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ALMOST HIGHER ORDER STOCHASTIC DOMINANCE*

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Abstract. In this paper, we develop the concept of almost stochastic dominance for higher order preferences and investigate the related properties of this concept.

Keywords. Almost stochastic dominance, expected-utility maximization, higher-order preferences..

Mathematics Subject Classification. 90B50, 91B06.

1. Introduction

Leshno and Levy [1] develop the theory of almost stochastic dominance (ASD) as a relaxation of the stochastic dominance (SD). This theory plays an important role in several fields particularly in financial research. There are numerous applications based on this concept, see, *e.g.*, [2–5]. Lizyayev and Ruszczynski [6] propose a new almost stochastic dominance concept that is computationally tractable and enjoys many favourable features. Lizyayev and Ruszczynski [6] define the almost first and second order stochastic dominance. However, in the economic and financial literature higher-order preferences are believed to be important, see, *e.g.*, [7,8]. In this paper, we aim to extend Lizyayev and Ruszczynski's work [6] to higher order and study the related properties of the almost higher order stochastic

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dominance. For completeness of the presentation, we next introduce Lizyayev and Ruszczynski's Almost Stochastic Dominance concept [6].

Random variables, denoted by X and Y, defined on $\Omega = [a, b]$ are considered together with their corresponding distribution functions F and G, and their corresponding probability density functions f and g, respectively. The following notations will be used throughout this paper: $\mu_F = \mu_X = E(X) = \int_a^b t\, d\, F(t), \mu_G = \mu_Y = E(Y) = \int_a^b t\, d\, G(t), H^0(x) = h(x)$, where $h = f$ or g and $H = F$ or G. In addition, we define

$$H^{(j)}(x) = \int_a^x H^{(j-1)}(y)\, \mathrm{d}y, \quad \text{for} \quad H = F \quad \text{or} \quad G,$$

Lizyayev and Ruszczynski [6] give the following definition for almost second order stochastic dominance.

Definition 1.1. For $0 \leq \epsilon < 1/2$,

ϵ-ASSD: X is said to dominate Y by ϵ-ASSD, denoted by $X \succeq_2^{almost(\epsilon)} Y$, if and only if $E(X) \geq E(Y)$ and

$$F^{(2)}(t) - G^{(2)}(t) \leq \epsilon, \quad \forall t \in \Omega.$$

Inspired by the above definition and notice the fact that $E(X) - E(Y) = G^{(2)}(b) - F^{(2)}(b)$, we develop the concept of almost stochastic dominance for higher order preferences as follows:

Definition 1.2. For $0 \leq \epsilon < 1/2$,

ϵ-AkSD: X is said to dominate Y by ϵ-AkSD, denoted by $X \succeq_k^{almost(\epsilon)} Y$, if and only if $G^{(n)}(b) \geq F^{(n)}(b), n = 2, 3, \ldots, k, k \geq 3$ and

$$F^{(k)}(t) - G^{(k)}(t) \leq \epsilon, \quad \forall t \in \Omega. \tag{1.1}$$

Note that if ϵ is taken to be zero, then we return to the classical kth order stochastic dominance(kSD) concept, see [2] for more details.

2. Main results

We are now ready to present the main results related to the properties of the almost higher order stochastic dominance defined above.

Theorem 2.1. *If X dominates Y by ϵ-AkSD $\left(X \succeq_k^{almost(\epsilon)} Y\right)$, there exists a nonnegative random variables Z such that $E(Z^{k-1}) \leq \epsilon(k-1)!$ and $X + Z$ dominates Y by kSD, here $k \geq 3$.*

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Proof. From Proposition 1 in Ogryczak and Ruszczynski [9], we know that

$$F^{(k)}(\eta) = \frac{1}{(k-1)!}\int_{-\infty}^{\eta}(\eta - x)^{k-1}\mathrm{d}F(x) = \frac{1}{(k-1)!}E(\eta - X)_+^{k-1}$$

here the function $t \mapsto (t)_+ = \max(0, t)$ and $k \geq 3$.

When X is said to dominate Y by ϵ-AkSD, we can have

$$E(\eta - X)_+^{k-1} \leq E(\eta - Y)_+^{k-1} + \epsilon(k-1)!, \quad \forall \eta \in R.$$

Let d be such that $E(d - X)_+^{k-1} = \epsilon(k-1)!$. Defining $Z = (d - X)_+$, we can see that $X + Z = \max(d, X)$ and $\eta - (X + Z) = \eta - \max(d, X)$.

If $\eta \leq d$, then $(\eta - (X + Z))_+ = (\eta - \max(d, X))_+ = 0$ and thus

$$E(\eta - (X + Z))_+^{k-1} \leq E(\eta - Y)_+^{k-1},$$

and the Theorem holds.

