Math. Proc. Camb. Phil. Soc. (1992), **111**, 455
Printed in Great Britain

Ends of graphs. II

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(*Received* 31 *May* 1991; *revised* 25 *September* 1991)

Abstract

This paper deals with locally finite connected graphs with infinitely many ends. The structure of such graphs with a transitive group of automorphisms that fixes an end is investigated.

Introduction

This note is a sequel to [**2**]. For notation and basic definitions the reader is referred to that paper.

Let Γ be a locally finite connected graph with infinitely many ends. Put $G := \mathrm{Aut}(\Gamma)$. We will write $d_\Gamma(-,-)$ to denote the usual distance in Γ. We are interested in gaining information about the structure of Γ if we know that there is an end ω such that G_ω (the stabilizer of ω in G) acts transitively on $V\Gamma$, the set of vertices of Γ. In [**6**] it is shown that this property is equivalent with Γ allowing a transitive amenable group action. These graphs also have some special properties relating to random walks: see [**7**]. In [**8**], Woess asks for a classification of such graphs and conjectures that they 'look like' trees. More precisely, he conjectures that a graph with these properties is quasi-isometric to a tree. Two graphs Γ_1 and Γ_2 are said to be *quasi-isometric* if there are maps $\phi: V\Gamma_1 \to V\Gamma_2$ and $\psi: V\Gamma_2 \to V\Gamma_1$ and constants a_1, a_2, b_1, b_2, c, d such that for all $\alpha_1, \alpha_2 \in V\Gamma_1$ and all $\beta_1, \beta_2 \in V\Gamma_2$ the following conditions hold:

(*a*) $d_{\Gamma_2}(\phi(\alpha_1), \phi(\alpha_2)) \leqslant a_1 d_{\Gamma_1}(\alpha_1, \alpha_2) + a_2$;
(*b*) $d_{\Gamma_1}(\psi(\beta_1), \psi(\beta_2)) \leqslant b_1 d_{\Gamma_2}(\beta_1, \beta_2) + b_2$;
(*c*) $d_{\Gamma_1}(\psi\phi(\alpha_1), \alpha_1) \leqslant c$;
(*d*) $d_{\Gamma_2}(\phi\psi(\beta_1), \beta_1) \leqslant d$.

We prove the following results:

Theorem 1. *Let Γ be a locally finite connected graph with infinitely many ends. Put $G := \mathrm{Aut}(\Gamma)$. Assume that there is an end ω of Γ such that G_ω acts transitively on $V\Gamma$. Then Γ is quasi-isometric to a tree.*

Theorem 2. *Let Γ and G be as in the above theorem. Then G acts transitively on $\mathscr{E}\Gamma\backslash\{\omega\}$, where $\mathscr{E}\Gamma$ is the set of ends of Γ. In particular, if ω is not fixed by G then G acts transitively on $\mathscr{E}\Gamma$.*

For an example of a graph Γ where the end ω is fixed by G see example 2 in [**4**]. Our results imply that all such graphs must arise in a similar way from semi-regular trees. But it is difficult to see how this knowledge would enable us to get a more

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https://doi.org/10.1017/S0305004100075551 Published online by Cambridge University Press

precise description. In the first section we look more closely at quasi-isometries, and in the second section we prove Theorems 1 and 2.

1. *Quasi-isometries*

Let Γ_1 and Γ_2 be locally finite connected graphs. Assume that the maps $\phi: V\Gamma_1 \to V\Gamma_2$ and $\psi: V\Gamma_2 \to V\Gamma_1$ satisfy conditions (a) to (d) above for some constants a_1, a_2, b_1, b_2, c, d.

PROPOSITION 1. *There is a natural bijection* $\Phi: \mathscr{E}\Gamma_1 \to \mathscr{E}\Gamma_2$.

To prove this we need yet another way to handle the ends of a graph. The following definition can be found in [3].

