Ergod. Th. & Dynam. Sys., (2026), **46**, 2047–2075 © The Author(s), 2026. Published by Cambridge University Press.
doi:10.1017/etds.2026.10277

A saturated theorem along cubes for a measure and applications

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(*Received* 31 *July* 2025 *and accepted in revised form* 8 *January* 2026)

Abstract. We show that for a minimal system (X, T), the set of saturated points along cubes with respect to its maximal ∞-step pro-nilfactor X_∞ has a full measure. As an application, it is shown that if a minimal system (X, T) has no non-trivial $(k+1)$-tuples with arbitrarily long finite IP-independence sets, then it has only at most k ergodic measures and is an almost k' to one extension of X_∞ for some $k' \leqslant k$. In particular, for $k = 1$, we prove that (X, T) is uniquely ergodic (even regular with respect to X_∞), which answers a conjecture stated by Dong *et al* [Infinite-step nilsystems, independence and complexity. *Ergod. Th. & Dynam. Sys.* **33**(1) (2013), 118–143].

Key words: independence, almost finite to one extension, finite ergodic measures
2020 Mathematics Subject Classification: 37A25 (Primary); 37B05 (Secondary)

1. *Introduction*

In this section, we will provide the background of the research and state the main results of the paper.

1.1. *Characteristic factors.*

The study of convergence of the *multiple ergodic averages* began with Furstenberg's elegant proof of the Szemerédi theorem [28] via an ergodic theoretical analysis [5]. After nearly three decades of work by many researchers, the problem (for L^2 convergence) was finally solved in [13, 29]. In their proof, the key idea due to Furstenberg is to find an appropriate factor, called a *characteristic factor*, that controls the behavior of the limit in the sense of L^2 norm. For the origin of these ideas and this terminology, see [5]. The next step is to obtain a specific description for some well-chosen characteristic factors to prove convergence. The result in [13, 29] shows that such a characteristic factor can be described as an inverse limit of nilsystems, which is also called a *pro-nilfactor*.

The counterpart of characteristic factors for topological dynamics was first studied by Glasner in [7]. To state the result, we need a notion called saturated subset. Given a map $\pi : X \to Y$ of sets X and Y, a subset L of X is called π-*saturated* if $L = \pi^{-1}(\pi(L))$.

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Next is the definition of topological characteristic factors. Let $\pi : (X, T) \to (Y, T)$ be a factor map of topological dynamical systems and $d \in \mathbb{N}$. Here, (Y, T) is said to be a *d-step topological characteristic factor* if there exists a dense G_δ subset Ω of X such that for each $x \in \Omega$, the orbit closure

$$L_x^d(X) := \overline{\mathcal{O}}(\underbrace{(x, \dots, x)}_{d \text{ times}}, T \times \cdots \times T^d)$$

is $\underbrace{\pi \times \cdots \times \pi}_{d \text{ times}}$-saturated. That is, $(x_1, \dots, x_d) \in L_x^d(X)$ if and only if $(x_1', \dots, x_d') \in L_x^d(X)$ whenever $\pi(x_i) = \pi(x_i')$ for $i = 1, \dots, d$.

In [7], it was shown that for minimal systems, up to a canonically defined proximal extension, a characteristic family for $T \times \cdots \times T^d$ is the family of canonical proximal isometric (PI) flows of class $(d-1)$. In particular, if (X, T) is distal, then its largest class $(d-1)$ distal factor is its topological characteristic factor along $T \times \cdots \times T^d$. Moreover, if (X, T) is weakly mixing, then the trivial system is its topological characteristic factor. For more related results, we refer the reader to [7].

However, to get the corresponding pro-nilfactors for topological dynamics, in a pioneer work, Host, Kra, and Maass [15] introduced the notion of *regionally proximal relation of order d* for a topological dynamical system (X, T), denoted by $\mathbf{RP}^{[d]}$. For $d \in \mathbb{N}$, we say that a minimal system (X, T) is a *d-step pro-nilsystem* if $\mathbf{RP}^{[d]} = \Delta$ and this is equivalent for (X, T) being an inverse limit of minimal d-step nilsystems. For a minimal distal system (X, T), it was proved that $\mathbf{RP}^{[d]}$ is an equivalence relation and $X/\mathbf{RP}^{[d]}$ is the maximal d-step pro-nilfactor [15]. Later, Shao and Ye [27] showed that in fact for any minimal system, the regionally proximal relation of order d is an equivalence relation. Furthermore, for any minimal system, $\mathbf{RP}^{[\infty]} = \bigcap_{d \geqslant 1} \mathbf{RP}^{[d]}$ is also an equivalence relation.

The recent result in [10] improves Glasner's result significantly to pro-nilsystems. That is, they proved the following theorem.

THEOREM 1.1. (Glasner–Huang–Shao–Weiss–Ye) *Let (X, T) be a minimal system and let $\pi : X \to X/\mathbf{RP}^{[\infty]} = X_\infty$ be the factor map. Then, there exist minimal systems X^* and X_∞^* which are almost one to one extensions of X and X_∞, respectively, and a commuting diagram below such that X_∞^* is a d-step topological characteristic factor of X^* for all $d \geqslant 2$.*

$$\begin{array}{ccc} X & \xleftarrow{\sigma} & X^* \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi^*} \\ X_\infty & \xleftarrow{\tau} & X_\infty^* \end{array}$$

An equivalent form of Theorem 1.1 is the following.

THEOREM 1.2. *Let (X, T) be a minimal system and let $\pi : X \to X_\infty$ be the factor map. Then, there exists a dense G_δ subset Ω of X such that for every $x \in \Omega$ and every $d \geqslant 2$,*

$$(\pi^{-1}(\pi(x)))^d \subseteq L_x^d(X).$$

That is, if $\pi(x_i) = \pi(x)$ for $i = 1, \dots, d$, then $(x_1, \dots, x_d) \in L_x^d(X)$.

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Motivated by Theorem 1.2, the following question is natural.

Question 1.3. Let (X, T) be a minimal system and let $\pi : X \to X_\infty$ be the factor map. Given an ergodic measure μ on (X, T), is there a subset Ω of X with $\mu(\Omega) = 1$ such that $(\pi^{-1}(\pi(x)))^d \subseteq L_x^d(X)$ for every $x \in \Omega$ and every $d \geqslant 2$?

We are unable to address this question in the present paper. Still, we answer a similar question by considering the face cubegroup instead of the group generated by $T \times T^2 \times \cdots \times T^d$, and derive some interesting applications.

As we know, the face cubegroup introduced in [15] plays an important role when studying the pro-nilfactors. See §2.3 for the precise definition. We first study the topological characteristic factors under the action of face cubegroups. Following a recurrence theorem obtained by Bergelson and McCutcheon [1], and performing a detailed analysis of the Host–Kra cubic measures, we prove the following theorem.

THEOREM 1.4. *Let (X, T) be a minimal system and let $\pi : X \to X/\mathbf{RP}^{[\infty]}$ be the factor map. Let*

$$\Omega = \{x \in X : \{x\} \times (\pi^{-1}(\pi(x)))^{2^d-1} \subseteq \overline{\mathcal{F}^{[d]}}(x^{[d]}) \text{ for all } d \in \mathbb{N}\}. \tag{1.1}$$

Then, $\mu(\Omega) = 1$ for any ergodic measure μ on (X, T).

1.2. *Applications.* Following the *local entropy theory*, for a survey, see [12], each dynamical system admits a maximal zero topological entropy factor and this factor is induced by the smallest closed invariant equivalence relation containing *entropy pairs* [2]. In [22], entropy pairs are characterized as those pairs that admit an *interpolating set* of positive density. Later on, the notions of *sequence entropy pairs* [18] and *untame pairs* (called *scrambled pairs* in [16]) were introduced. In [24], the concept of *independence* was extensively studied and used to unify the notions mentioned above.

