J. Plasma Physics (2006), *vol.* 72, *part* 3, *pp.* 409–418. © 2005 Cambridge University Press
doi:10.1017/S0022377805004125 First published online 7 December 2005 Printed in the United Kingdom

Nonlinear gain and threshold current variations in a periodically corrugated waveguide

B. SHOKRI[1,2] and S. HOUSHMANDYAR[1]

[1]Physics Department and Laser-Plasma Research Institute of Shahid Beheshti University, Tehran, Iran

[2]Institute for Studies in Theoretical Physics and Mathematics, PO Box 19395-5531, Tehran, Iran
(B-shokri@cc.sbu.ac.ir)

(Received 15 November 2004 and in revised form 15 June 2005; accepted 29 July 2005)

Abstract. A rectangular periodically corrugated waveguide is considered in the exponential gain regime. In general, gain obeys a nonlinear cubic equation. In this paper it is solved numerically. Also the behavior of gain diagrams versus beam velocities for various parameters of the problem is surveyed. Furthermore, in the context of a slow-wave periodic structure oscillator, the threshold current which cause the system to be self-sustained, is obtained from a nonlinear cubic equation similar to the one obtained for gain, and the variation of its diagrams with other parameters is also considered.

1. Introduction

Slow-wave periodic structures such as periodically corrugated waveguides and Smith–Purcell free-electron lasers were first found experimentally and introduced by Smith and Purcell [1], where they showed that electromagnetic (EM) radiation can be produced by an electron beam skimming over a periodic structure. This periodic structure is a metallic corrugated surface with a spatial periodicity L and corrugation height a which is usually called a grating. This problem was solved by a model based on induced image charges; this model is still being used and reported [2–5]. After that various methods were presented in order to justify this phenomenon [6–8], but the first attempt to calculate the total power exchange between the EM wave and an electron beam skimming over an open corrugated surface was reported by Watchel based on kinetic theory and the Vlasov equation [9]. Later Shachter and Ron [10, 11] viewed this problem based on the dynamics of ensemble equations and a simplification using a reflection matrix formulation [12–15]. They released some of the constraints in Watchel's analysis, namely that the electron beam is not assumed to be very thin. It is also assumed to be at a finite height above the grating and finally the coupling between the interaction harmonics and the radiating ones is not introduced as a parameter of the problem. Recently, such a structure has been used for wakefield production [16]. The periodically corrugated waveguide amplifier (PCWA) and periodically

https://doi.org/10.1017/S0022377805004125 Published online by Cambridge University Press

corrugated waveguide oscillator (PCWO) are the main focus in this work. A PCWA is composed of a corrugated surface, an electron beam with an average velocity of V_0 parallel to the surface and an EM plane wave which is scattered by the grating into a wide spatial spectrum of propagating and evanescent waves. Assuming that the beam is cold, there is only a single evanescent wave which effectively exchanges energy with it; this is the synchronous wave (SW). On the other hand, PCWO has another component compared with the PCWA case, as well as a feedback system, which returns part of the scattered waves back to the grating for intensification. In this case, it is possible to formulate the boundary problem in terms of a reflection matrix which couples the amplitude of the outgoing and incoming waves. This formulation is convenient in a periodic structure since the component of the wave vector parallel to the grating can be decomposed and is conserved in the scattering process. This means that the periodic structure, or in other words the reflection matrix, couples between all the different harmonics. In the present work, in obtaining the equation for the gain and the threshold current for different beam-grating parameters, and treating the condition for the amplified incident wave to overcome the losses, we obtain the gain and the threshold current, numerically.

The present work has four sections. First, we introduce the formulation in Sec. 2. Then we will dedicate Sec. 3 to the gain and the threshold current. In Sec. 4 we finally summarize the results of this study.