Now we turn to consider the case with $\eta > d$, in this case, we can have

$$(\eta - \max(d, X))_+ = \begin{cases} (\eta - X)_+ & \text{if } X \geq d, \\ (\eta - X)_+ - (d - X) & \text{if } X < d. \end{cases}$$

As a result, we can have

$$\begin{aligned} E(\eta - (X + Z))_+^{k-1} &= \int_d^{\infty}(\eta - X)_+^{k-1}\mathrm{d}F(x) + \int_{-\infty}^{d}[(\eta - X)_+ - (d - X)]^{k-1}\mathrm{d}F(x) \\ &\leq \int_d^{\infty}(\eta - X)_+^{k-1}\mathrm{d}F(x) + \int_{-\infty}^{d}(\eta - X)_+^{k-1}\mathrm{d}F(x) \\ &\quad - \int_{-\infty}^{d}(d - X)^{k-1}\mathrm{d}F(x) \\ &= E(\eta - X)_+^{k-1} - \epsilon(k-1)! \leq E(\eta - Y)_+^{k-1}. \end{aligned}$$

The first inequality follows from the fact that for $k \geq 2$, $(d_1 - d_2)^k \leq {d_1}^k - {d_2}^k$ when $0 < d_2 < d_1$. The proof is finished. □

The above Theorem provides a characteristic of almost higher order stochastic dominance. It links the almost stochastic dominance concept with the traditional stochastic dominance. Besides, it gives an interesting interpretation of the value of ϵ. To be precise, it is the smallest value of the $k-1$ order moment of a random variable over $(k-1)!$ that needs to be added to a random variable X in order for it to dominate a given benchmark Y.

Below, we prove an equivalent ϵ-AkSD formulation in terms of utility functions. We first define the utility function set U_k which represents high order preferences. It's defined as follows:

$$U_k = \{u : (-1)^i u^{(i)} \leq 0\,,\ i = 1, \ldots, k\}$$

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where $u^{(i)}$ is the ith derivative of the utility function u. Since scaling of a utility function does not change the optimal portfolio that maximizes the expected value of that function, we may without loss of generality restrict the set U_k to the following set:

$$\tilde{U}_k = \{u \in U_k : (-1)^k u^{(k-1)}(t) \leq 1\} \tag{2.1}$$

here indeed, any function $u \in U_k$ defined on $\Omega = [a, b]$, can be substituted with $\tilde{u}(t) = u(t)/u^{(k-1)}(a)$ which will preserve the optimal solution.

Theorem 2.2. *Under the condition that $G^{(n)}(b) \geq F^{(n)}(b), n = 2, 3, \ldots, k, k \geq 3$, a random variable X ϵ-AkSD dominates a random variable Y if and only if*

$$E[u(X)] + \epsilon(k-1)! \geq E[u(Y)], \quad \forall u \in \tilde{U}_k.$$

Proof. We first consider the if part. For a fixed $\eta \in R$, define the following utility function

$$u_\eta(t) = -(\eta - t)_+^{k-1}, \quad t \in R.$$

From Definition 1.2, we can know that the ϵ-AkSD dominance is equivalent to the relation:

$$\begin{aligned} E[u_\eta(X)] + \epsilon(k-1)! &= -E[(\eta - X)_+^{k-1}] + \epsilon(k-1)! \\ &\geq -E[(\eta - X)_+^{k-1}] \\ &= E[u_\eta(Y)]. \end{aligned} \tag{2.2}$$

As $u_\eta \in \tilde{U}_k$, thus the sufficiency is proved.

To prove the necessity, consider an arbitrary $u \in \tilde{U}_k$. For every $\delta > 0$, we can find a finite collection of numbers η_l and $\alpha_l \geq 0, l = 1, \ldots, L$, and a constant c such that $\sum_{l=1}^L \alpha_l = 1$ and the function

$$\omega(t) = c + \sum_{l=1}^{L} \alpha_l u_{\eta_l}(t)$$

has the following properties:

$$\begin{aligned} E[|u(X) - \omega(X)|] \leq \delta, \\ E[|u(Y) - \omega(Y)|] \leq \delta. \end{aligned}$$

This collection can be constructed by a sufficiently accurate piecewise polynomial approximation of the function $u(\cdot)$. Since the α_l's are nonnegative and total 1, $\omega \in \tilde{U}_k$. Adding inequalities (2.2) multiplied by α_l for each $u_{\eta_l}(t)$, we obtain

$$E[\omega(X)] + \epsilon(k-1)! \geq E[\omega(Y)].$$

Then

$$E[u(X)] + \epsilon(k-1)! + 2\delta \geq E[u(Y)].$$

As $\delta > 0$ is arbitrary, the necessity is proved. □

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