Definition 1. Let Γ be a locally finite connected graph. If Δ is a finite subset of $V\Gamma$ and ϵ is an end of Γ then let $C(\Delta, \epsilon)$ denote the component of $\Gamma\backslash\Delta$ that contains ϵ. If $\{\Delta_i\}_{i\geqslant 1}$ is a sequence of finite subsets of $V\Gamma$ such that

$$C(\Delta_i, \epsilon) \supseteq \Delta_{i+1} \cup C(\Delta_{i+1}, \epsilon),$$

then we say that $\{\Delta_i\}_{i\geqslant 1}$ *contracts towards* ϵ.

Clearly every end can be described by sequence of finite sets that contracts towards it. Moreover, if we have a sequence that contracts towards an end, then that end is uniquely determined.

Proof of Proposition 1. Let $L := \{\alpha_1, \alpha_2, \ldots\}$ be some half-line in Γ_1. The set $\phi(L)$ need not be a half-line, but if S is a minimal spanning tree of $\phi(L)$ then S must contain some half-line Λ. Denote by ϵ_L the end of Γ_2 to which Λ belongs. Take a sequence $\{\Delta_i\}_{i\geqslant 1}$ of finite subsets of $V\Gamma_2$ that contracts towards ϵ_L. Clearly $C(\Delta_i, \epsilon_L) \cap \phi(L)$ is infinite for all i. The next step is to show that ϵ_L does not depend on the choice of S and Λ. To do this it is enough to prove that $\phi(L)\backslash C(\Delta_i, \epsilon_L)$ is finite for all i. We know that $S \cap C(\Delta_i, \epsilon_L)$ is infinite. Both Γ_1 and Γ_2 are locally finite, so, by condition (c), no fibre of ϕ is infinite. Again because Γ_2 is locally finite, we know that the set of vertices in Γ_2 of distance less than $a_1 + a_2$ from Δ_i is finite. Thus there is a number j_i such that if $j \geqslant j_i$ then $d_{\Gamma_2}(\phi(\alpha_j), \Delta_i) > a_1 + a_2$ and $\phi(\alpha_{j_i}) \in C(\Delta_i, \epsilon_L)$. Now

$$d_{\Gamma_2}(\phi(\alpha_{j_i}), \phi(\alpha_{j_i+1})) \leqslant a_1 + a_2$$

(by (a)), and thus $\phi(\alpha_{j_i+1}) \in C(\Delta_i, \epsilon_L)$. By induction, $\phi(\{\alpha_{j_i}, \alpha_{j_i+1}, \ldots\}) \subseteq C(\Delta_i, \epsilon_L)$.

Suppose that L_1 and L_2 are two half-lines in Γ_1 that belong to the same end. Two half-lines L_1 and L_2 belong to the same end if and only if there is a third half-line L_3 that contains infinitely many vertices from both L_1 and L_2. From the above argument we see that $\epsilon_{L_1} = \epsilon_{L_3} = \epsilon_{L_2}$. Thus ϵ_L depends only on the end that L belongs to but not on the particular half-line L. Now we have a well defined function

$$\Phi: \mathscr{E}\Gamma_1 \to \mathscr{E}\Gamma_2,$$

which maps an end ϵ to the end ϵ_L where L is some half-line in ϵ. One can now define

$$\Psi: \mathscr{E}\Gamma_2 \to \mathscr{E}\Gamma_1$$

in the same way.