2. General formulation

Let us consider a metallic grating with its surface in the XY plane, with periodicity L in the X direction and with the Y axis parallel to the grooves which are uniform in this direction (Fig. 1). The peak of the grating is taken to be in the $z = 0$ plane. The grating is assumed to be very long so that spatial transients, which decay after a few periods, have a negligible effect. An electron beam is sent through the system at height h above the plane of $z = 0$ with an average velocity of $\mathbf{V} = \mathbf{V}_0\mathbf{x}$, a density of n_0 and a current density of $\mathbf{J} = \mathbf{J}_x\mathbf{x}$. A monochromatic electromagnetic plane wave incident upon the grating at angle θ and a time dependence of $e^{i\omega t}$, is described by the y component of the magnetic field as

$$H_y^{\text{inc}}(x, z, \omega) = H_0 \exp\left[i\frac{\omega}{c}(z\cos\theta - x\sin\theta)\right], \tag{1}$$

where H_0 is the amplitude of the magnetic field. The total EM field due to the presence of the current will satisfy the inhomogeneous wave equation

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) H_y(x, z, \omega) = -\frac{\partial}{\partial z} J_x(x, z, \omega). \tag{2}$$

By considering the incident and scattered wave as the combination of different harmonics by the Floquet decomposition of the wave vector along the X direction (Fig. 1), one can obtain the reflection matrix [17–21].

It can be shown that, instead of all harmonics, only one harmonic named the Mth harmonic, is important and it is named the synchronous harmonic (SH) or synchronous wave (SW) [10].

https://doi.org/10.1017/S0022377805004125 Published online by Cambridge University Press

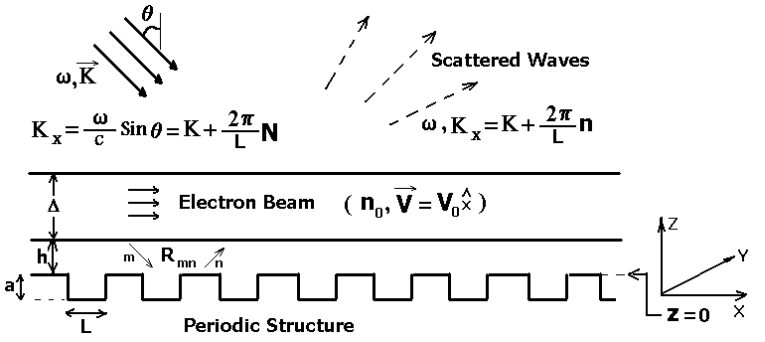


Figure 1. The basic configuration of a periodically corrugated waveguide.

3. Gain and threshold current

Let us consider Fig. 1. Bearing in mind the solution of (2) in regions (a) between the grating and the beam ($0 < z < h$), (b) within the beam ($h \leqslant z \leqslant h + \Delta$) and (c) above the beam $z > h+\Delta$, and using the continuity of the tangential component of the EM field at $z = h$ and $z = h+\Delta$, and modeling the grating as a dielectric slab on a conductor, we will obtain the dispersion relation of the amplifier, with its poles giving the gain [10]. Since only the Mth harmonic is important, its wave vector can be considered with a perturbation factor q in the presence of the beam as

$$k_M = k_e + q, \quad |q| \ll \frac{\omega}{V_0}, \tag{3}$$

where $k_n = k + (2\pi/L)n$ is the Floquet decomposition of wave vector along the x direction, n is an integer, $2\pi/L$ is the wave vector of the grating and k is defined in the first Brillouin zone ($|k| < \pi/L$).

Making use of the dispersion [10], with some algebraic calculations we finally find

$$q^3 = -\frac{ne^2\omega}{\epsilon_0 mc^2\beta^3\gamma^5} \exp\left(-\frac{2\omega h}{\beta c\gamma}\right) \frac{\sinh(\tau\Delta\omega/\beta c\gamma)}{\tau\cosh(\tau\Delta\omega/\beta c\gamma) + \frac{1}{2}(1+\tau^2)\sinh(\tau\Delta\omega/\beta c\gamma)} \bar{Q}, \tag{4}$$

where γ is the relativistic factor, m is the electron mass, ϵ_0 is the vacuum permittivity, c is the light speed and

$$\tau = \left(1 - \frac{ne^2}{\epsilon_0 mc^2\beta^3\gamma^3 q^2}\right)^{1/2}, \quad \beta = \sqrt{k_n^2 - (\omega/c)^2}.$$