Let (X, T) be a topological dynamical system and let $\mathcal{U} = (U_1, \ldots, U_k)$ be a tuple of subsets of X. We say that a subset $F \subseteq \mathbb{N}$ is an *independence set* for $\mathcal{U}$ if for any non-empty finite subset $J \subseteq F$ and any $s = (s(j) : j \in J) \in \{1, \ldots, k\}^J$, we have $\bigcap_{j\in J} T^{-j} U_{s(j)} \neq \emptyset$. It was shown that a tuple of points $x_1, \ldots, x_k$ in X is a sequence entropy k-tuple (if $k = 2$, a tuple is said to be a pair) if and only if each $\mathcal{U} = (U_1, \ldots, U_k)$, where $U_1, \ldots, U_k$ are neighborhoods of $x_1, \ldots, x_k$, respectively, has arbitrarily long finite independence sets. Also, the pair is an untame pair if and only if each $\mathcal{U} = (U_1, U_2)$ as before has infinite independence sets. It was shown [9, 18, 24] that a minimal null (respectively tame) system is an almost one-to-one extension of its maximal equicontinuous factor and is also uniquely ergodic.

For $d \in \mathbb{N}$ and $n_1, \ldots, n_d \in \mathbb{N}$, we call the set

$$\mathrm{IP}(n_1, \ldots, n_d) := \{n_{i_1} + \cdots + n_{i_k} : 1 \leqslant i_1 < \cdots < i_k \leqslant d, 1 \leqslant k \leqslant d\}$$

an IP_d*-set*. The notion of Ind_{fip}*-k-tuple* was introduced in [17]: a tuple of points $x_1, \ldots, x_k$ in X is an Ind_{fip}-k-tuple if and only if the independence sets for each $\mathcal{U} = (U_1, \ldots, U_k)$ as before contain an IP_d-set for every $d \in \mathbb{N}$. It was shown that a minimal system without any non-trivial Ind_{fip}-pairs is an almost one-to-one extension of

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its maximal ∞-step pro-nilfactor [3]. They also asked whether a minimal system without any non-trivial Ind_{fip}-pairs is uniquely ergodic [3, Conjecture 6.6]. In this paper, we give a positive answer to this question.

It is also interesting to study the structure of minimal systems without any non-trivial sequence entropy tuples (respectively Ind_{fip}-tuples). This problem was also studied in [26]. In the recent work, Huang *et al* [19] proved that if a minimal system has no non-trivial sequence entropy $(k+1)$-tuples, then the system is an almost k'-to-one extension of its maximal equicontinuous factor for some $k' \in \{1, \ldots, k\}$, and the system has at most k^k ergodic measures.

In this paper, we study the structure of minimal systems without any non-trivial Ind_{fip}-tuples. As an application of Theorem 1.4, we prove the following theorem.

THEOREM 1.5. *Let (X, T) be a minimal system without any non-trivial Ind_{fip}-$(k+1)$-tuples and let $\pi : X \to X/\mathbf{RP}^{[\infty]} = Y$ be the factor map. Then, there exists $j \in \{1, \ldots, k\}$ such that $m(Y_j) = 1$, where m is the unique ergodic measure on Y and $Y_i = \{y \in Y : |\pi^{-1}(y)| = i\}$, $i \in \mathbb{N}$. (It was proved in [3] that Y is an inverse limit of minimal nilsystems, thus (Y, T) is uniquely ergodic.) Consequently, X has at most j ergodic measures and X is an almost j'-to-one extension of Y for some $j' \in \{1, \ldots, j\}$, that is, $Y_{j'}$ contains a dense G_δ set.*

Remark 1.6. Notice that if any Ind_{fip}-tuple is a sequence entropy tuple, Theorem 1.5 implies that if a minimal system has no non-trivial sequence entropy $(k+1)$-tuples, then the system has at most k ergodic measures, which gives a sharp bound on the result obtained in [19].

Remark 1.7. It was shown in [4] that every minimal tame system is a regular extension of its maximal equicontinuous factor (the set of points with trivial fibers has full measure). Theorem 1.5 implies that any minimal system without any non-trivial Ind_{fip}-pairs is a regular extension of its maximal ∞-step pro-nilfactor.

The paper is organized as follows. In §2, the basic notions used in the paper are introduced. In §3, we prove Theorem 1.4 and, as an application, we prove Theorem 1.5 in §4.

2. *Preliminaries*

In this section, we gather definitions and preliminary results that will be necessary later on. Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of all positive integers and integers, respectively. For a finite set F, denote by $|F|$ the number of elements of F.

2.1. *Dynamical systems.*

A *topological dynamical system* (TDS) is a pair (X, T), where X is a compact metric space with a metric ρ and $T : X \to X$ is a homeomorphism. For $x \in X$, $\mathcal{O}(x, T) = \{T^n x : n \in \mathbb{Z}\}$ denotes the *orbit* of x. A TDS (X, T) is called *minimal* if every point has dense orbit in X. A *homomorphism* between TDSs (X, T) and (Y, T) is a continuous onto map $\pi : X \to Y$ which intertwines the actions; one says that (Y, T) is a *factor* of (X, T) and that (X, T) is an *extension* of (Y, T). One also refers to π as a *factor*

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map or an *extension* and one uses the notation $\pi : (X, T) \to (Y, T)$. An extension π is determined by the corresponding closed invariant equivalence relation $R_\pi = \{(x, x') \in X \times X : \pi(x) = \pi(x')\}$.

A *measure preserving probability system* (MPS) is a quadruple $(X, \mathcal{X}, \mu, T)$, where $(X, \mathcal{X}, \mu)$ is a standard Borel probability space (in particular, X is a Polish space and $\mathcal{X}$ is its Borel σ-algebra) and T is an invertible Borel measure-preserving map ($\mu(TA) = \mu(A)$ for all $A \in \mathcal{X}$). An MPS $(X, \mathcal{X}, \mu, T)$ is *ergodic* if for every set $A \in \mathcal{X}$ such that $TA = A$, one has $\mu(A) = 0$ or 1.

Let (X, T) be a TDS and $M(X)$ be the set of Borel probability measures on X. Denote by $M(X, T)$ the set of invariant probability measures. For $x_0 \in X$ and $\mu \in M(X, T)$, we say that x_0 is *generic for* μ if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x_0) = \int_X f \, d\mu$$

for any continuous function f on X.

PROPOSITION 2.1. [6, Proposition 3.7] *Let (X, T) be a TDS and $\mu \in M(X, T)$ be ergodic. Then, almost every point of X with respect to μ is generic for μ.*

2.2. *Discrete cubes.* Let $d \geqslant 1$ be an integer and write $[d] = \{1, \ldots, d\}$. We view the element in $\{0, 1\}^d$ in one of two ways, either as a sequence $\epsilon = (\epsilon_1, \ldots, \epsilon_d)$ of 0s and 1s, or as a subset of $[d]$. A subset ϵ corresponds to the sequence $(\epsilon_1, \ldots, \epsilon_d) \in \{0, 1\}^d$ such that $i \in \epsilon$ if and only if $\epsilon_i = 1$ for $i \in [d]$. For example, $\vec{0} = (0, \ldots, 0) \in \{0, 1\}^d$ is the same to $\emptyset \subseteq [d]$. Let $\mathcal{V}_d = \{0, 1\}^d$ and $\mathcal{V}_d^* = \{0, 1\}^d \backslash \{\vec{0}\}$. If $\vec{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $\epsilon \in \mathcal{V}_d$, we define $\vec{n} \cdot \epsilon = \sum_{i=1}^{d} n_i \cdot \epsilon_i$. If we consider ϵ as $\epsilon \subseteq [d]$, then $\vec{n} \cdot \epsilon = \sum_{i\in\epsilon} n_i$.

