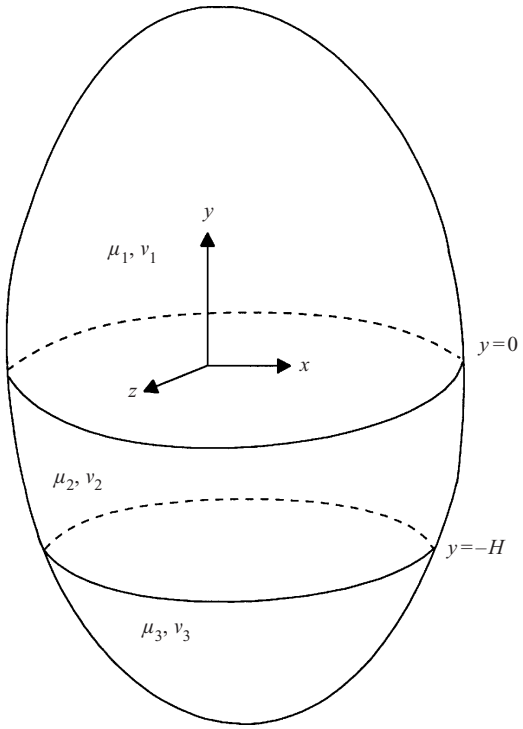



Fig. 1. Geometry of problem.

These equations are to be accompanied by the continuity conditions

$$\left.\begin{aligned} u_x^{(1)} &= u_x^{(2)}, \quad u_y^{(1)} = u_y^{(2)}, \\ \sigma_{xy}^{(1)} &= \sigma_{xy}^{(2)}, \quad \sigma_{yy}^{(1)} = \sigma_{yy}^{(2)}, \end{aligned}\right\} \tag{3}$$

at the interface $y = 0$, for all $x$ and $z$, as well as the continuity conditions

$$\left.\begin{aligned} u_x^{(2)} &= u_x^{(3)}, \quad u_y^{(2)} = u_y^{(3)}, \\ \sigma_{xy}^{(2)} &= \sigma_{xy}^{(3)}, \quad \sigma_{yy}^{(2)} = \sigma_{yy}^{(3)}, \end{aligned}\right\} \tag{4}$$

at the other interface $y = -H$, for all $x$ and $z$, where we are distinguishing the elastic fields in the regions $y > 0, -H < y < 0$ and $y < -H$ by the superscripts 1, 2 and 3 in parentheses, respectively.

For the sake of brevity of notation, it is henceforth analytically convenient to introduce the auxiliary parameters

$$\left.\begin{aligned} \alpha_i &= \frac{\mu_{i+1} - \mu_i}{\mu_{i+1}(3 - 4\nu_i) + \mu_i}, \\ \bar{\alpha}_i &= \frac{\mu_i - \mu_{i+1}}{\mu_i(3 - 4\nu_{i+1}) + \mu_{i+1}}, \\ \beta_i &= \frac{\mu_{i+1}(3 - 4\nu_i) - \mu_i(3 - 4\nu_{i+1})}{\mu_i(3 - 4\nu_{i+1}) + \mu_{i+1}}, \\ \bar{\beta}_i &= \frac{\mu_i(3 - 4\nu_{i+1}) - \mu_{i+1}(3 - 4\nu_i)}{\mu_{i+1}(3 - 4\nu_i) + \mu_i}, \end{aligned}\right\} \tag{5}$$

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where the subscript $i$ ranges from 1 to 2. It may be verified that

$$\bar{\alpha}_i\bar{\beta}_i = \alpha_i\beta_i, \quad 1 - \alpha_i\beta_i = (1 + \alpha_i)(1 + \bar{\alpha}_i), \tag{6}$$

which permit both $\bar{\alpha}_i$ and $\bar{\beta}_i$ to be expressed in terms of both $\alpha_i$ and $\beta_i$. It will be seen in what follows that these relations enable the stress fields induced in the three phases to be expressed in terms of only the four parameters $\alpha_i$ and $\beta_i$ although six elastic constants are involved in the formulation of our boundary-value problem.

## 3. *Analysis*

We suppose that an arbitrary line singularity passes through the point $(0, h)$ in the elastic half-space of moduli $(\mu_1, \nu_1)$ which is separated from another half-space of moduli $(\mu_3, \nu_3)$ by the thick layer of moduli $(\mu_2, \nu_2)$. It is desired to find the Airy stress function induced in each medium.