By using similar ideas to those above and conditions (c) and (d) we get that $\Psi\Phi = \mathrm{id}_{\mathscr{E}\Gamma_1}$ and $\Phi\Psi = \mathrm{id}_{\mathscr{E}\Gamma_2}$. ∎

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Let Γ be a locally finite, transitive, connected graph with infinitely many ends. Put $G := \mathrm{Aut}(\Gamma)$. Take an element $e_0 \in \mathscr{B}\Gamma$ that is both infinite and co-infinite and such that $E := \{e_0 G, e_0^* G\}$ is a tree set. Set $T := T(E)$. Define

$$\phi : V\Gamma \to VT; \alpha \mapsto \alpha \mid E.$$

LEMMA 1. *If the fibres of ϕ are all finite and the tree T is locally finite then Γ is quasi-isometric to T.*

Proof. We know that either $\mathrm{Im}\,\phi = VT$ or $\mathrm{Im}\,\phi$ is one of the natural bipartite blocks of T. For each vertex $\alpha \in \mathrm{Im}\,\phi$ choose a vertex $s(\alpha)$ from $\phi^{-1}(\alpha)$. If $\alpha \in VT$ is in $\mathrm{Im}\,\phi$ then let $t(\alpha) = \alpha$; otherwise let $t(\alpha)$ be some vertex in T adjacent to α. Define a map $\psi : VT \to V\Gamma$ by

$$\psi(\alpha) := \begin{cases} s(\alpha) & \text{if } \alpha \in \mathrm{Im}\,\phi, \\ s(t(\alpha)) & \text{if } \alpha \notin \mathrm{Im}\,\phi. \end{cases}$$

Now we just go through the conditions in the definition of a quasi-isometry one by one, showing that for each condition there is a choice of the relevant constants so that the condition is satisfied. Let $\alpha_1, \alpha_2 \in V\Gamma$ and $\beta_1, \beta_2 \in VT$.

(*a*) Because Γ is transitive and locally finite we know that G has only finitely many orbits on adjacent vertices. It follows now from $\phi(\alpha g) = \phi(\alpha) g$ that

$$d_T(\phi(\alpha_1), \phi(\alpha_2)) = d_T(\phi(\alpha_1 g), \phi(\alpha_2 g)),$$

for all $\alpha_1, \alpha_2 \in V\Gamma$ and $g \in G$. Therefore we can make the following definition:

$$a_1 := \max\{d_T(\phi(\alpha_1), \phi(\alpha_2)) \mid d_\Gamma(\alpha_1, \alpha_2) = 1\}.$$

Then, by the triangle inequality, we get

$$d_T(\phi(\alpha_1), \phi(\alpha_2)) \leqslant a_1 d_\Gamma(\alpha_1, \alpha_2).$$

(*b*) Let us assume that ϕ is not onto. Define

$$b_1 := \max\{d_\Gamma(\nu, \mu) \mid \nu \in \phi^{-1}(\beta_1), \mu \in \phi^{-1}(\beta_2) \text{ and } d_T(\beta_1, \beta_2) = 2\}.$$

Here we use that T is locally finite and that the fibres of ϕ all have the same finite diameter. If $\beta_1, \beta_2 \in \mathrm{Im}\,\phi$ then we get

$$d_\Gamma(\psi(\beta_1), \psi(\beta_2)) \leqslant b_1 d_T(\beta_1, \beta_2).$$

By the way we defined the function t we have

$$d_T(t(\beta_1), t(\beta_2)) \leqslant d_T(\beta_1, \beta_2) + 2.$$

Therefore we have in general

$$d_\Gamma(\psi(\beta_1), \psi(\beta_2)) \leqslant b_1 d_T(\beta_1, \beta_2) + 2b_1.$$

(*c*) Let $\gamma \in \mathrm{Im}\,\phi$ and set

$$c := \mathrm{diam}\,(\phi^{-1}(\gamma)).$$

Now $\psi\phi(\alpha_1)$ is in the same ϕ-fibre as α_1. Because Γ is transitive, we then have

$$d_\Gamma(\psi\phi(\alpha_1), \alpha_1) \leqslant c.$$

(*d*) If $\beta_1 \in \mathrm{Im}\,\phi$ then $\phi\psi(\beta_1) = \beta_1$. If $\beta_1 \notin \mathrm{Im}\,\phi$ then $\phi\psi(\beta_1) = \phi\psi(t(\beta_1)) = t(\beta_1)$, and $d_T(\beta_1, t(\beta_1)) = 1$. Thus $d_T(\phi\psi(\beta_1), \beta_1) \leqslant 1$. ∎

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Under the above conditions Proposition 1 yields a natural bijection between the set $\mathscr{E}T$ and the set $\mathscr{E}\Gamma$. The reader can easily check that the injection from $\mathscr{E}T$ into $\mathscr{E}\Gamma$ given in lemma 2 in [2] is indeed a bijection.