Furthermore,

$$\bar{Q} = \left[1 + \frac{1}{\gamma^2(\varepsilon\beta^2 - 1)} + \frac{\omega}{c} d \frac{1}{\varepsilon\beta\gamma^3} \frac{\epsilon^2 + \gamma^2(\epsilon\beta^2 - 1)}{(\epsilon\beta^2 - 1)^{3/2}}\right]^{-1},$$

depending on the beam–grating parameters, is a number between zero and one. This nonlinear and cubic equation is the equation describing the variation of the propagation constant or in other words, the dispersion relation which is dependent on the interaction of the beam and the wave. It is the positive imaginary part of q that gives us the gain per unit length.

https://doi.org/10.1017/S0022377805004125 Published online by Cambridge University Press

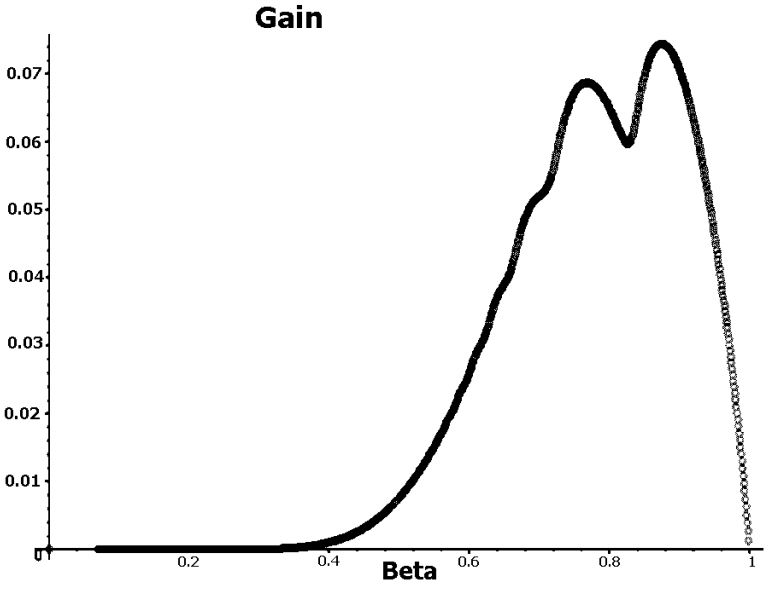


Figure 2. Gain diagram.

3.1. Gain

In the case of a dilute beam, (4) will be cast into a simple and linear equation, which can be solved analytically in the strong coupling regime [10]. Since (4) is a nonlinear complicated equation we are going to solve it numerically using the iteration method. The amazing point is that for any primary quantity chosen for q, the final result after several iteration cycles will be the same. Another point is the selection of signs for the imaginary and real parts of q; this means that if we choose both parts with the same sign, the final result for q will be positive for both signs of real and imaginary parts of q. Otherwise the imaginary part of the final q will be negative.

If we plot the behavior of the gain for various beam velocities β, then Fig. 2 is obtained. The most interesting thing in this figure is the two peaks which are almost equal, and the dip between them. Now if we vary other parameters of the beam such as its thickness Δ, Fig. 3 is obtained. Similar to the atomic level saturation in conventional lasers, we find a resemblance to the 'Lamb-dip' property [22] in which with over-pumping at a specified frequency, the peak of the gain diagram versus frequency, falls down and makes a dip.

The parameters used in Figs 2 and 3 are: $I = 10$ mA, $\Delta_y = 0.1$ m and $\Delta = 7.4 \times 10^{-5}$ m, where I is the fixed current intensity and Δ_y is the beam width in the Y direction.