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{V}_n$ and $\beta = (\beta_1, \ldots, \beta_d) \in \mathcal{V}_d$, we define

$$\underline{\alpha\beta} = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_d). \tag{2.1}$$

Given a set X, we denote X^{2^d} by $X^{[d]}$. A point $\mathbf{x} \in X^{[d]}$ can be written in one of two equivalent ways, depending on the context:

$$\mathbf{x} = (x_\epsilon)_{\epsilon\in\mathcal{V}_d} = (x_\epsilon)_{\epsilon\subseteq[d]}.$$

Hence, $x_\emptyset = x_{\vec{0}}$ is the first coordinate of $\mathbf{x}$. Sometimes, $\mathbf{x}(\epsilon)$ will be used to denote the ϵ-component of $\mathbf{x}$. For $x \in X$, we write $x^{[d]} = (x, \ldots, x) \in X^{[d]}$. The *diagonal* of $X^{[d]}$ is $\Delta^{[d]} = \Delta^{[d]}(X) = \{x^{[d]} : x \in X\}$. Usually, when $d = 1$, we denote the diagonal by Δ_X or Δ instead of $\Delta^{[1]}$. We can isolate the first coordinate, writing $X_*^{[d]} = X^{2^d-1}$ and writing $\mathbf{x} \in X^{[d]}$ as $\mathbf{x} = (\mathbf{x}(\vec{0}), \mathbf{x}_*)$, where $\mathbf{x}_* \in X_*^{[d]}$.

2.3. *Dynamical cubespaces.* Let (X, T) be a TDS and $d \in \mathbb{N}$. We define $\mathbf{Q}^{[d]}(X)$ to be the closure in $X^{[d]}$ of elements of the form

$$(T^{\vec{n}\cdot\epsilon} x)_{\epsilon\in\mathcal{V}_d},$$

where $\vec{n} \in \mathbb{Z}^d$ and $x \in X$. We call this set the *dynamical cubespace of dimension d* of the system.

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Face transformations are defined inductively as follows. Let $T_1^{[1]} = \mathrm{id} \times T$. If $\{T_j^{[d-1]}\}_{j=1}^{d-1}$ is defined already, then set

$$T_j^{[d]} = T_j^{[d-1]} \times T_j^{[d-1]}, \quad j \in [d-1],$$
$$T_d^{[d]} = \mathrm{id}^{[d-1]} \times T^{[d-1]}.$$

It is easy to see that for $j \in [d]$, the face transformation $T_j^{[d]} : X^{[d]} \to X^{[d]}$ can be defined by, for every $\mathbf{x} \in X^{[d]}$ and $\epsilon \in \mathcal{V}_d$,

$$T_j^{[d]}\mathbf{x} = \begin{cases} (T_j^{[d]}\mathbf{x})(\epsilon) = T\mathbf{x}(\epsilon), & \epsilon_j = 1; \\ (T_j^{[d]}\mathbf{x})(\epsilon) = \mathbf{x}(\epsilon), & \epsilon_j = 0. \end{cases}$$

The *face cubegroup* of dimension d is the group $\mathcal{F}^{[d]}(X)$ of transformations of $X^{[d]}$ spanned by the face transformations. The *parallelepiped group* of dimension d is the group $\mathcal{G}^{[d]}(X)$ spanned by the diagonal transformation and the face transformations. We often write $\mathcal{F}^{[d]}$ and $\mathcal{G}^{[d]}$ instead of $\mathcal{F}^{[d]}(X)$ and $\mathcal{G}^{[d]}(X)$, respectively. For convenience, we denote the orbit closure of $\mathbf{x} \in X^{[d]}$ under $\mathcal{F}^{[d]}$ by $\overline{\mathcal{F}^{[d]}}(\mathbf{x})$, instead of $\overline{\mathcal{O}(\mathbf{x}, \mathcal{F}^{[d]})}$. For $x \in X$, let $\mathbf{Q}_x^{[d]}(X) = \mathbf{Q}^{[d]}(X) \cap (\{x\} \times X^{2^d-1})$.

THEOREM 2.2. [27] *Let (X, T) be a minimal system and $d \in \mathbb{N}$. Then:*

(1) $(\mathbf{Q}^{[d]}(X), \mathcal{G}^{[d]})$ *is a minimal system;*
(2) $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ *is minimal for all $x \in X$;*
(3) $\overline{\mathcal{F}^{[d]}}(x^{[d]})$ *is the unique $\mathcal{F}^{[d]}$-minimal subset in $\mathbf{Q}_x^{[d]}(X)$ for all $x \in X$.*

2.4. *Regionally proximality of higher order.*

Definition 2.3. [15] Let (X, T) be a TDS and $d \in \mathbb{N}$. The *regionally proximal relation of order d* is the relation $\mathbf{RP}^{[d]}$ defined by: $(x, y) \in \mathbf{RP}^{[d]}$ (or $\mathbf{RP}^{[d]}(X)$ in case of ambiguity) if and only if for every $\delta > 0$, there exist $x', y' \in X$ and $n_1, \ldots, n_d \in \mathbb{N}$ such that $\rho(x, x') < \delta$, $\rho(y, y') < \delta$ and

$$\rho(T^n x', T^n y') < \delta \quad \text{for all } n \in \mathrm{IP}(n_1, \ldots, n_d).$$

We say (X, T) is a *d-step pro-nilsystem* if $\mathbf{RP}^{[d]}$ is trivial.

THEOREM 2.4. [27] *Let (X, T) be a minimal system and $d \in \mathbb{N}$. Then:*

(1) $(x, y) \in \mathbf{RP}^{[d]}$ *if and only if* $(x, y_*^{[d+1]}) = (x, y, \ldots, y) \in \mathbf{Q}^{[d+1]}$ *if and only if* $(x, y_*^{[d+1]}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$;
(2) $\mathbf{RP}^{[d]}$ *is a closed invariant equivalence relation.*

The regionally proximal relation of order d allows us to construct the maximal d-step pro-nilfactor of a minimal system. That is, any factor of order d factorizes through this system.

THEOREM 2.5. [27] *Let $\pi : (X, T) \to (Y, T)$ be the factor map of minimal systems and $d \in \mathbb{N}$. Then:*

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(1) $(\pi \times \pi)\mathbf{RP}^{[d]}(X) = \mathbf{RP}^{[d]}(Y)$;
(2) (Y, T) *is a d-step pro-nilsystem if and only if* $\mathbf{RP}^{[d]}(X) \subseteq R_\pi$.

In particular, the quotient of (X, T) *under* $\mathbf{RP}^{[d]}(X)$ *is the maximal d-step pro-nilfactor of X.*

It follows from Theorem 2.4 that for any minimal system (X, T),

$$\mathbf{RP}^{[\infty]} = \bigcap_{d \geqslant 1} \mathbf{RP}^{[d]}$$

is also a closed invariant equivalence relation.

Now, we formulate the definition of ∞-step pro-nilsystems.

Definition 2.6. [3] A *minimal system* (X, T) is an *∞-step pro-nilsystem*, if the equivalence relation $\mathbf{RP}^{[\infty]}$ is trivial, that is, coincides with the diagonal. We say the quotient under $\mathbf{RP}^{[\infty]}$ is the *maximal ∞-step pro-nilfactor* of X.

PROPOSITION 2.7. [20, Proposition 8.1.5] *Let* (X, T) *be a minimal system and* $(x, y) \in X \times X \backslash \Delta$. *Then,* $(x, y) \in \mathbf{RP}^{[\infty]}$ *if and only if for every neighborhood V of y and any* $d \in \mathbb{N}$, *there exists an* IP_d*-set* I_d *such that* $T^n x \in V$ *for all* $n \in I_d$.

2.5. *Nilpotent groups, nilmanifolds, and nilsystems.* Let G be a Lie group. Let $G_1 = G$ and $G_k = [G, G_{k-1}]$ for $k \geqslant 2$, where $[G, H] = \{[g, h] : g \in G, h \in H\}$ and $[g, h] = g^{-1}h^{-1}gh$. If there exists some $d \geqslant 1$ such that $G_{d+1} = \{e\}$, G is called a *d-step nilpotent Lie group*. We say that a discrete subgroup Γ of a Lie group G is *cocompact* if G/Γ, endowed with the quotient topology, is compact. We say that the quotient $X = G/\Gamma$ is a *d-step nilmanifold* if G is a d-step nilpotent Lie group and Γ is a discrete, cocompact subgroup. The group G acts on X by left translations and we write this action as $(g, x) \mapsto gx$. Let $\tau \in G$ and T be the transformation $x \mapsto \tau x$ of X. Then, (X, T) is called a *d-step nilsystem*.