2. *The results*

For the rest of this note we will assume that there is an end ω of Γ such that G_ω acts transitively on $V\Gamma$. By lemma 3 in [2] we know that ω appears as an end of T.

LEMMA 2. *The fibres of ϕ are finite.*

Before proving the lemma we must state a result of Watkins [5].

Definition 2. If L is a line and $\alpha, \beta \in VL$ then let $[\alpha, \beta]_L$ denote the path (in L) between α and β. If ϵ is an end of L then $[\alpha, \epsilon)_L$ denotes the half-line in L that starts at α and belongs to ϵ. A line L in Γ is called an *axis* if for all $\alpha, \beta \in VL$ we have $d_\Gamma(\alpha, \beta) = d_L(\alpha, \beta)$.

THEOREM 3 ([5], theorem 3·3). *Let Γ be a locally finite connected graph with more than one end. If ϵ_1 and ϵ_2 are two distinct ends of Γ then there is an axis L in Γ such that the ends of L are ϵ_1 and ϵ_2.*

Proof of Lemma 2. Let $\alpha \in V\Gamma$. We aim to show that there is a number n_0 such that if $d_\Gamma(\alpha, \beta) \geqslant n_0$ then there is an element $e \in E$ with $\alpha \in e$ but $\beta \notin e$. Because Γ is locally finite, this suffices to show that the fibres of ϕ are finite. For $e \in E$ define

$$\partial_v e := \{\gamma \in V\Gamma \mid \gamma \notin e \text{ but } \gamma \text{ adjacent to a vertex in } e\}.$$

Since G_ω is transitive we can find $e \in E$ such that $\alpha \in \partial_v e$ and $\omega \in e$. An application of theorem 1 in [1] gives us a hyperbolic element $g \in G_\omega$ with $\mathscr{D}(g^{-1}) = \omega$ and $\epsilon := \mathscr{D}(g) \in e^*$. Replacing g with a suitable power of g we can assume that $\partial_v e \subset eg$. By the theorem of Watkins above we can find an axis L in Γ with ends ϵ and ω.

Claim 1. Continue with the above notation and assume furthermore that $\alpha \in \partial_v e \cap L$. Then there is a number k_0 such that if $\beta \in L \cap e^*$ and $d_\Gamma(\alpha, \beta) \geqslant k_0$ then $\alpha \in eg$ and $\beta \in e^* g$.

We can find $\gamma \in \partial_v eg \cap L$ such that $[\gamma, \epsilon)_L \subset e^* g$. If $d_\Gamma(\alpha, \beta) \geqslant d_\Gamma(\alpha, \gamma)$ then $\alpha \in eg$ but $\beta \in e^* g$. This establishes Claim 1, with $k_0 := d(\alpha, \gamma)$.

Claim 2. Let e and L be as above. Then there is a number l_0 such that if $d_\Gamma(\beta, \partial_v e) \geqslant l_0$ and $h \in G_\omega$ such that $\beta \in Lh \cap e^*$ then $\epsilon h \in e^*$.

