From (1) and (2) we then have

$$mX^2 + Y^2 + 2qZ^2 \equiv (1 + 2q)(px + 2qy)^2 \equiv 0 \pmod{m}$$

$$mX^2 + Y^2 + 2qZ^2 \equiv (m + p^2)x^2 \equiv 0 \pmod{2q}.$$

Since $2q$ and m are relatively prime, $mX^2 + Y^2 + 2qZ^2 \equiv 0 \pmod{2qm}$ and so

$$mX^2 + Y^2 + 2qZ^2 = 2qm. \tag{3}$$

Now suppose the prime power factorisation of $mX^2 + Y^2$ contained a prime r to an odd power, where $r \equiv 3 \pmod 4$. Let $X = gX_1$ and $Y = gY_1$ where g is the g.c.d. of X and Y. Then

$$r \neq q,\ mX_1^2 + Y_1^2 \equiv 0 \pmod{r},\ Z^2 \equiv m \pmod{r}. \tag{4}$$

If r is not a factor of m then $\left(\frac{-m}{r}\right) = \left(\frac{m}{r}\right) = +1$ and $r \equiv 1 \pmod 4$, a contradiction. If r is a factor of m then it is also a factor of Y and Z. Since m is square-free, (1) and (3) give $X^2 \equiv 2q \equiv -1 \pmod{r}$ and, again, $r \equiv 1 \pmod 4$.

Hence $m - Z^2 = \dfrac{mX^2 + Y^2}{2q}$ is a sum of two integer squares, as required.

References

1. C. F. Gauss, *Disquisitiones arithmeticae*, Yale Univ. Press, New Haven, Conn., and London (1966).
2. P. Pollack and P. Schorn, Dirichlet's proof of the three-square theorem, http://pollack.uga.edu/finding3squares-6.pdf
3. N. C. Ankeny, Sums of three squares, *Proc. Amer. Math. Soc.* **8** (1957) pp. 316-319.
4. S. W. Dolan, Nint reciprocity, *Math. Gaz.* **98** (July 2014), pp. 317-319.
5. S. W. Dolan, Primes of the form $x^2 + z^2$, *Math. Gaz.* **101** (November 2017) p. 465.
6. S. W. Dolan, A very simple proof of the two-squares theorem, Math. Gaz. 105 (November 2021) p. 511.
7. H. Davenport, The geometry of numbers, *Math. Gaz.* **31** (October 1947) pp. 206-210.

STAN DOLAN
4 Orchard Close, Charmouth DT6 6RS
e-mail: *stan@standolan.co.uk*

10.1017/mag.2021.121
© The Mathematical Association 2021

105.40 A tale of two cubics

While dealing with quadratics one observes that if $x^2 + ax + b$ factorises into linear polynomials with integer coefficients for some choice of integers a and b then so does $x^2 - ax + b$ because

$$x^2 - ax + b = (-x)^2 + a(-x) + b$$

https://doi.org/10.1017/mag.2021.122 Published online by Cambridge University Press

and its roots differ from those of $x^2 + ax + b$ by a negative sign. But this is not always the case for two cubics of the form $x^3 + ax + b$ and $x^3 - ax + b$ with a and b integers. For instance,

$$x^3 - 2x + 1 = (x - 1)(x^2 + x - 1)$$

but $x^3 + 2x + 1$ cannot be reduced into a product of two non-constant polynomials with integer coefficients of smaller degree. To see why this is true, observe that if we could write this polynomial as a product of two non-constant polynomials with integer coefficients, then one of the polynomials must be linear, implying that $x^3 + 2x + 1$ has an integer root. Since the product of roots is -1, the integer root can be either 1 or -1. But at $x = \pm 1$, $x^3 + 2x + 1 \neq 0$.