If $\Psi(x, y)$ denotes the corresponding Airy stress function due to the presence of the same line singularity in an unbounded homogeneous solid of moduli $(\mu_1, \nu_1)$, then without any loss in generality $\Psi$ may be expressed in the integral form

$$\Psi(x, y) = \int_0^\infty (f + gyt)\, e^{ty} \cos tx\, dt, \tag{7}$$

where $f$ and $g$ are suitable functions of the integration parameter $t$. This integral will be convergent provided that $y < h$. Hence, for the interior of the layer and the half-space of moduli $(\mu_3, \nu_3)$, this integral is convergent.

In view of the theory of Aderogba [**1**] in the special case of two perfectly bonded half-spaces without an interface layer, each desired Airy stress function will be equal to that due to a system of imaginary singularities placed at certain points along the normal to the interface layer through $(0, h)$.

We therefore invoke this theory to select the Airy functions $\Psi_1, \Psi_2$ and $\Psi_3$ for the respective regions $y > 0, -H < y < 0$ and $y < -H$ as follows:

$$\left.\begin{aligned}
\Psi_1 &= \Psi(x, y) + \alpha_1 \left[1 - 2y\frac{\partial}{\partial y} + y^2\nabla^2\right] \Psi(x, -y) \\
&\quad + \tfrac{1}{4}(\alpha_1 - \beta_1) \iint \nabla^2\Psi(x, -y)\, dy\, dy + (1 + \bar{\alpha}_1)\, \Phi_1(x, y) \\
&\quad + \tfrac{1}{4}(\bar{\alpha}_1 - \bar{\beta}_1) \left[\iint \nabla^2\Phi_1(x, y)\, dy\, dy - 2y \int \nabla^2\Phi_1(x, y)\, dy\right], \\
\Psi_2 &= \Phi_1(x, y) + \bar{\alpha}_1 \left[1 - 2y\frac{\partial}{\partial y} + y^2\nabla^2\right] \Phi_1(x, -y) \\
&\quad + \tfrac{1}{4}(\bar{\alpha}_1 - \bar{\beta}_1) \iint \nabla^2\Phi_1(x, -y)\, dy\, dy + (1 + \alpha_1)\, \Psi(x, y) \\
&\quad + \tfrac{1}{4}(\alpha_1 - \beta_1) \left[\iint \nabla^2\Psi(x, y)\, dy\, dy - 2y \int \nabla^2\Psi(x, y)\, dy\right], \\
\Psi_3 &= \Phi_2(x, y).
\end{aligned}\right\} \tag{8}$$

The perturbation functions $\Phi_1$ and $\Phi_2$ represent the effects of the two parallel plane

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interfaces and they must be constructed cognizant of the fact that the only permissible inducing singularity is that due to $\Psi(x, y)$ alone. We will ultimately seek their functional forms *explicitly* in terms of the basic function $\Psi$ for the unbounded homogeneous solid, noting from the outset that the representations for $\Psi_1$ and $\Psi_2$ in (8) ensure automatic satisfaction of the continuity conditions (3) at the first interface $y = 0$.

Guided by (7), we select the perturbation functions $\Phi_1$ and $\Phi_2$ in the forms

$$\left.\begin{aligned} \Phi_1(x, y) &= \int_0^\infty [A_1 + (y + H)tB_1]\, e^{-t(y+2H)} \cos tx\, dt, \\ \Phi_2(x, y) &= \int_0^\infty [A_2 + (y + H)tB_2]\, e^{ty} \cos tx\, dt, \end{aligned}\right\} \tag{9}$$

where the weighting functions $A_i(t)$ and $B_i(t)$ are to be determined from the matching of displacement and traction at the second interface $y = -H$. This tedious but straightforward algebra yields the explicit expressions

$$\left.\begin{aligned} A_1 &= \tfrac{1}{2}(\alpha_2\Delta_2 + \beta_2\Delta_1)\Delta, \\ B_1 &= \alpha_2\Delta_2\Delta, \\ A_2 &= \tfrac{1}{2}[(1 + \alpha_2)\Delta_2 + (1 + \beta_2)\Delta_1]\Delta, \\ B_2 &= -(1 - \beta_2)\Delta_1\Delta, \end{aligned}\right\} \tag{10}$$

where

$$\left.\begin{aligned} \Delta_1 &= 2tH\bar{\alpha}_1\alpha_2(1 + \alpha_1)E(2f + g) + (1 + \beta_1)(\alpha_2\bar{\beta}_1E - 1)g, \\ \Delta_2 &= -(1 + \alpha_1)(\bar{\alpha}_1\beta_2E - 1)(2f + g) - 2tHg(1 + \beta_1), \\ \Delta &= \left[1 - (\bar{\alpha}_1\beta_2 + \alpha_2\bar{\beta}_1)E + \alpha_1\alpha_2\beta_1\beta_2E^2 - (2tH)^2\bar{\alpha}_1\alpha_2E\right]^{-1}, \\ E &= e^{-2tH}. \end{aligned}\right\} \tag{11}$$

Exact integral representations for the Airy functions $\Psi_1, \Psi_2$ and $\Psi_3$ may now be deduced from (8), in conjunction with (9) to (11). Integral representations for the associated displacements and stresses appropriate to the layer and the half-spaces are then immediately obtainable from (1). This concludes the formal solution of our problem.