Assume that β and h satisfy the conditions in Claim 2. Put

$$l_0 := \tfrac{1}{2} \operatorname{diam}(\partial_v e) + 2.$$

Take β and h as above. Suppose that $\epsilon h \in e$. Then we can find $\gamma_1, \gamma_2 \in \partial_v e^*$ such that $[\gamma_1, \omega)_{Lh} \subset e$ and $[\gamma_2, \epsilon)_{Lh} \subset e$. Of course

$$d_\Gamma(\gamma_1, \gamma_2) \leqslant \operatorname{diam}(\partial_v e^*) \leqslant \operatorname{diam}(\partial_v e) + 2,$$

but, at the same time, we have

$$d_\Gamma(\gamma_1, \gamma_2) = d_{Lh}(\gamma_1, \gamma_2) \geqslant 2 d_\Gamma(\beta, \partial_v e).$$

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Thus it is impossible that $\epsilon h \in e$ and therefore $\epsilon h \in e^*$, so Claim 2 is now proved.

Clearly we can now choose n_0 such that if $d(\alpha, \beta) \geqslant n_0$ with $\beta \in e^*$ then $d(\beta, \partial_v e) \geqslant \max\{l_0, k_0\}$. Take $h \in G_\omega$ such that β lies on Lh. Now we can apply Claim 1 with L replaced by Lh and g replaced by $h^{-1}gh$. Note that the value of k_0 is not changed when g is replaced by $h^{-1}gh$. Thus the lemma is established. ∎

LEMMA 3. *The tree T is locally finite.*

Proof. Let $\alpha \in VT$. Define $u(\alpha)$ as the vertex in T adjacent to α such that the half-line originating at α and belonging to ω goes through $u(\alpha)$. If β_1, β_2 are in $\operatorname{Im}\phi$ with $d(\alpha, \beta_1) = d(\alpha, \beta_2)$ and $d(u(\alpha), \beta_1) = d(u(\alpha), \beta_2) = d(\alpha, \beta_1) + 1$ then every element in G_ω moving β_1 to β_2 must fix α. Let $U(\alpha)$ be the set of vertices γ in T such that the α, γ geodesic passes through $u(\alpha)$. Because G_ω acts transitively on $\operatorname{Im}\phi$ we can conclude that $G_{\alpha,\omega}$ acts transitively on the set

$$\{\beta \in T \mid d_T(\alpha, \beta) = i\} \setminus U(\alpha),$$

for all $i \geqslant 1$.

The group $G_{\alpha,\omega}$ fixes the half-line starting at α and going into ω. Thus $G_{\alpha,\omega}$ must fix some vertex belonging to $\operatorname{Im}\phi$, and therefore $G_{\alpha,\omega}$ must leave some non-empty finite set of $V\Gamma$ invariant. Hence all orbits on $V\Gamma$, and therefore also on VT, are finite. It follows that T is locally finite. ∎

Proof of Theorem 1. Theorem 1 follows straight from Lemmas 1, 2 and 3. ∎

Proof of Theorem 2. The argument given in the proof of Lemma 3 indeed proves more. The group G is closed in the permutation group topology given by the action of G on VT. Thus we see that G_ω must act transitively on $\mathscr{E}T \setminus \{\omega\}$. The result then follows from Proposition 1. ∎

If the end ω is not fixed by G then the classification given in theorem 10 in [2] applies. We can also give the following characterization in the same spirit.

THEOREM 4. *Let Γ be a locally finite connected graph with infinitely many ends. Put $G := \operatorname{Aut}(\Gamma)$. Assume that there is an end ω of Γ such that G_ω acts transitively on $V\Gamma$. Then there is a G-congruence π with finite classes on $V\Gamma$ such that Γ/π is a poly-orbital graph arising from the action of a transitive group H on a locally finite distance-transitive graph Γ_0. We can also find an end ω_0 in Γ_0 such that H_{ω_0} is transitive on $V\Gamma_0$.*

This theorem can also be formulated in terms of group actions on semi-regular trees. The problem of obtaining a more precise classification thus comes down to classifying those transitive groups acting on distance-transitive graphs and fixing an end. This seems to be a very hard problem.

This work forms a part of a D.Phil. thesis written at Oxford University under the supervision of Peter M. Neumann. The author would like to thank him for many helpful suggestions. He also wants to express his gratitude for financial support received under the ORS and FCO awa d schemes.

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