3.2. Oscillations

Figure 4 is a structure which is usually used to demonstrate periodically corrugated waveguide oscillators. With the presence of mirrors, a feedback system of optical length $d_{\rm opt}$, returns this wave back towards the grating with its phase shifted by an amount $\exp[-i(\omega/c)d_{\rm opt}]$. Mirror losses and energy extraction from the oscillator decrease the field amplitude. We denote these losses by r. The total reflection coefficient of the feedback system can be introduced as $r\exp[-i(\omega/c)d_{\rm opt}] = \exp(-\Gamma)$,

https://doi.org/10.1017/S0022377805004125 Published online by Cambridge University Press

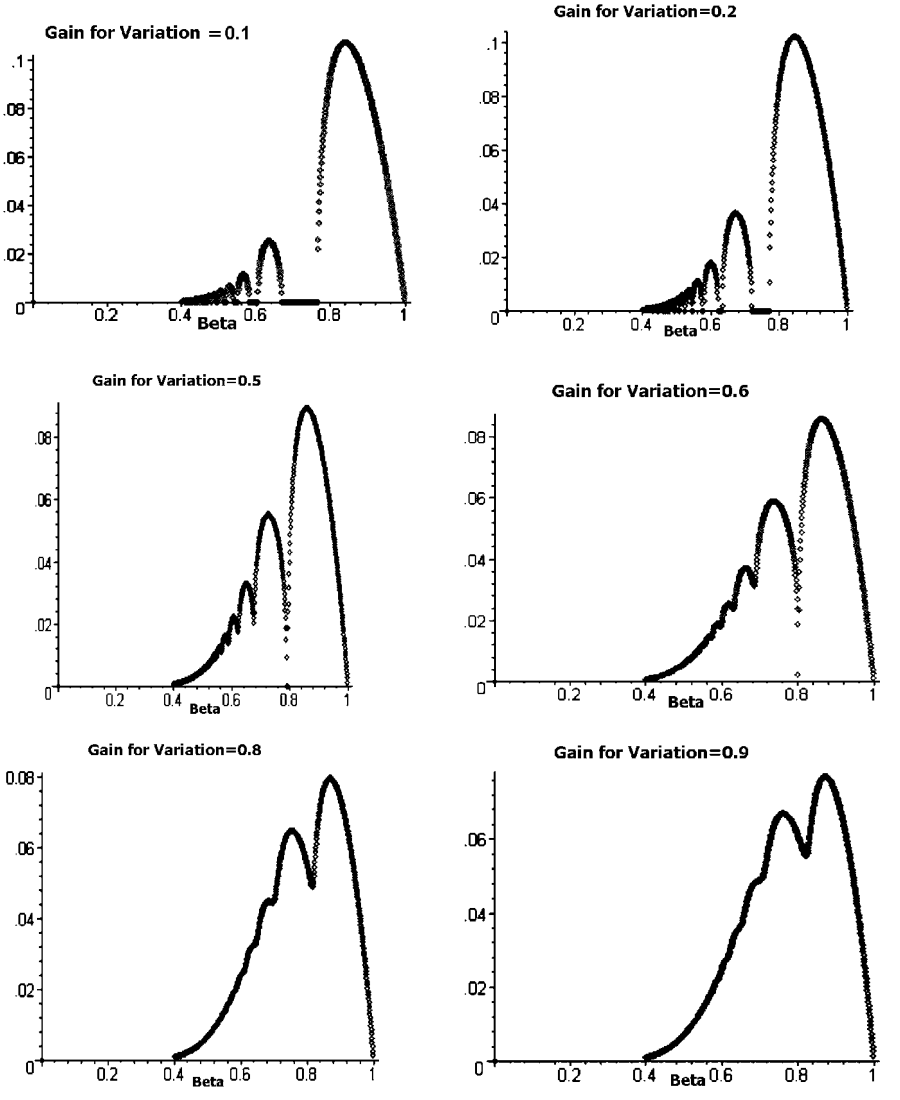


Figure 3. Gain diagrams for various widths of electron beam.

where Γ is a complex quantity. Here, we use a method which is common in studying oscillators. We consider the imaginary part of the frequency $\omega_{\rm i}$, to be $\omega_{\rm i} = \omega_{\rm i}^{+} + \omega_{\rm i}^{-}$ where $\omega_{\rm i}^{-}$ is a part which comes from different loss mechanisms and causes a decrease in the field amplitude, while $\omega_{\rm i}^{+}$ is due to the extraction of kinetic energy from the beam and may cause an increase in the field amplitude [10]. It is common that each of $\omega_{\rm i}^{+}$ and $\omega_{\rm i}^{-}$ is calculated ignoring the other. Before, we found that the solution of (4) gave us the gain, or in other words energy extraction. Therefore, we use this equation to calculate $\omega_{\rm i}^{+}$. In this case we should replace q with (Ω/c) and $k_{\rm e}$ with (ω_0/v_0), and as a result we obtain