## 4. *Equal phase rigidities*

Although the integral solution obtained in the last section is complete, it is decidedly difficult to investigate its physical characteristics in its general form. Therefore, we first proceed directly to the special results for the Airy functions when the phases have equal rigidities but different compressibilities [**5**], that is $\mu_1 = \mu_2 = \mu_3$.

From (5) follows

$$\alpha_i = \bar{\alpha}_i = 0, \quad \beta_i = \frac{\nu_{i+1} - \nu_i}{1 - \nu_{i+1}} = \lambda_i. \tag{12}$$

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Substituting for these parameters in (8) to (11) we arrive at the results

$$\left.\begin{aligned}
\Psi_1 &= \Psi(x,y) - \tfrac{1}{4}\lambda_1 \iint \nabla^2\Psi(x,-y)\,dy\,dy \\
&\quad - \tfrac{1}{4}\lambda_2(1+\lambda_1)\iint \nabla^2\Psi(x,-y-2H)\,dy\,dy, \\
\Psi_2 &= \Psi(x,y) - \tfrac{1}{4}\lambda_1 \iint \nabla^2\Psi(x,y)\,dy\,dy - 2y\int \nabla^2\Psi(x,y)\,dy \\
&\quad - \tfrac{1}{4}\lambda_2(1+\lambda_1)\iint \nabla^2\Psi(x,-y-2H)\,dy\,dy, \\
\Psi_3 &= \Psi(x,y) - \tfrac{1}{4}(\lambda_1+\lambda_2+\lambda_1\lambda_2)\iint \nabla^2\Psi(x,y)\,dy\,dy \\
&\quad + \tfrac{1}{2}[(1+\lambda_1)\{(1+\lambda_2)(y+H)-H\}-y]\int \nabla^2\Psi(x,y)\,dy,
\end{aligned}\right\} \tag{13}$$

which explicitly express the desired Airy functions in terms of the corresponding function for the unbounded homogeneous solid, whatever the nature of the loading mechanism.

Hence, if the layered system is under the influence of a concentrated line force $(f_1, f_2)$ through the points $(0, h)$, characterized in the unbounded homogeneous solid by the Airy function

$$\Psi(x,y) = \left\{\begin{array}{l} \dfrac{1-2\nu_1}{4\pi(1-\nu_1)}[f_1 x + f_2(y-h)]\log r_1, \\ +\dfrac{1}{2\pi}[f_2 x - f_1(y-h)]\tan^{-1}\left(\dfrac{y-h}{x}\right), \end{array}\right. \tag{14}$$

with $r_1^2 = x^2 + (y-h)^2$, *the image system for generating the elastic field in the first medium of moduli* $(\mu_1, \nu_1)$ *consists of the superposition of two semi-infinite planes* $\{x = 0, y < -h, |z| < \infty\}$ *and* $\{x = 0, y < -h - 2H, |z| < \infty\}$ *of centres of dilatation and concentrated couples*, that is, the complete Airy stress function in the first medium is

$$\begin{aligned}
\Psi_1 = \Psi(x,y) &- \frac{\lambda_1}{8\pi(1-\nu_1)}\int\left[f_2\log r_2 - f_1\tan^{-1}\left(\frac{y+h}{x}\right)\right]dy \\
&- \frac{\lambda_2(1+\lambda_1)}{8\pi(1-\nu_1)}\int\left[f_2\log r_3 - f_1\tan^{-1}\left(\frac{y+h+2H}{x}\right)\right]dy,
\end{aligned} \tag{15}$$

where $\Psi(x,y)$ is once again given by (14), while

$$\left.\begin{aligned}
&r_2^2 = x^2 + (y+h)^2, \quad r_3^2 = x^2 + (y+h+2H)^2, \\
&\int \log(x^2+y^2)\,dy = y\log(x^2+y^2) - 2y + 2x\tan^{-1}\left(\frac{y}{x}\right), \\
&\int \tan^{-1}\left(\frac{y}{x}\right)dy = y\tan^{-1}\left(\frac{y}{x}\right) - \tfrac{1}{2}x\log(x^2+y^2).
\end{aligned}\right\} \tag{16}$$

