A CHARACTERIZATION OF FRONTIER

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Let X be a set and let f be a mapping of $P(X)$, the set of all subsets of X, into $P(X)$. Conditions are given which are necessary and sufficient for there to exist a topology on the set X with respect to which, for all $S \in P(X)$, the frontier of S is fS.

We consider a topology on a set X to be a closure operation on X, i.e. a mapping k of $P(X)$ into $P(X)$ such that, for all S, $T \in P(X)$, (i) $k\phi = \phi$ (where ϕ is the empty subset of X), (ii) $kS \supset S$, (iii) $kkS = kS$ and (iv) $k(S \cup T) = k(S) \cup k(T)$.

For each $S \in P(X)$, we put $fS = (kS) \cap (kcS)$ (where cS is the complement of S in X) and call fS the frontier of the set S (relative to the topology k on X). f is a mapping of $P(X)$ into $P(X)$, and it is easy to verify that the properties I–V given below hold for all $S, T \in P(X)$.

I. $f\phi = \phi$.

II. $fcS = fS$.

III. $ffS \subset fS$.

IV. $f(S \cup T) \subset f(S) \cup f(T)$.

V. If $S \subset T$ then $fS \subset T \cup fT$.

Further, the mapping f of $P(X)$ into $P(X)$ determines the closure operation k, since, for all $S \in P(X)$, $kS = S \cup fS$.

Theorem. *Let X be a set and let f be a mapping of $P(X)$ into $P(X)$ such that the properties* I–V* *hold for all $S, T \in P(X)$. Then there is a topology on X with respect to which, for all $S \in P(X)$, the frontier of S is fS.*

Proof. We show that (i) the mapping k of $P(X)$ into $P(X)$ which maps S into $kS = S \cup fS$ is a closure operation on X and (ii) the frontier of S relative to the topology k on X is fS, i.e. $(kS) \cap (kcS) = fS$ for all $S \in P(X)$.

Proof of (i):

$$k\phi = \phi \cup f\phi = \phi. \quad kS = S \cup fS \supset S \quad \text{for all} \quad S \in P(X).$$

$$kkS \supset kS \quad \text{and} \quad kkS = (kS) \cup (fkS) = (S \cup fS) \cup f(S \cup fS) \subset S \cup fS \cup fS \cup ffS$$

$$= S \cup fS = kS,$$

i.e.
$$kkS = kS \quad \text{for all } S \in P(X).$$

$$k(S \cup T) = (S \cup T) \cup f(S \cup T) \subset S \cup T \cup fS \cup fT = (kS) \cup (kT)$$

* It is not difficult to show that no four of the properties I–V entails the remaining property.

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and

$$S \subset S \cup T, \quad S \cup fS \subset (S \cup T) \cup f(S \cup T), \quad kS \subset k(S \cup T), \quad (kS) \cup (kT) \subset k(S \cup T),$$

i.e. $$k(S \cup T) = k(S) \cup k(T) \quad \text{for all} \quad S, T \in P(X).$$

Proof of (ii):

$$(kS) \cap (kcS) = (S \cup fS) \cap ((cS) \cup (fcS)) = (S \cup fS) \cap ((cS) \cup (fS)) = fS \quad \text{for all} \quad S \in P(X).$$

This completes the proof of the theorem.

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A NOTE ON LINEAR DERIVATES OF MEASURABLE FUNCTIONS

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By the measure of a set of directions in the plane we mean the linear measure of the set of points P on a unit circle for which the direction from the centre to P belongs to the set.

The ray issuing from a point P in direction θ will be denoted by $P(\theta)$.

THEOREM. *Assuming the axiom of choice, we can construct a plane set E of measure zero with the property that at almost all points P of the plane, the set of directions θ such that P is a limit point of $EP(\theta)$ is of full outer measure, and so also is the set of directions θ such that $EP(\theta)$ is empty.*

Proof. Let R denote the set of all points in the plane at least one of whose coordinates is rational, and let M denote the complement of R; thus M is of full measure.

Let Ω denote the least ordinal with the power of the continuum, and let

$$(P_1, \Theta_1), (P_2, \Theta_2), \ldots, (P_\alpha, \Theta_\alpha), \ldots \quad (1 \leqslant \alpha < \Omega)$$

be a transfinite sequence consisting of all possible different pairs (P, Θ) with first element a point P of M and second element Θ a non-empty perfect set of directions.

By transfinite induction we shall define for each ordinal α $(1 \leqslant \alpha < \Omega)$ two distinct directions θ_α, θ'_α both in the set Θ_α, in such a way that

$$[P_\alpha(\theta_\alpha) + P_\alpha(\theta'_\alpha)] \sum_{1 \leqslant \beta < \alpha} [P_\beta(\theta_\beta) + P_\beta(\theta'_\beta)]\, R = 0. \tag{1}$$

We may choose as θ_1, θ'_1 any two distinct directions in Θ_1.

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