We also make use of inverse limits of nilsystems and so we recall the definition of an inverse limit of systems (restricting ourselves to the case of sequential inverse limits). If $\{(X_i, T_i)\}_{i\in\mathbb{N}}$ are systems with $\mathrm{diam}(X_i) \leqslant 1$ and $\phi_i : X_{i+1} \to X_i$ are factor maps, the *inverse limit* of the systems is defined to be the compact subset of $\prod_{i\in\mathbb{N}} X_i$ given by $\{(x_i)_{i\in\mathbb{N}} : \phi_i(x_{i+1}) = x_i, i \in \mathbb{N}\}$, which is denoted by $\lim_{\leftarrow}\{X_i\}_{i\in\mathbb{N}}$. It is a compact metric space endowed with the distance $\rho(x, y) = \sum_{i\in\mathbb{N}} 1/2^i \rho_i(x_i, y_i)$. We note that the maps $\{T_i\}$ induce a transformation T on the inverse limit.

The following structure theorems characterize pro-nilsystems.

THEOREM 2.8. [15] *A minimal system is a d-step pro-nilsystem if and only if it is an inverse limit of d-step minimal nilsystems.*

THEOREM 2.9. [3] *A minimal system is an ∞-step pro-nilsystem if and only if it is an inverse limit of minimal nilsystems.*

2.6. *Independence.* The notion of *independence* was first introduced and studied in [24, 25]. It corresponds to a modification of the notion of *interpolating set* studied in [11, 22].

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Definition 2.10. Let (X, T) be a TDS. Given a tuple $\mathcal{U} = (U_1, \ldots, U_k)$ of subsets of X, we say that a subset $F \subseteq \mathbb{N}$ is an *independence set* for $\mathcal{U}$ if for any non-empty finite subset $J \subseteq F$ and any $s = (s(j) : j \in J) \in \{1, \ldots, k\}^J$, we have

$$\bigcap_{j \in J} T^{-j} U_{s(j)} \neq \emptyset.$$

We shall denote the collection of all independence sets for $\mathcal{U}$ by $\mathrm{Ind}(U_1, \ldots, U_k)$ or $\mathrm{Ind}(\mathcal{U})$.

The following definition of an Ind_{fip}-k-tuple comes from [3].

Definition 2.11. Let (X, T) be a TDS and $k \geqslant 2$. A tuple $(x_1, \ldots, x_k)$ of points in X is called an Ind_{fip}-*k-tuple* if for any neighborhoods $U_1, \ldots, U_k$ of $x_1, \ldots, x_k$, respectively, and any $d \in \mathbb{N}$, there exists an IP_d-set I_d such that I_d is an independence set for $(U_1, \ldots, U_k)$.

Denote by $\mathrm{Ind}_{fip}^{(k)}(X)$ the set of all Ind_{fip}-k-tuples of (X, T).

The following characterization of Ind_{fip}-tuples using dynamical cubespaces was first given in [3] for $k = 2$.

LEMMA 2.12. *Let (X, T) be a minimal system, $k \geqslant 2$, and $x_1, \ldots, x_k \in X$. Then, we have $\{x_1, \ldots, x_k\}^{2^d} \subseteq \mathbf{Q}^{[d]}(X)$ for all $d \in \mathbb{N}$ if and only if $(x_1, \ldots, x_k)$ is an Ind_{fip}-k-tuple.*

Proof. Let $x_1, \ldots, x_k \in X$ be distinct points. First, we assume $\{x_1, \ldots, x_k\}^{2^d} \subseteq \mathbf{Q}^{[d]}(X)$ for all $d \in \mathbb{N}$.

Fix $d \in \mathbb{N}$. Let $\{1, \ldots, k\}^{\mathcal{V}_d} = \{\omega_1, \ldots, \omega_{k^{2^d}}\}$. For $1 \leqslant i \leqslant k^{2^d}$, write $\omega_i = (\omega_{i,\epsilon})_{\epsilon \in \mathcal{V}_d}$. Choose $m \in \mathbb{N}$ with $2^m \geqslant k^{2^d}$ and let $\omega_i = (1, \ldots, 1)$ (2^d times) for $k^{2^d} + 1 \leqslant i \leqslant 2^m$. Let $\mathcal{V}_m = \{\alpha_1, \ldots, \alpha_{2^m}\}$. Notice that for every $\eta \in \mathcal{V}_{m+d}$, there exists unique $i \in \{1, \ldots, 2^m\}$ and $\beta \in \mathcal{V}_d$ such that $\eta = \underline{\alpha_i \beta}$.

Let $U_1, \ldots, U_k$ be neighborhoods of $x_1, \ldots, x_k$, respectively. Now, we define $\mathbf{x} = (x_\eta)_{\eta \in \mathcal{V}_{m+d}} \in \{x_1, \ldots, x_k\}^{\mathcal{V}_{m+d}} = \{x_1, \ldots, x_k\}^{2^{m+d}}$ and $\mathbf{U} = (U_\eta)_{\eta \in \mathcal{V}_{m+d}} \in \{U_1, \ldots, U_k\}^{\mathcal{V}_{m+d}}$ such that $x_\eta = x_{\omega_{i,\beta}}$ and $U_\eta = U_{\omega_{i,\beta}}$ if $\eta = \underline{\alpha_i \beta}$. Then, $\mathbf{U}$ is a neighborhood of $\mathbf{x}$ and $\mathbf{x} \in \mathbf{Q}^{[m+d]}(X)$ by our assumption. Thus, by Theorem 2.2, there exist $a \in \mathbb{N}$ and $\vec{a} = (a_1, \ldots, a_m, n_1, \ldots, n_d) \in \mathbb{N}^{m+d}$ such that $T^{a+\vec{a}\cdot\eta} x_1 \in U_\eta$ for all $\eta \in \mathcal{V}_{m+d}$.

We claim that the IP_d-set generated by $n_1, \ldots, n_d$ belongs to $\mathrm{Ind}(U_1, \ldots, U_k)$. As a matter of fact, for any $s \in \{1, \ldots, k\}^{\mathcal{V}_d}$, there exists ω_i such that $\omega_{i,\epsilon} = s(\epsilon)$ for all $\epsilon \in \mathcal{V}_d$. Set $\vec{b} = (a_1, \ldots, a_m)$ and $\vec{n} = (n_1, \ldots, n_d)$. Then, for any $\epsilon \in \mathcal{V}_d$, we have

$$T^{\vec{n}\cdot\epsilon}(T^{a+\vec{b}\cdot\alpha_i} x_1) = T^{a+\vec{a}\cdot\underline{\alpha_i \epsilon}} x_1 \in U_{\underline{\alpha_i \epsilon}} = U_{\omega_{i,\epsilon}} = U_{s(\epsilon)},$$

which implies that

$$T^{a+\vec{b}\cdot\alpha_i} x_1 \in \bigcap_{\epsilon \in \mathcal{V}_d} T^{-\vec{n}\cdot\epsilon} U_{s(\epsilon)}.$$

As d is arbitrary, we conclude that $(x_1, \ldots, x_k)$ is an Ind_{fip}-k-tuple.

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Conversely, assume that $(x_1, \dots, x_k)$ is an Ind_{fip}-k-tuple. That is, for any neighborhoods $U_1, \dots, U_k$ of $x_1, \dots, x_k$, respectively, and any $d \in \mathbb{N}$, there exists $\vec{m}_0 = (m_1, \dots, m_d, m_{d+1}) \in \mathbb{N}^{d+1}$ such that for any $t \in \{1, \dots, k\}^{\mathcal{V}_{d+1}}$,

$$\bigcap_{\omega \in \mathcal{V}_{d+1}} T^{-\vec{m}_0 \cdot \omega} U_{t(\omega)} \neq \emptyset.$$

Now, fix $s \in \{1, \dots, k\}^{\mathcal{V}_d}$. There exists $t \in \{1, \dots, k\}^{\mathcal{V}_{d+1}}$ such that $t(\underline{\epsilon 1}) = s(\epsilon)$ for all $\epsilon \in \mathcal{V}_d$. Let $x \in \bigcap_{\omega \in \mathcal{V}_{d+1}} T^{-\vec{m}_0 \cdot \omega} U_{t(\omega)}$ and set $\vec{m} = (m_1, \dots, m_d)$. Then, we have

$$T^{\vec{m} \cdot \epsilon}(T^{m_{d+1}} x) = T^{\vec{m}_0 \cdot \underline{\epsilon 1}} \in U_{t(\underline{\epsilon 1})} = U_{s(\epsilon)}$$

for all $\epsilon \in \mathcal{V}_d$, which implies $\{x_1, \dots, x_k\}^{2^d} \subseteq \mathbf{Q}^{[d]}(X)$.