It may be verified that the same physical interpretation holds good when an edge dislocation with Burgers vector $(b_1, b_2)$ passes through the point $(0, h)$, since in this

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case the Airy function for the unbounded homogeneous solid is

$$\left.\begin{aligned} \Psi(x,y) &= [D_1(y-h) + D_2 x]\,\log r_1, \\ (D_1, D_2) &= \mu_1 \frac{(-b_1, b_2)}{2\pi(1-\nu_1)}. \end{aligned}\right\} \tag{17}$$

Now, for the concentrated line force $(0, f_2)$ passing through the point $(0, h)$, equations (13) and (14) reveal that the elastic field in the third medium of moduli $(\mu_3, \nu_3)$ is generated by the Airy function

$$\begin{aligned} \Psi_3 = f_2 \left[\frac{1-2\nu_1}{4\pi(1-\nu_1)}(y-h)\log r_1 \frac{1}{2\pi} x \tan^{-1}\left(\frac{y-h}{x}\right)\right] - \frac{f_2(\lambda_1+\lambda_2+\lambda_1\lambda_2)}{4\pi(1-\nu_1)} \\ \times (y-h)\,\log r_1 - \frac{f_2}{4\pi(1-\nu_1)}[(\lambda_1+\lambda_2+\lambda_1\lambda_2)h + \lambda_2(1+\lambda_1)\,H]\log r_1 \\ + \frac{f_2(\lambda_1+\lambda_2+\lambda_1\lambda_2)}{8\pi(1-\nu_1)} \int \log r_1\, dy. \end{aligned} \tag{18}$$

Hence, *the elastic field in the third medium is produced by a combination of a concentrated line force parallel to the y-axis, an edge dislocation with Burgers vector parallel to the x-axis, a line of centres of dilatation, all through the point* $(0, h)$, *and a semi-infinite plane* $\{x = 0, y > h, |z| < \infty\}$ *of centres of dilatation.*

On the other hand, it may be similarly established with the aid of (13) and (14) that for the influencing concentrated line force $(f_1, 0)$ passing through the point $(0, h)$, the elastic field in the third medium is produced by the combined influence in an unbounded homogeneous solid of a concentrated line force parallel to the $x$-axis, an edge dislocation with Burgers vector parallel to the $y$-axis, a line of concentrated couples, all through the point $(0, h)$, and a semi-infinite plane $\{x = 0, y > h, |z| < \infty\}$ of concentrated couples. Thus, there is a fundamental difference in the elastic field produced by a line force acting normal to the interface layer and that produced by a line force acting parallel to the interface layer.

Far away in the first and third media, the Airy stress functions in (13) take the closed forms

$$\left.\begin{aligned} \Psi_1 &= \Psi_o(x,y) - \frac{1}{4}\frac{\nu_3-\nu_1}{1-\nu_3}\iint \nabla^2\Psi_o(x,-y)\,dy\,dy, \\ \Psi_3 &= \Psi_o(x,y) - \frac{1}{4}\frac{\nu_3-\nu_1}{1-\nu_3}\iint \nabla^2\Psi_o(x,y)\,dy\,dy \\ &\quad + \tfrac{1}{2} y \frac{\nu_3-\nu_1}{1-\nu_3}\int \nabla^2\Psi_o(x,y)\,dy, \end{aligned}\right\} \tag{19}$$

where $\Psi_o(x, y)$ is the corresponding Airy function for the unbounded homogeneous solid when the singularity passes through the origin. One could therefore say that the far fields in the first and third media do not feel the presence of the interface layer. In particular, far away in the first medium, the elastic field of an influencing line force of arbitrary orientation is produced by the application at the origin of a concentrated line of force of arbitrary orientation, together with a semi-infinite plane through the negative $y$-axis of concentrated couples and centres of dilatation.

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Similarly, by substituting (17) into (13), the far field in the first medium due to an influencing edge dislocation with Burgers vector of arbitrary orientation is characterized by the Airy function

$$\Psi_1 = (D_1 y + D_2 x)\,\log r - \frac{1}{2}\frac{\nu_3 - \nu_1}{1 - \nu_3}\int \left[D_1\,\log r + D_2\,\tan^{-1}\left(\frac{y}{x}\right)\right] dy \tag{20}$$

with $r^2 = x^2 + y^2$. Thus, the far field is a combination of an edge dislocation with Burgers vector of arbitrary orientation applied at the origin, together with a semi-infinite plane through the negative $y$-axis of concentrated couples and centres of dilatation.

## 5. *An elastostatic interface layer theorem*

A general result which may be of considerable importance in the theory of composite materials may now be stated as follows.