$$\Omega^3 = \frac{\omega_{\rm p}^2}{\beta^4\gamma^4}\omega_0 e^{-2\eta}\frac{\sinh(\tau\delta)}{\tau\cosh(\tau\delta) + \frac{1}{2}(1+\tau^2)\sinh(\tau\delta)}\bar{Q}, \tag{5}$$

https://doi.org/10.1017/S0022377805004125 Published online by Cambridge University Press

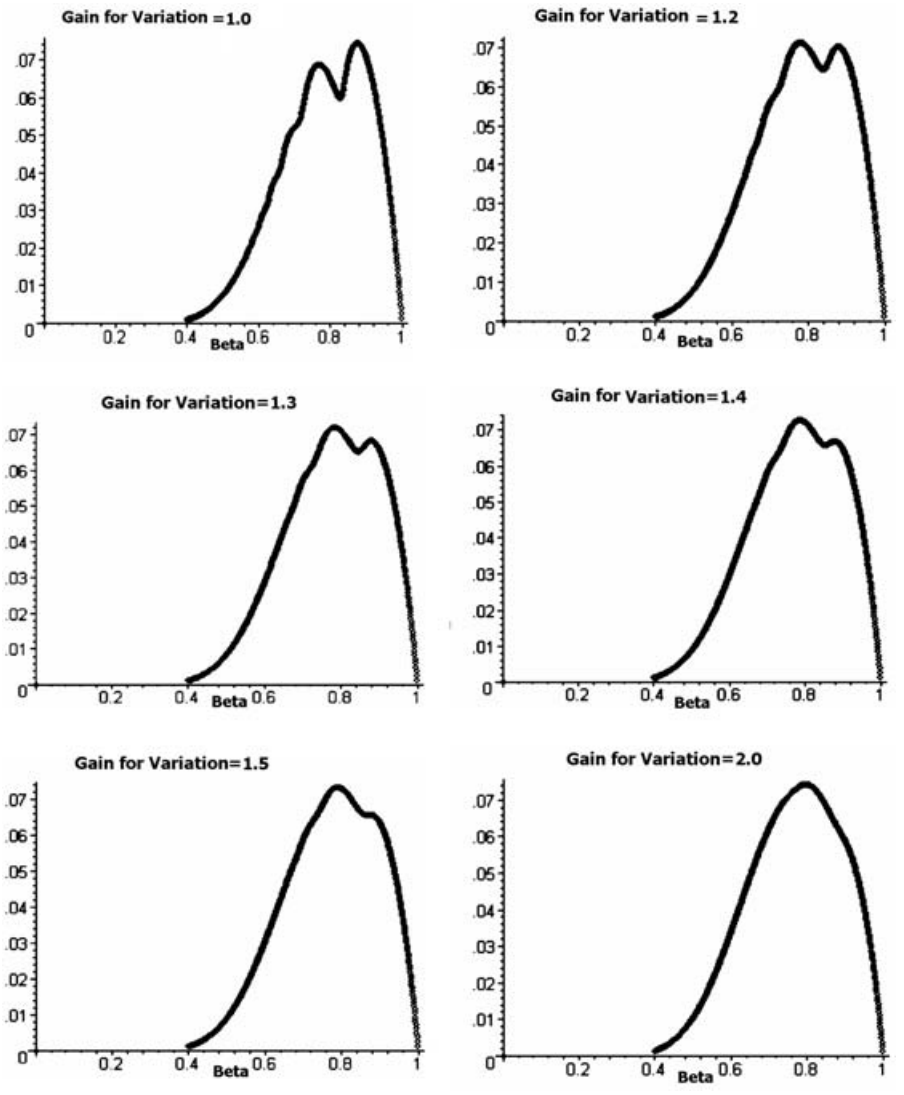


Figure 3. Continued.

where

$$\Omega = \omega - \omega_0, \quad \tau = [1 - \omega_p^2/(\beta^3\gamma^2\Omega^2)]^{1/2},$$

and Ω is a negative parameter. The analytical solution of this equation can be obtained in the strong coupling regime ($\omega_p \ll \Omega$). Here, solving the aforementioned nonlinear equation numerically, we try to find its nonlinear results. But before that, let us add some more information to our problem.