This completes the proof. □

2.7. *Some facts about hyperspaces.* Let X be a compact metric space with a metric ρ. Let 2^X be the collection of non-empty closed subsets of X. We may define a metric on 2^X as follows:

$$\begin{aligned} \rho_H(A, C) &= \inf\{\epsilon > 0 : A \subseteq B(C, \epsilon), C \subseteq B(A, \epsilon)\} \\ &= \max\left\{ \max_{a \in A} \rho(a, C), \max_{c \in C} \rho(c, A) \right\}, \end{aligned}$$

where $\rho(x, A) = \inf_{y \in A} \rho(x, y)$ and $B(A, \epsilon) = \{x \in X : \rho(x, A) < \epsilon\}$. The metric ρ_H is called the *Hausdorff metric* on 2^X.

Let X, Y be two compact metric spaces. Let $F : Y \to 2^X$ be a map and $y \in Y$. We say that F is *lower semi-continuous* (*LSC*) at y if for any $\epsilon > 0$, there exists a neighborhood U of y such that $F(y) \subseteq B(F(y'), \epsilon)$ for all $y' \in U$; and F is *upper semi-continuous* (*USC*) at y if for any $\epsilon > 0$, there exists a neighborhood U of y such that $F(U) \subseteq B(F(y), \epsilon)$. If F is LSC (respectively USC) at every point of Y, then we say that F is LSC (respectively USC).

If $f : X \to Y$ is a continuous map, then it is easy to verify that

$$F = f^{-1} : Y \to 2^X, \quad y \mapsto f^{-1}(y),$$

is USC.

LEMMA 2.13. *Let (X, T) be a TDS and $d \in \mathbb{N}$. The map $F_d : X \to 2^{X^{[d]}}$, $x \mapsto \overline{\mathcal{F}}^{[d]}(x^{[d]})$ is LSC.*

Proof. Let Θ_d be the metric on $X^{[d]}$ defined by $\Theta_d(\mathbf{u}, \mathbf{v}) = \max_{\epsilon \in \mathcal{V}_d} \rho(\mathbf{u}(\epsilon), \mathbf{v}(\epsilon))$ for $\mathbf{u}, \mathbf{v} \in X^{[d]}$, where ρ is the metric on X.

Let $x \in X$ and $\delta > 0$. Since $\overline{\mathcal{F}}^{[d]}(x^{[d]})$ is compact and $\{(T^{\vec{n} \cdot \epsilon} x)_{\epsilon \in \mathcal{V}_d} : \vec{n} \in \mathbb{Z}^d\}$ is dense in $\overline{\mathcal{F}}^{[d]}(x^{[d]})$, there exist $\vec{n}_1, \dots, \vec{n}_k \in \mathbb{Z}^d$ such that for any $\mathbf{x} \in \overline{\mathcal{F}}^{[d]}(x^{[d]})$, one can choose $i \in [k]$ such that $\Theta_d((T^{\vec{n}_i \cdot \epsilon} x)_{\epsilon \in \mathcal{V}_d}, \mathbf{x}) < \delta/2$. Then, there exists $\eta > 0$ with $\eta < \delta/2$ such that whenever $x' \in X$ with $\rho(x, x') < \eta$, one has $\rho(T^{\vec{n}_i \cdot \epsilon} x, T^{\vec{n}_i \cdot \epsilon} x') < \delta/2$ for all $i \in [k]$ and $\epsilon \in \mathcal{V}_d$.

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We claim that $F_d(x) \subseteq B(F_d(x'), \delta)$ for all $x' \in B(x, \eta)$. Let $x' \in B(x, \eta)$ and $\mathbf{x} \in F_d(x) = \overline{\mathcal{F}}^{[d]}(x^{[d]})$. There exists $j \in [k]$ such that $\Theta_d((T^{\vec{n}_j \cdot \epsilon} x)_{\epsilon \in \mathcal{V}_d}, \mathbf{x}) < \delta/2$. Thus, we have

$$\begin{aligned}
&\Theta_d((T^{\vec{n}_j \cdot \epsilon} x')_{\epsilon \in \mathcal{V}_d}, \mathbf{x}) \\
&\quad \leqslant \Theta_d((T^{\vec{n}_j \cdot \epsilon} x')_{\epsilon \in \mathcal{V}_d}, (T^{\vec{n}_j \cdot \epsilon} x)_{\epsilon \in \mathcal{V}_d}) + \Theta_d((T^{\vec{n}_j \cdot \epsilon} x)_{\epsilon \in \mathcal{V}_d}, \mathbf{x}) \\
&\quad < \max_{\epsilon \in \mathcal{V}_d} \rho(T^{\vec{n}_j \cdot \epsilon} x', T^{\vec{n}_j \cdot \epsilon} x) + \delta/2 \\
&\quad < \delta,
\end{aligned}$$

which implies $\mathbf{x} \in B(\overline{\mathcal{F}}^{[d]}(x'^{[d]}), \delta) = B(F_d(x'), \delta)$, as was to be shown. □

2.8. *Almost finite-to-one extensions.* Let $\pi : (X, T) \to (Y, T)$ be a factor map of TDSs and $k \in \mathbb{N}$. One says that π is an *almost k-to-one* extension if there exists a dense G_δ subset Ω of Y such that $|\pi^{-1}(y)| = k$ for every $y \in \Omega$.

PROPOSITION 2.14. [19, Proposition 2.14] *Let $\pi : (X, T) \to (Y, T)$ be a factor map of TDSs with (Y, T) being minimal. If $Y_f = \{y \in Y : |\pi^{-1}(y)| < \infty\}$ is non-empty, then $Y_0 = \{y \in Y : |\pi^{-1}(y)| = N\}$ contains a dense G_δ subset of Y, where $N = \min_{y \in Y_f} |\pi^{-1}(y)|$.*

2.9. *A recurrence theorem.* The following recurrence theorem plays an important role in our proof. We recall that a subset R of $\mathbb{Z}^k$ is an *IP_s-set* if there exist $\vec{n}_1, \ldots, \vec{n}_s \in \mathbb{Z}^k$ such that

$$R = \mathrm{IP}(\vec{n}_1, \ldots, \vec{n}_s) = \{\vec{n}_{i_1} + \cdots + \vec{n}_{i_k} : 1 \leqslant i_1 < \cdots < i_k \leqslant s, 1 \leqslant k \leqslant s\}.$$

We shall call a subset of $\mathbb{Z}^k$ an *IP_s^*-set* if it intersects every non-trivial IP_s-set.

THEOREM 2.15. [1, Theorem 6.15] *Suppose $t \in \mathbb{N}$ and $\delta > 0$ are given. There exists $s = s(t, \delta) \in \mathbb{N}$ and $\xi = \xi(t, \delta) > 0$ having the property that for all $r, k \in \mathbb{N}$, if r commuting measure preserving transformations $T_1, \ldots, T_r$ of a probability space $(X, \mathcal{B}, \mu)$ are given, as well as linear polynomials $p_{i,j}(n_1, \ldots, n_k) \in \mathbb{Z}[n_1, \ldots, n_k]$ with $p_{i,j}(0, \ldots, 0) = 0$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant t$, then for every $B \in \mathcal{B}$ with $\mu(B) \geqslant \delta$, the set*

$$R_{B,\xi} = \left\{(n_1, \ldots, n_k) \in \mathbb{Z}^k : \mu\left(\bigcap_{j=1}^{t} \left(\prod_{i=1}^{r} T_i^{p_{i,j}(n_1, \ldots, n_k)}\right)^{-1} B\right) > \xi\right\}$$

is an IP_s^*-set in $\mathbb{Z}^k$.

The following corollary follows from Theorem 2.15 directly.