Suppose there is a general two-dimensional deformation of an isotropic homogeneous elastic solid of moduli $(\mu_1, \nu_1)$ occupying the whole space $|x| < \infty, |y| < \infty$, characterized by the Airy function $\Psi(x, y)$, where the singularities of $\Psi$ are all confined to the region $|x| < \infty, |y| < h$, with $h$ standing for a positive constant. If two other elastic materials of moduli $(\mu_2, \nu_2)$ and $(\mu_3, \nu_3)$ are introduced into the regions $\{|x| < \infty, -H < y < 0\}$ and $\{|x| < \infty, y < -H\}$, respectively, retaining the material of moduli $(\mu_1, \nu_1)$ in the region $\{|x| < \infty, y > 0\}$ and keeping the three materials perfectly bonded together at the parallel interfaces $y = 0$ and $y = -H$, the Airy stress function in the first medium of moduli $(\mu_1, \nu_1)$ is

$$\begin{aligned}\Psi_1 = \Psi(x, y) &+ \alpha_1\left[1 - 2y\frac{\partial}{\partial y} + y^2\nabla^2\right]\Psi(x, -y)\\ &+ \tfrac{1}{4}(\alpha_1 - \beta_1)\iint \nabla^2\Psi(x, -y)\,dy\,dy + (1 + \bar{\alpha}_1)\,\Phi_1(x, y)\\ &+ \tfrac{1}{4}(\bar{\alpha}_1 - \bar{\beta}_1)\left[\iint \nabla^2\Phi_1(x, y)\,dy\,dy - 2y\int \nabla^2\Phi_1(x, y)\,dy\right],\end{aligned} \tag{21}$$

that in the second medium of moduli $(\mu_2, \nu_2)$ is

$$\begin{aligned}\Psi_2 = \Phi_1(x, y) &+ \bar{\alpha}_1\left[1 - 2y\frac{\partial}{\partial y} + y^2\nabla^2\right]\Phi_1(x, -y)\\ &+ \tfrac{1}{4}(\bar{\alpha}_1 - \bar{\beta}_1)\iint \nabla^2\Phi_1(x, -y)\,dy\,dy + (1 + \alpha_1)\,\Psi(x, y)\\ &+ \tfrac{1}{4}(\alpha_1 - \beta_1)\left[\iint \nabla^2\Psi(x, y)\,dy\,dy - 2y\int \nabla^2\Psi(x, y)\,dy\right]\end{aligned} \tag{22}$$

and that in the third medium of moduli $(\mu_3, \nu_3)$ is

$$\Psi_3 = \Phi_2(x, y), \tag{23}$$

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where the auxiliary biharmonic functions $\Phi_1$ and $\Phi_2$ are defined as follows:

$$\begin{aligned}\Phi_1(x,y) &= \alpha_2(1+\alpha_2)[1-2(y+H)L]\sum\sum\sum\Delta_{mnk}L^{2k}\\
&\times\left[\Psi(x,-\bar{y}_1)-\tfrac{1}{2}\bar{y}_1\int\nabla^2\Psi(x,\bar{y}_1)\,dy+\tfrac{1}{4}\iint\nabla^2\Psi(x,-\bar{y}_1)\,dy\,dy\right]\\
&-\bar{\alpha}_1\beta_2\left\{\Psi(x,-\bar{y}_2)-\tfrac{1}{2}\bar{y}_2\int\nabla^2\Psi(x,-\bar{y}_2)\,dy+\tfrac{1}{4}\iint\nabla^2\Psi(x,-\bar{y}_2)\,dy\,dy\right\}\\
&+\tfrac{1}{2}H\alpha_2(1+\beta_1)[1-2(y+H)L]\\
&\times\sum\sum\sum\Delta_{mnk}L^{2k+1}\iint\nabla^2\Psi(x,-\bar{y}_1)\,dy\,dy\\
&-2H\bar{\alpha}_1\alpha_2\beta_2(1+\alpha_1)\sum\sum\sum\Delta_{mnk}L^{2k+1}\\
&\times\left[\Psi(x,-\bar{y}_2)-\tfrac{1}{2}\bar{y}_2\int\nabla^2\Psi(x,-\bar{y}_2)\,dy+\tfrac{1}{4}\iint\nabla^2\Psi(x,-\bar{y}_2)\,dy\,dy\right]\\
&-\tfrac{1}{4}\beta_2(1+\beta_1)\sum\sum\sum\Delta_{mnk}L^{2k}\iint\nabla^2[\Psi(x,-\bar{y}_1)-\alpha_2\bar{\beta}_1\Psi(x,-\bar{y}_2)]\,dy\,dy\end{aligned}\tag{24}$$