If we ignore the beam momentarily, we can define three different mechanisms for losses: (a) radiation losses by scattering at the periodic structure, (b) losses due to the finite conductivity of the metallic surfaces of mirrors and the grating, and (c) radiation losses due to extraction of energy through one of the mirrors. As we are considering threshold conditions, we can assume that no energy is extracted out

https://doi.org/10.1017/S0022377805004125 Published online by Cambridge University Press

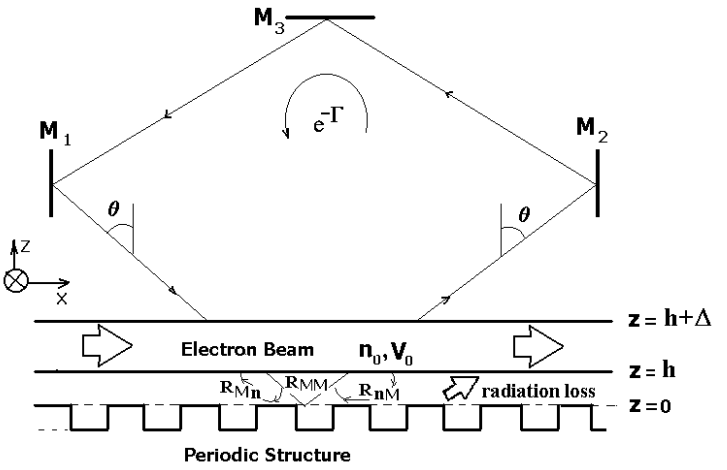


Figure 4. The configuration for a high-frequency oscillator. The role of different R terms of the oscillator dispersion relation is illustrated.

of the oscillator. Also in contrast between the other two, the loss due to scattering from the surface is more important than the others. This loss can be written as

$$P_{\mathrm{r,loss}} = P^{\mathrm{inc}} - P^{\mathrm{sc}}, \tag{6}$$

where P^{inc} and P^{sc} are the losses due to the incident and the scattered field, respectively. So, the decay rate in the small loss regime is estimated by

$$2/\tau_{\mathrm{loss}} = P_{\mathrm{r,loss}}/W_{\mathrm{em}},$$

where W_{em} is the total energy stored in the oscillator. Therefore, the necessary condition for self-sustained oscillations will be found to be

$$\omega_{\mathrm{i}}^{+} \geqslant \omega_{\mathrm{i}}^{-}, \tag{7}$$

where

$$\omega_{\mathrm{i}}^{-} = 1/\tau_{\mathrm{loss}}.$$

This relation in turn, gives us the threshold current. So we use the following relations:

$$\mathrm{Im}(f_2) = \mathrm{Im}(\Omega) = \alpha, \quad \alpha = \omega_{\mathrm{i}}^{-}, \quad \mathrm{Re}(f_2) = Re(\Omega), \tag{8}$$

where with some algebraic calculations we find

$$f_2 = \frac{I}{\epsilon_0 m\beta^4\gamma^5} \exp\left(-\frac{2\omega h}{\beta c\gamma}\right) \frac{\sinh(\tau\Delta\omega/\beta c\gamma)}{\tau\cosh(\tau\Delta\omega/\beta c\gamma) + \frac{1}{2}(1+\tau^2)\sinh(\tau\Delta\omega/\beta c\gamma)} \bar{Q}, \tag{9}$$

$$\tau = \left(1 - \frac{I}{\epsilon_0 m\beta^3\gamma^3\Omega^2}\right)^{1/2}.$$

Bearing in mind (7), the current I in (9) gives us the threshold current.