COROLLARY 2.16. *Assume that $T_1, \ldots, T_d$ are commuting measure preserving transformations of a probability space $(X, \mathcal{B}, \mu)$ and $B \in \mathcal{B}$ with $\mu(B) > 0$, then for any $s \in [d]$, there exists $N \in \mathbb{N}$ such that for any $\mathbf{n}_1, \ldots, \mathbf{n}_N \in (\mathbb{Z}^d)^s$, one can find some*

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$\mathbf{n} = (\vec{n}_1, \dots, \vec{n}_s) \in \mathrm{IP}(\mathbf{n}_1, \dots, \mathbf{n}_N)$ *such that*

$$\mu\Big(\bigcap_{\alpha\subseteq[s]}\Big(\prod_{i\in\alpha} T^{-\vec{n}_i}\Big)B\Big) > 0,$$

where $T^{\vec{n}} = T_1^{n_1}\cdots T_d^{n_d}$ *for* $\vec{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$.

We give some examples to explain this corollary.

Assume that T_1, T_2 are commuting measure preserving transformations of a probability space $(X, \mathcal{B}, \mu)$ and $B \in \mathcal{B}$ with $\mu(B) > 0$.

- There exists $N_1 \in \mathbb{N}$ such that for any $n_1, \dots, n_{N_1} \in \mathbb{Z}$, one can find some $n \in \mathrm{IP}(n_1, \dots, n_{N_1})$ such that
$$\mu(B \cap T_1^{-n}B) > 0.$$
- There exists $N_2 \in \mathbb{N}$ such that for any $\vec{n}_1, \dots, \vec{n}_{N_2} \in \mathbb{Z}^2$, one can find some $\vec{n} = (a, b) \in \mathrm{IP}(\vec{n}_1, \dots, \vec{n}_{N_2})$ such that
$$\mu(B \cap T_1^{-a}B \cap T_2^{-b}B \cap T_1^{-a}T_2^{-b}B) > 0.$$
- There exists $N_3 \in \mathbb{N}$ such that for any $\mathbf{n}_1, \dots, \mathbf{n}_{N_3} \in (\mathbb{Z}^2)^2$, one can find some $\mathbf{n} = (\vec{u}, \vec{v}) \in \mathrm{IP}(\mathbf{n}_1, \dots, \mathbf{n}_{N_3})$ with $\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2)$ such that
$$\begin{aligned}&\mu(B \cap T^{-\vec{u}}B \cap T^{-\vec{v}}B \cap T^{-\vec{u}-\vec{v}}B)\\ &\quad= \mu(B \cap T_1^{-u_1}T_2^{-u_2}B \cap T_1^{-v_1}T_2^{-v_2}B \cap T_1^{-u_1-u_2}T_2^{-v_1-v_2}B) > 0.\end{aligned}$$

2.10. *A useful lemma.* To end this section, we give a useful lemma which can be derived from [15, 21]. For readability, we move the complicated proof to Appendix A.

For $d \in \mathbb{N}$ and $k \in [d]$, we say a map $p : \mathcal{V}_k \to \mathcal{V}_d$ is a *morphism* if there exist pairwise disjoint non-empty subsets $\epsilon_1, \dots, \epsilon_k$ of $[d]$ such that $p(\emptyset) = \emptyset$ and $p(\eta) = \bigcup_{i\in\eta}\epsilon_i$ for $\eta \in \mathcal{V}_k^*$. Given a set X, then p determines a projection $\bar{p} : X^{[d]} \to X^{[k]}$ such that

$$\bar{p}(\mathbf{x})(\eta) = \mathbf{x}(p(\eta))$$

for $\mathbf{x} \in X^{[d]}$ and $\eta \in \mathcal{V}_k$.

LEMMA 2.17. *Let (X, T) be a minimal system and $\mu \in M(X, T)$ be an ergodic measure. Then, there exists a family $\{\mu_x^{[d]} = \delta_x \times \mu_x^{[d]_*}\}_{x\in X}$ of Borel probability measures on $X^{[d]}$ such that: for any measurable subset M of X with positive measure, there exists a measurable subset M_0 of M with $\mu(M_0) = \mu(M)$ such that for any $x \in M_0$ and any $d \in \mathbb{N}$:*

(1) *$\mu_x^{[d]}$ is $\mathcal{F}^{[d]}$-ergodic and $\mu_x^{[1]} = \delta_x \times \mu$;*

(2) *$\bar{p}_*(\mu_x^{[d]}) = \mu_x^{[k]}$ for any $k \in [d]$ and any morphism $p : \mathcal{V}_k \to \mathcal{V}_d$. In particular, for measurable subsets $W_\epsilon, \epsilon \in \mathcal{V}_d$ of X, one has*
$$\mu_x^{[k]}\Big(\prod_{\eta\in\mathcal{V}_k} W_{p(\eta)}\Big) \geqslant \mu_x^{[d]}\Big(\prod_{\epsilon\in\mathcal{V}_d} W_\epsilon\Big);$$

(3) *$\mu_x^{[d]}((M \cap V)^{[d]}) > 0$ for any neighborhood V of x.*

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3. *Proof of Theorem 1.4*

In this section, we prove Theorem 1.4. Throughout this section, we assume that (X, T) is a minimal system and $\mu \in M(X, T)$ is an ergodic measure. Let $\pi : X \to X/\mathbf{RP}^{[\infty]} = Y$ be the factor map.

3.1. *An intermediate proposition of proving Theorem 1.4.*

We start with the following definition.

Definition 3.1. For a non-empty subset κ of $\mathcal{V}_d^*$, we say that π has *the property* $F_d(\kappa)$ if, for any measurable subset V of X with $\mu(V) > 0$ and any open subsets $V_\epsilon, \epsilon \in \kappa$ of X with

$$\pi(V) \subseteq \bigcap_{\epsilon\in\kappa} \pi(V_\epsilon),$$

there exists $\vec{n} \in \mathbb{Z}^d$ such that

$$\mu\left(\bigcap_{\epsilon\in\mathcal{V}_d} T^{-\vec{n}\cdot\epsilon} V_\epsilon\right) > 0,$$

where $V_\epsilon = V$ for $\epsilon \in \mathcal{V}_d \backslash \kappa$.

In this section, we focus on proving the following proposition, from which we can directly derive Theorem 1.4. As the proof of this implication is straightforward, we will present the proof of Theorem 1.4, assuming Proposition 3.2 for the remainder of this subsection.

PROPOSITION 3.2. *For every $d \in \mathbb{N}$, π has the property $F_d(\mathcal{V}_d^*)$. That is, for any measurable subset $V_{\vec{0}}$ of X with $\mu(V_{\vec{0}}) > 0$ and any open subsets $V_\epsilon, \epsilon \in \mathcal{V}_d^*$ of X with*

$$\pi(V_{\vec{0}}) \subseteq \bigcap_{\epsilon\in\mathcal{V}_d^*} \pi(V_\epsilon),$$

there exists $\vec{n} \in \mathbb{Z}^d$ such that

$$\mu\left(\bigcap_{\epsilon\in\mathcal{V}_d} T^{-\vec{n}\cdot\epsilon} V_\epsilon\right) > 0.$$

Proof of Theorem 1.4 assuming Proposition 3.2. Let $\pi : X \to X/\mathbf{RP}^{[\infty]}$ be the factor map and let μ be an ergodic measure on (X, T). For $d \in \mathbb{N}$, let Θ_d be the metric on $X^{[d]}$ defined by $\Theta_d(\mathbf{u}, \mathbf{v}) = \max_{\epsilon\in\mathcal{V}_d} \rho(\mathbf{u}(\epsilon), \mathbf{v}(\epsilon))$ for $\mathbf{u}, \mathbf{v} \in X^{[d]}$, where ρ is the metric on X. Let $\Theta_{d,H}$ be the Hausdorff metric on $2^{X^{[d]}}$. Consider the maps

$$\Pi : X \to 2^X, \quad x \mapsto \pi^{-1}(\pi(x)),$$

and

$$F_d : X \to 2^{X^{[d]}}, \quad x \mapsto \overline{\mathcal{F}}^{[d]}(x^{[d]}).$$

Then, Π is upper semi-continuous and F_d is lower semi-continuous by Lemma 2.13.