$$\begin{aligned}\Phi_2(x,y) &= -\tfrac{1}{4}(1+\beta_1)(1+\beta_2)[1-2(y+H)L]\\
&\times\sum\sum\sum\Delta_{mnk}L^{2k}\iint\nabla^2[\Psi(x,y_0)-\alpha_2\bar{\beta}_1\Psi(x,y_1)]\,dy\,dy\\
&+2H\bar{\alpha}_1\alpha_2(1+\alpha_1)(1+\beta_2)[1-2(y+H)L]\\
&\times\sum\sum\sum\Delta_{mnk}L^{2k+1}\left[\Psi(x,y_1)-\tfrac{1}{2}y_1\int\nabla^2\Psi(x,y_1)\,dy\right.\\
&\left.+\tfrac{1}{4}\iint\nabla^2\Psi(x,y_1)\,dy\,dy\right]+(1+\alpha_1)(1+\alpha_2)\sum\sum\sum\Delta_{mnk}L^{2k}\\
&\times\left[\Psi(x,y_0)-\tfrac{1}{2}y_0\int\nabla^2\Psi(x,y_0)\,dy+\tfrac{1}{4}\iint\nabla^2\Psi(x,y_0)\,dy\,dy\right.\\
&\left.-\bar{\alpha}_1\beta_2\left\{\Psi(x,y_1)-\tfrac{1}{2}y_1\int\nabla^2\Psi(x,y_1)\,dy+\tfrac{1}{4}\iint\nabla^2\Psi(x,y_1)\,dy\,dy\right\}\right]\\
&-\tfrac{1}{2}H(1+\alpha_2)(1+\beta_1)\sum\sum\sum\Delta_{mnk}L^{2k+1}\iint\nabla^2\Psi(x,y_0)\,dy\,dy.\end{aligned}\tag{25}$$

In equations (24) and (25) the symbol $\Sigma\Sigma\Sigma$ serves to indicate triple summation over the non-negative values of the integers $m, n$, and $k$, while

$$\Delta_{mnk}=\frac{(m+n+k)!}{m!\,n!\,k!}(-\alpha_1\alpha_2\beta_1\beta_2)^m(\bar{\alpha}_1\beta_2+\alpha_2\bar{\beta}_1)^n(1+H^2\bar{\alpha}_1\alpha_2)^k,$$

$$y_p=y-2H(2m+n+k+p),\quad \bar{y}_p=y+2H(2m+n+k+p),\quad L=\frac{\partial}{\partial y}.\tag{26}$$

Far away from the inhomogeneous interface layer, the perturbation potentials $\Phi_i$

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pass over into the simpler expressions

$$\Phi_1(x,y) = \alpha_2(1+\alpha_1)(1-\alpha_2\bar{\beta}_1)^{-1}\left[1-2y\frac{\partial}{\partial y}\right]$$
$$\times\left[\Psi_o(x,-y) - \tfrac{1}{2}y\int \nabla^2\Psi_o(x,-y)\,dy + \tfrac{1}{4}\iint \nabla^2\Psi_o(x,-y)\,dy\,dy\right]$$
$$-\tfrac{1}{4}\beta_2(1+\beta_1)(1-\bar{\alpha}_1\beta_2)^{-1}\iint \nabla^2\Psi_o(x,-y)\,dy\,dy, \tag{27}$$

$$\Phi_2(x,y) = (1+\alpha_1)(1+\alpha_2)(1-\alpha_2\bar{\beta}_1)^{-1}$$
$$\times\left[\Psi_o(x,y) - \tfrac{1}{2}y\int \nabla^2\Psi_o(x,y)\,dy + \tfrac{1}{4}\iint \nabla^2\Psi_o(x,y)\,dy\,dy\right]$$
$$-\tfrac{1}{4}(1+\beta_1)(1+\beta_2)(1-\bar{\alpha}_1\beta_2)^{-1}\left(1-2y\frac{\partial}{\partial y}\right)\iint \nabla^2\Psi_o(x,y)\,dy\,dy, \tag{28}$$

where $\Psi_o(x,y)$ is the corresponding Airy function when the influencing singularity is operative at the origin. The substitution of (14) into (27) and (28) immediately leads to the conclusion that an influencing line force aligned normal to the interface layer produces a far-field effect of an edge dislocation with Burgers vector parallel to the interface layer, together with a semi-infinite plane of centres of dilatation, whereas a line force aligned parallel to the interface layer produces a far field effect of an edge dislocation with Burgers vector perpendicular to the interface layer, together with a semi-infinite plane of concentrated couples.