In solving (9), the role of ω is taken to be as a constant because with the presence of a feedback system at a specified angle the frequency and ω will be fixed. Moreover, the roles of all fixed parameters and their dependence on velocity, are separated.

https://doi.org/10.1017/S0022377805004125 Published online by Cambridge University Press

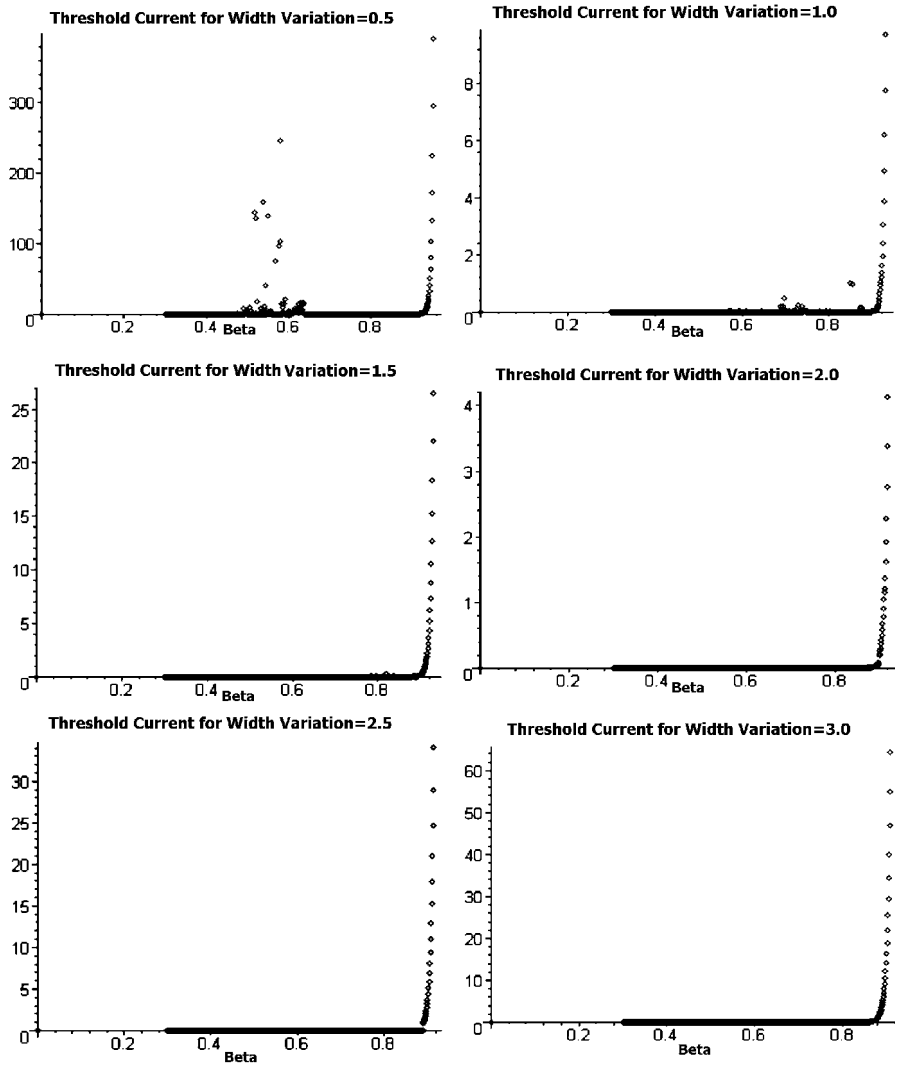


Figure 5. Diagrams of the threshold current for various width parameters.

In Fig. 5 the variation of the threshold current versus different velocities for different beam widths are shown, where we have used the following data:

$$\Delta_y = 10^{-2} \text{ m}, \quad \Delta = 10^{-4} \text{ m}.$$

As an example of this structure, we find $I_{\text{th}} = 0.17 \times 10^{-3}$ A.

4. Summary and conclusion

In the present study we have investigated the energy exchange between electrons and EM waves in the presence of a periodic structure. Here we have concentrated on the interaction of an electron beam with EM waves in an open periodic structure, i.e. in a periodically corrugated waveguide system. This will open up the way to tunable radiation sources at shorter wavelengths, from millimeters to soft X-rays.

https://doi.org/10.1017/S0022377805004125 Published online by Cambridge University Press

In this work, it was assumed that an ideal beam is injected into the system. The longitudinal momentum distribution is extremely packed (Dirac δ function) and the transverse electron motion is ignored.