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For $d \in \mathbb{N}$, $\delta > 0$, and $x \in X$, let

$$A_x^{[d]} = \{(x_\epsilon)_{\epsilon \in \mathcal{V}_d} \in X^{[d]} : x_{\vec{0}} = x,\ \pi(x_\epsilon) = \pi(x), \epsilon \in \mathcal{V}_d^*\}$$

and

$$\Lambda_{d,\delta} = \{x \in X : \text{there exists } \mathbf{x} \in A_x^{[d]},\ \text{such that } \Theta_d(\mathbf{x}, F_d(x)) \geqslant \delta\}.$$

CLAIM 1. *We have that $\Lambda_{d,\delta}$ is closed for any $d \in \mathbb{N}$ and $\delta > 0$.*

Proof of Claim 1. Fix $d \in \mathbb{N}$ and $\delta > 0$. Let $\gamma > 0$ with $4\gamma < \delta$. Assume that $\{x_i\}_{i\in\mathbb{N}}$ is a sequence in $\Lambda_{d,\delta}$ such that $\lim_{i\to\infty} x_i = x$ for some $x \in X$. Then, for every $i \in \mathbb{N}$, there exists $\mathbf{x}_i \in A_{x_i}^{[d]}$ such that $\Theta_d(\mathbf{x}_i, F_d(x_i)) > \delta - \gamma$. Without loss of generality, we may assume that $\lim_{i\to\infty} \mathbf{x}_i = \mathbf{x}$ for some $\mathbf{x} \in X^{[d]}$. Then, $\mathbf{x}(\vec{0}) = x$ and

$$\pi^{[d]}(\mathbf{x}) = \lim_{i\to\infty} \pi^{[d]}(\mathbf{x}_i) = \lim_{i\to\infty} \pi(x_i)^{[d]} = \pi(x)^{[d]},$$

which implies $\mathbf{x} \in A_x^{[d]}$. To show this claim, it suffices to show $\mathbf{x} \in \Lambda_{d,\delta}$.

Recall that the map F_d is lower semi-continuous, there exists a neighborhood U of x such that $F_d(x) \subseteq B(F_d(x'), \gamma)$ for all $x' \in U$. Choose $i \in \mathbb{N}$ such that $x_i \in U$ and $\Theta_d(\mathbf{x}_i, \mathbf{x}) < \gamma$. Then, $F_d(x) \subseteq B(F_d(x_i), \gamma)$. Let $\mathbf{y} \in F_d(x)$ and choose $\mathbf{y}_i \in F_d(x_i)$ with $\Theta_d(\mathbf{y}_i, \mathbf{y}) < \gamma$. Hence, we have

$$\Theta_d(\mathbf{x}_i, \mathbf{x}) + \Theta_d(\mathbf{x}, \mathbf{y}) + \Theta_d(\mathbf{y}, \mathbf{y}_i) \geqslant \Theta_d(\mathbf{x}_i, \mathbf{y}_i) \geqslant \Theta_d(\mathbf{x}_i, F_d(x_i)) > \delta - \gamma,$$

and thus, $\Theta_d(\mathbf{x}, \mathbf{y}) > \delta - 3\gamma$. This implies that

$$\Theta_d(\mathbf{x}, F_d(x)) = \inf_{\mathbf{y}\in F_d(x)} \Theta_d(\mathbf{x}, \mathbf{y}) \geqslant \delta - 3\gamma.$$

As γ is arbitrary, we deduce $\Theta_d(\mathbf{x}, F_d(x)) \geqslant \delta$ and thus, $\mathbf{x} \in \Lambda_{d,\delta}$, as was to be shown. □

CLAIM 2. *We have $\mu(\Lambda_{d,\delta}) = 0$ for any $d \in \mathbb{N}$ and $\delta > 0$.*

Proof of Claim 2. Suppose for a contradiction that $\mu(\Lambda_{d,\delta}) > 0$ for some $d \in \mathbb{N}$ and $\delta > 0$. Here, Π is upper semi-continuous and F_d is lower semi-continuous; hence, both maps Π and F_d are Borel measurable. By Lusin's theorem, there exists $K \subseteq \Lambda_{d,\delta}$ with $\mu(K) > 0$ such that both maps $F_d|_K$ and $\Pi|_K$ are continuous.

Now, let $x \in \text{supp}(\mu|_K) \subseteq K \subseteq \Lambda_{d,\delta}$ and fix $\gamma > 0$. Choose $\mathbf{x} = (x_\epsilon)_{\epsilon\in\mathcal{V}_d} \in A_x^{[d]}$ such that

$$\Theta_d(\mathbf{x}, F_d(x)) > \delta - \gamma.$$

As both the map $F_d|_K$ and $\Pi|_K$ are continuous and $x \in K$, there exists $\eta > 0$ with $\eta < \gamma$ such that whenever $x' \in K \cap B(x, \eta)$, one has

$$\Theta_{d,H}(F_d(x), F_d(x')) < \gamma \tag{3.1}$$

and

$$\Theta_{d,H}(\Pi(x), \Pi(x')) < \gamma. \tag{3.2}$$

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Now, set $V_{\vec{0}} = K \cap B(x, \eta)$ and $V_\epsilon = B(x_\epsilon, \gamma)$ for $\epsilon \in \mathcal{V}_d^*$. Then, we have $\mu(V_{\vec{0}}) > 0$ as $x \in \mathrm{supp}(\mu|_K)$ and by (3.2),

$$\pi(V_{\vec{0}}) \subseteq \bigcap_{\epsilon \in \mathcal{V}_d^*} \pi(V_\epsilon).$$

It follows from Proposition 3.2 that there is some $\vec{n} \in \mathbb{Z}^d$ such that

$$\mu\left(\bigcap_{\epsilon \in \mathcal{V}_d} T^{-\vec{n}\cdot\epsilon} V_\epsilon\right) > 0.$$

Let $w \in \bigcap_{\epsilon \in \mathcal{V}_d} T^{-\vec{n}\cdot\epsilon} V_\epsilon \subseteq V_{\vec{0}} = K \cap B(x, \eta)$ and set $\mathbf{w} = (T^{\vec{n}\cdot\epsilon} w)_{\epsilon \in \mathcal{V}_d}$. Then, we have $\Theta_d(\mathbf{x}, \mathbf{w}) < \gamma$ and $\Theta_{d,H}(F_d(x), F_d(w)) < \gamma$ by (3.1). Since $\mathbf{w} \in F_d(w)$, by the definition of the Hausdorff metric, one can choose $\mathbf{x}' \in F_d(x)$ with $\Theta_d(\mathbf{w}, \mathbf{x}') < \gamma$. This implies that

$$\Theta_d(\mathbf{x}, F_d(x)) \leqslant \Theta_d(\mathbf{x}, \mathbf{x}') \leqslant \Theta_d(\mathbf{x}, \mathbf{w}) + \Theta_d(\mathbf{w}, \mathbf{x}') < 2\gamma.$$

As γ is arbitrary, we conclude that $\mathbf{x} \in F_d(x)$, which is a contradiction. □

Let

$$\Omega = \{x \in X : \{x\} \times (\pi^{-1}(\pi(x)))^{2^d - 1} \subseteq \overline{\mathcal{F}^{[d]}}(x^{[d]}) \text{ for all } d \in \mathbb{N}\}.$$

Notice that $X \backslash \Omega \subseteq \bigcup_{d,k \in \mathbb{N}} \Lambda_{d,(1/k)}$ and $\mu(\Lambda_{d,(1/k)}) = 0$ for any $d, k \in \mathbb{N}$ by Claim 2, we conclude that $\mu(\Omega) = 1$.

This finishes the proof. □

3.2. *Proof of Proposition 3.2.* We note that the proof of Proposition 3.2 is quite lengthy and we will structure it into three parts. First, we prove that π has the property $F_d(\{\{1\}\})$ for every $d \in \mathbb{N}$, which is a critical step in the overall proof (see Proposition 3.3). Next, we show that π also has the property $F_d(\kappa)$ for any $d \in \mathbb{N}$ and any $\kappa \subseteq \mathcal{V}_d^*$ with $|\kappa| = 1$ (see Proposition 3.4). Finally, we use this property to demonstrate Proposition 3.2 by employing induction on the number of elements in κ.