Analytically, far away in the third medium, the dominant distant Airy stress function due to an influencing line force $(f_1, f_2)$ through the point $(0, h)$ is

$$\Psi_3 = -\frac{1}{4\pi(1-\nu_1)}\frac{(1+\beta_1)(1+\beta_2)}{1-\bar{\alpha}_1\beta_2}(f_1x + f_2y)\log r$$
$$-\frac{1}{8\pi(1-\nu_1)}\left[\frac{(3-4\nu_1)(1+\alpha_1)(1+\alpha_2)}{1-\alpha_2\bar{\beta}_1} + \frac{(1+\beta_1)(1+\beta_2)}{1-\bar{\alpha}_1\beta_2}\right]$$
$$\times\int\left[f_1\tan^{-1}\left(\frac{y}{x}\right) - f_2\log r\right]dy \tag{29}$$

representing the superposition of an edge dislocation with Burgers vector of arbitrary orientation, applied at the origin, and a distribution of concentrated couples and centres of dilatation on the plane $\{x = 0, y < 0, |z| < \infty\}$.

The simplest possible application of the interface layer theorem is to the case of a line dilatation of strength $\Gamma$ through the point $(0, h)$ in the first medium. In the unbounded homogeneous solid, this influencing singularity is characterized by the harmonic Airy function

$$\Psi(x,y) = -\mu_1\Gamma\log\,[x^2 + (y-h)^2], \tag{30}$$

and the interface theorem shows that when this singularity acts near an interface layer the desired Airy function induced in the material of moduli $(\mu_1, \nu_1)$ containing

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the singularity is

$$\Psi_1 = \Psi(x,y) + \alpha_1\left[1 - 2y\frac{\partial}{\partial y}\right]\Psi(x,-y) + (1+\bar{\alpha}_1)\Phi(x,y)$$
$$+\tfrac{1}{4}(\bar{\alpha}_1 - \bar{\beta}_1)\left[\iint \nabla^2\Phi(x,y)\,dy\,dy - 2y\int \nabla^2\Phi(x,y)\,dy\right], \quad (31)$$

where

$$\Phi(x,y) = \alpha_2(1+\alpha_1)\left[1 - 2(y+H)\frac{\partial}{\partial y}\right]\sum\sum\sum\Delta_{mnk}L^{2k}$$
$$\times[\Psi(x,-\bar{y}_1) - \bar{\alpha}_1\beta_2\Psi(x,-\bar{y}_2)]$$
$$-2H\bar{\alpha}_1\alpha_2\beta_2(1+\alpha_1)\sum\sum\sum\Delta_{mnk}L^{2k+1}\Psi(x,-\bar{y}_2). \quad (32)$$

Since

$$\nabla^2\Phi(x,y) = -4\alpha_2(1+\alpha_1)\sum\sum\sum\Delta_{mnk}L^{2k+2}[\Psi(x,-\bar{y}_1) - \bar{\alpha}_1\beta_2\Psi(x,-\bar{y}_2)], \quad (33)$$

equations (31) and (32) show that the desired Airy function can be obtained by differentiation of the corresponding result for the whole space, as in the solution of Mindlin and Cheng [**6**] for a centre of dilatation in the interior of an elastic half-space with a traction-free boundary.

The effect of the interface layer on the force acting on the line dilatation can be determined by means of Eshelby's [**7**] general formula

$$F = 2\pi\Gamma\frac{(1-\nu)}{(1+\nu)}L\nabla^2\Psi_1, \quad (34)$$

evaluated at the point $(0,h)$. Thus, this normal component of force takes the explicit form

$$F = -8\pi\Gamma\frac{1-\nu_1}{1+\nu_1}L^3\big[\alpha_1\Psi(x,-y) + \alpha_2(1+\alpha_1)(1+\bar{\beta}_1)$$
$$\times\sum\sum\sum\Delta_{mnk}L^{2k}\{\Psi(x,-\bar{y}_1) - \bar{\alpha}_1\beta_2\Psi(x,-\bar{y}_2)\}\big]. \quad (35)$$

At large distances from the interface inhomogeneity, this force is dominated by the term

$$F = -8\pi\Gamma\frac{1-\nu_1}{1+\nu_1}\frac{\mu_3-\mu_1}{\mu_3(3-4\nu_1)+\mu_1}L^3\Psi(x,-y), \quad (36)$$

which reveals that the effect of the interface layer is unnoticeable, the dilatation feeling only the presence of the materials of moduli $(\mu_1,\nu_1)$ and $(\mu_3,\nu_3)$.