We may summarize the main conclusions of the our present study of PCWs, as follows.

(1) The influence of the beam height h on the gain and the threshold current is shown to be considerable (an exponential factor).

(2) Here, considering previous work [10] mainly based on a reflection matrix formulation, we used a simple iteration method to calculate the gain and the threshold current. In our analysis of the gain, for various parameters such as the beam width Δ, we found a dip property in PCW. It resembles the Lamb dip in conventional lasers [21], although the latter effect, being in power versus frequency, takes place only in standing wave cavities. The Lamb dip claim may seem to be irrational since it is defined in frequency space, but here we are in velocity space. Our idea is an absolute equivalence of both with the following justification. For the monoenergetic beam used in this case, we have the dispersion relation $\omega = kV_0$, which verifies that there is a direct relation between frequency and velocity spaces. On the other hand, increasing the width of the beam equals a sum of sheet beams over each other, which of course cancels the effect of the dip, because the last dispersion relation is no longer valid and we have velocity diffusion there.

(3) The effect of increasing the beam thickness on the threshold current is seen in Fig. 5, where the point at which the curve goes to infinity, is pushed toward lower velocities.

References

[1] Smith, S. J. and Purcell, E. M. 1953 *Phys. Rev.* **92**, 1069.
[2] Doucas, G., Mulvey, J. H., Omori, M., Walsh, J. and Kimmitt, M. F. 1992 *Phys. Rev. Lett.* **69**, 1761.
[3] Brownell, J. H., Walsh, J. and Doucas, G. 1998 *Phys. Rev.* E **57**, 1075.
[4] Trotz, S. R., Brownell, J. H., Walsh, J. and Doucas, G. 2000 *Phys. Rev.* E. **61**, 7057.
[5] Doucas, G., Kimmitt, M. F., Doria, A., Gallerano, G. P., Giovenale, E., Messina, G., Andrews, H. L. and Brownell, J. H. 2002 *Phys. Rev. Spec. Topics Accl. Beam* **5**.
[6] Toraldo di Francia, G. 1984 *Nuovo Cimento* **16**, 61.
[7] Yariv, A. and Shih, C. C. 1978 *Opt. Commun.* **24**, 233.
[8] Gover, A. and Livni, Z. 1978 *Opt. Commun.* **26**, 375.
[9] Watchel, J. M. 1979 *J. Appl. Phys.* **50**, 49.
[10] Shachter, L. and Ron, A. 1989 *Phys. Rev.* A **40**, 876.
[11] Shachter, L. 1987 *J. Appl. Phys.* **61**, 2718.
[12] Petit, R. 1980 *Electromagnetic Theory of Gratings*. Berlin: Springer.
[13] Lippmann, B. A. 1953 *J. Opt. Soc. Am.* **43**, 408.
[14] Uretsky, J. L. 1965 *Ann. Phys., NY* **3**, 400.
[15] Waterman, P. C. 1975 *J. Acoust. Soc. Am.* **57**, 791.
[16] Mostacci, A. and Ruggiero, F. 2002 *Phys. Rev. Spec. Topics, Accel. Beams* **5**, 044401-1.
[17] Ishimaru, A. 1991 *Electromagnetic Wave Propagation, Radiation and Scattering*. New York: Prentice-Hall.

https://doi.org/10.1017/S0022377805004125 Published online by Cambridge University Press

[18] Van den Berg, P. M. 1971 *Appl. Sci. Res.* **24**, 261.
[19] Van den Berg, P. M. 1973 *J. Opt. Soc. Am.* **63**, 689.
[20] Van den Berg, P. M. 1973 *J. Opt. Soc. Am.* **63**, 1588.
[21] Van den Berg, P. M. and Tan, T. H. 1974 *J. Opt. Soc. Am.* **64**, 325.
[22] Milonni, P. W. and Eberly, J. H. 1988 *Lasers*. New York: Wiley.

https://doi.org/10.1017/S0022377805004125 Published online by Cambridge University Press