However, as we shall see in this Note, one can construct infinitely many pairs (a, b) of integers with $ab \neq 0$ such that both $x^3 + ax + b$ and $x^3 - ax + b$ can be resolved into factors consisting of non-constant polynomials with integer coefficients. *

To this end, suppose

$$x^3 - ax + b = (x - u)(x^2 + tx + v), \tag{1}$$

and

$$x^3 + ax + b = (x - w)(x^2 + lx + m), \tag{2}$$

with u, t, v, w, l, m integers. Equate the coefficients to obtain $t = u$, $l = w$, $uv = mw = -b$ and $u^2 - v = m - w^2 = a$. From this we further obtain

$$w^3 + aw = u^3 - au \tag{3}$$

which leads to

$$(w - u)(w^2 + wu + u^2) + a(w + u) = 0. \tag{4}$$

Now observe that since $b \neq 0$, neither $u = 0$ nor $w = 0$. So, $w \neq u$ otherwise $a = 0$, contrary to our assumption. If $w + u = 0$ then $w^2 + wu + u^2 = 0$. But

$$w^2 + wu + u^2 = \left(w + \frac{u}{2}\right)^2 + \frac{3u^2}{4} = 0$$

implies $w = u = 0$ leading to $b = 0$. Therefore $w + u \neq 0$. Hence

$$a = \frac{(u - w)(u^2 + uw + w^2)}{u + w} = (u - w)\left(u + \frac{w^2}{u + w}\right).$$

Now, if we set $u + w = w^2$, then $a = w(w - 2)(w^2 - w + 1)$ and

$$m = w^2 + a = w(w - 1)(w^2 - 2w + 2),$$

whence $b = -w^2(w - 1)(w^2 - 2w + 2)$. Further, if $w \notin \{0, 1, 2\}$ then

* It may be noted that if $a = 0$ then by choosing $b = k^3$ where k is a non-zero integer we may obtain factorisations of $x^3 \pm ax + b$. On the other hand, if $b = 0$, then evidently $x^3 \pm ax + b = x(x^2 \pm a)$ yields factorisations of the two cubics.

https://doi.org/10.1017/mag.2021.122 Published online by Cambridge University Press

$a \neq 0$ and $b \neq 0$. Thus with

$$(a, b) = \left(w(w-2)\left(w^2 - w + 1\right), -w^2(w-1)\left(w^2 - 2w + 2\right)\right)$$

where w is an integer different from 0, 1 and 2, we obtain infinitely many cubics $x^3 + ax + b$ and $x^3 - ax + b$, both of which can be resolved into factors.

The cubics considered here have a special form, in that the coefficient of x^2 is zero. It may appear that such a choice was motivated by the desire to subject the quadratics $x^2 \pm ax + b$ to a minimal modification but the reader may note that any cubic of the form $x^3 + px^2 + qx + r$ may be reduced to a cubic of the form $y^3 + ky + l$ where k and l depend on p, q and r, by changing the variable from x to $y - \frac{1}{3}p$.

PRITHWIJIT DE
Homi Bhabha Centre for Science Education,
Tata Institute of Fundamental Research,
Mumbai - 400088, India

10.1017/mag.2021.122
© The Mathematical Association 2021

105.41 Topology haiku matrix

	Searching for holes and voids	Metrics	Fine and coarse
Sometimes metrisable sometimes not	In homotopy we can quotient with great ease, as I search for voids.	In the Kansas plains we do move like an eagle, straight lines are enough.	There are but two sets: one HUGE and the other small The coarsest of all.
Spaces and points	Square, disk, triangle each its own unique space but basically a point.	and, in Manhattan we move block by block driving our geodesic.	The discrete is the topology of open points: all sets clopen.
Counter-intuitive but true!	A wedding ring is essentially a doughnut, But NOT to my wife!	Changing our metric allows us to make circles that look just like squares.	Topologies can be extreme, but excellent counterexamples!

JOSH HILLER
Adelphi University, One South Avenue,
Garden City, NY, 11053 USA
e-mail: *johiller@adelphi.edu*

10.1017/mag.2021.123
© The Mathematical Association 2021

105.42 Generalised binomial theorem via Laplace transform technique

Introduction

The binomial theorem is an important result which appears frequently in different areas of mathematics. It states that

$$(1 + t)^n = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\, t^k,$$

where t is any real number and n is a non-negative integer.

https://doi.org/10.1017/mag.2021.122 Published online by Cambridge University Press