Another application, not as readily treated in other ways, is the case of an influencing edge dislocation with Burgers vector of arbitrary orientation passing through the point $(0, h)$ in the first medium. In the unbounded homogeneous solid, the associated Airy stress function is given by equation (17) and equation (28) shows that at large distances from the interface inhomogeneity the induced Airy stress function in

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the third medium is dominated by

$$\Psi_3 = \frac{1}{2}\frac{(1+\alpha_1)(1+\alpha_2)}{1-\alpha_2\bar{\beta}_1}\int\left[D_1\log r - D_2\tan^{-1}\left(\frac{y}{x}\right)\right]dy$$
$$-\frac{1}{2}\frac{(1+\beta_1)(1+\beta_2)}{1-\bar{\alpha}_1\beta_2}\left[1-2y\frac{\partial}{\partial y}\right]\int\left[D_1\log r + D_2\tan^{-1}\left(\frac{y}{x}\right)\right]dy, \quad (37)$$

where the vector $(D_1, D_2)$ is once again given by (17).

It is thus seen that an influencing edge dislocation with Burgers vector parallel to the interface layer produces a far-field effect of an edge dislocation with Burgers vector of the same parallel orientation together with a semi-infinite plane of centres of dilatation, whereas an influencing edge dislocation with Burgers vector perpendicular to the interface layer produces a far-field effect of an edge dislocation with Burgers vector of the same normal orientation together with a semi-infinite plane of concentrated couples. An interesting conclusion is that the far-field effect of an influencing force singularity aligned normal to the interface layer has the same physical behaviour as that of an influencing edge dislocation with Burgers vector parallel to the interface layer, whereas the far-field effect of an influencing force singularity aligned parallel to the interface layer has the same physical behaviour as that of an influencing edge dislocation with Burgers vector perpendicular to the interface layer.

A final interesting application is to the case of a wedge or Weingarten [7] dislocation operative in the first medium. The interest in such an application lies in the fact that the results for the edge dislocations are often obtainable by differentiation from those for the wedge dislocations. In the unbounded homogeneous solid of moduli $(\mu_1, \nu_1)$, a wedge dislocation operative at the point $(0, h)$ is characterized by the Airy function

$$\Psi(x, y) = \Omega r_1^2 \log r_1, \quad (38)$$

where $\Omega$ is a constant, while $r_1^2 = x^2 + (y-h)^2$. Equation (8) then shows that when the wedge dislocation is operative at the point $(0, h)$ in the half-space $y > 0$ of moduli $(\mu_1, \nu_1)$ which is perfectly bonded to the half-space $y < 0$ of moduli $(\mu_2, \nu_2)$ along the plane $y = 0$, the complete Airy stress functions are, for the half-space $y > 0$,

$$\Psi_1 = \Omega r_1^2 \log r_1 + \Omega\alpha_1\left[1 - 2y\frac{\partial}{\partial y} + y^2\nabla^2\right] r_2^2 \log r_2 + \tfrac{1}{2}\Omega(\alpha_1 - \beta_1)$$
$$\times\left[r_2^2 \log r_2 - 2(y+h)\left\{(y+h)\log r_2 + x\tan^{-1}\frac{(y+h)}{x}\right\}\right] \quad (39)$$

and for the half-space $y < 0$,

$$\Psi_2 = \Omega(1+\alpha_1) r_1^2 \log r_1 - \tfrac{1}{2}\Omega(\alpha_1 - \beta_1)$$
$$\times\left[r_1^2 \log r_1 + 2(y+h)\left\{(y-h)\log r_1 + x\tan^{-1}\frac{(y-h)}{x}\right\}\right] \quad (40)$$

showing that the image system for the material occupying the half-space $y > 0$ consists of the superposition of a wedge dislocation at the image point $(0, -h)$ and a semi-infinite plane $x = 0, y < -h, |z| < \infty$ of centres of dilatation with uniform density and density proportional to $y$.

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Furthermore, equation (28) shows that when the wedge dislocation is located in the half-space $y > 0$ which is separated from the half-space $y < -H$ by a dissimilar layer occupying the region $-H < y < 0$, the dominant distant Airy stress function for the material in the half-space $y < -H$ is

$$\Psi_3 = \tfrac{1}{2}\Omega \left[ \frac{(1+\alpha_1)(1+\alpha_2)}{1-\alpha_2\bar{\beta}_1} + \frac{(1+\beta_1)(1+\beta_2)}{1-\bar{\alpha}_1\beta_2} \left(1 - 2y\frac{\partial}{\partial y}\right)\right] \times \left[ r^2 \log r - 2y \int (1+\log r)\, dy \right], \quad (41)$$

representing the superposition of a wedge dislocation at the origin and a semi-infinite plane of centres of dilatation whose density is proportional to $y$.

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