3.2.1. *Step 1: $\kappa = \{\{1\}\}$.*

PROPOSITION 3.3. *Let V_0 be a measurable subset of X with $\mu(V_0) > 0$, and let V_1 be an open subset of X such that $\pi(V_0) \subseteq \pi(V_1)$. Then, there exists a measurable set $W \subseteq V_0$ with $\mu(W) > 0$ and $(n_1, \ldots, n_d) \in \mathbb{Z}^d$ such that:*

- $T^{n_1} W \subseteq V_1$;
- $T^{\sum_{i \in \epsilon} n_i} W \subseteq V_0$ *for all* $\epsilon \subseteq [d], \epsilon \neq \{1\}$.

Proof. Let $\{\mu_x^{[d]}\}_{x \in X, d \in \mathbb{N}}$ be as in Lemma 2.17. It follows from Lemma 2.17 that there exists $x \in V_0$ such that $\mu_x^{[1]} = \delta_x \times \mu$ and for any $d \in \mathbb{N}$:

(E) $(X^{[d]}, \mu_x^{[d]}, \mathcal{F}^{[d]})$ is ergodic;

(P) $\mu_x^{[d]}((V_0 \cap B(x, \gamma))^{[d]}) > 0$ for any $\gamma > 0$;

(M) $\tilde{p}_*(\mu_x^{[d]}) = \mu_x^{[k]}$ for any $k \in [d]$ and any morphism $p : \mathcal{V}_k \to \mathcal{V}_d$.

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CLAIM. *For any $N \in \mathbb{N}$, there is an IP_N-set R and $W \subseteq V_0$ with $\mu(W) > 0$ such that:*
- *$T^n W \subseteq V_1$ for any $n \in R$;*
- *$\mu_x^{[d]}(W^{[d]}) > 0$ for every $d \in \mathbb{N}$.*

Proof of the claim. As $\pi(V_0) \subseteq \pi(V_1)$, we can choose $y \in V_1$ such that $(x, y) \in \mathbf{RP}^{[\infty]}$. Fix $N \in \mathbb{N}$. It follows from Proposition 2.7 that there is an IP_N-set R such that $T^n x \in V_1$ for all $n \in R$. Let $\delta > 0$ such that $T^n B(x, \delta) \subseteq V_1$ for all $n \in R$. Set $W = V_0 \cap B(x, \delta)$. Then, $\mu_x^{[d]}(W^{[d]}) > 0$ for every $d \in \mathbb{N}$ by condition (P). In particular, $\mu(W) = \mu_x^{[1]}(W^{[1]}) > 0$. □

Before giving the full proof, we will show that this proposition holds for $d = 2, 3$ as examples to illustrate our basic ideas.

The case $d = 2$. As $\mu(V_0) = \mu_x^{[1]}(V_0) > 0$, there exists $N \in \mathbb{N}$ such that for any IP_N-set R, one can find some $n \in R$ such that

$$\mu(V_0 \cap T^{-n} V_0) > 0.$$

By the claim above, there is an IP_N-set R and $W_0 \subseteq V_0$ with $\mu(W_0) > 0$ such that $T^n W_0 \subseteq V_1$ for all $n \in R$. Now, we can choose $n \in R$ such that

$$T^n W_0 \subseteq V_1 \quad \text{and} \quad \mu(V_0 \cap T^{-n} V_0) > 0.$$

Recall that μ is ergodic, so there exists $m \in \mathbb{N}$ such that

$$\mu(W_0 \cap T^{-m}(V_0 \cap T^{-n} V_0)) > 0.$$

Set $W = W_0 \cap T^{-m} V_0 \cap T^{-n-m} V_0$. Then, we have:
- $T^n W \subseteq T^n W_0 \subseteq V_1$;
- $W \subseteq W_0 \subseteq V_0$ and $T^m W,\ T^{n+m} W \subseteq V_0$.

This shows the case $d = 2$.

The case $d = 3$. We first define inductively positive integers m_3, m_2, m_1 and N_2, N_1 as follows. Set $m_3 = 1$. As $\mu(V_0) > 0$, it follows from Corollary 2.16 that there exists $N_2 \in \mathbb{N}$ such that for any $\vec{n}_1, \ldots, \vec{n}_{N_2} \in \mathbb{N}^2$, one can find some $(n, m) \in \mathrm{IP}(\vec{n}_1, \ldots, \vec{n}_{N_2})$ such that

$$\mu(V \cap T^{-n} V \cap T^{-m} V \cap T^{-n-m} V) > 0.$$

Set $m_2 = m_3 N_2 = N_2$. Let us consider the system $(X^{[N_2]}, \mu_x^{[N_2]}, \mathcal{F}^{[N_2]})$. As $\mu_x^{[N_2]}(V_0^{[N_2]}) > 0$ by condition (P), it follows from Corollary 2.16 that there exists $N_1 \in \mathbb{N}$ such that for any $\vec{n}_1, \ldots, \vec{n}_{N_1} \in \mathbb{N}^{m_2} = \mathbb{N}^{N_2}$, one can find some $\vec{n} \in \mathrm{IP}(\vec{n}_1, \ldots, \vec{n}_{N_1})$ such that $S = (T^{\vec{n}\cdot\epsilon})_{\epsilon \in \mathcal{V}_{N_2}}$ and

$$\mu_x^{[N_2]}(V_0^{[N_2]} \cap S^{-1} V_0^{[N_2]}) > 0.$$

Set $m_1 = m_2 N_1 = N_2 N_1$.

By the claim above, there is an IP_{m_1}-set R and $W_0 \subseteq V_0$ with $\mu(W_0) > 0$ such that:
- $T^n W_0 \subseteq V_1$ for any $n \in R$;
- $\mu_x^{[d]}(W_0^{[d]}) > 0$ for every $d \in \mathbb{N}$.

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Let $n_1^1, \dots, n_{m_1}^1 \in \mathbb{N}$ such that $R = \mathrm{IP}(n_1^1, \dots, n_{m_1}^1)$. As $m_1 = N_2 N_1$, we can divide $\{n_1^1, \dots, n_{N_2 N_1}^1\}$ into N_1 vectors of dimension N_2 as follows: set $\vec{n}_i^1 = (n_{(i-1)N_2+1}^1, \dots, n_{iN_2}^1)$ for $i \in [N_1]$. Then, there exists $\vec{m}^1 \in \mathrm{IP}(\vec{n}_1^1, \dots, \vec{n}_{N_1}^1)$ such that $S_1 = (T^{\vec{m}^1 \cdot \epsilon})_{\epsilon \in \mathcal{V}_{N_2}}$ and

$$\mu_x^{[N_2]}(V_0^{[N_2]} \cap S_1^{-1} V_0^{[N_2]}) > 0.$$

Recall that the system $(X^{[N_2]}, \mu_x^{[N_2]}, \mathcal{F}^{[N_2]})$ is ergodic by condition (E) and $\mu_x^{[N_2]}(W_0^{[N_2]}) > 0$, so there exists $\vec{m}^2 \in \mathbb{N}^{N_2}$ such that $S_2 = (T^{\vec{m}^2 \cdot \epsilon})_{\epsilon \in \mathcal{V}_{N_2}}$ and

$$\mu_x^{[N_2]}(W_0^{[N_2]} \cap S_2^{-1}(V_0^{[N_2]} \cap S_1^{-1} V_0^{[N_2]})) > 0. \tag{3.3}$$

Set $\vec{m}^i = (m_1^i, \dots, m_{N_2}^i)$ for $i = 1, 2$ and set $\vec{n}_j^2 = (m_j^1, m_j^2)$ for $j \in [N_2]$. Then, by the construction of N_2, there exists $(a, b) \in \mathrm{IP}(\vec{n}_1^2, \dots, \vec{n}_{N_2}^2)$ such that

$$\mu(V_0 \cap T^{-a} V_0 \cap T^{-b} V_0 \cap T^{-a-b} V_0) > 0.$$

Let $\omega \in \mathcal{V}_{N_2}^*$ such that

$$(a, b) = \sum_{j \in \omega} \vec{n}_j^2 = \left(\sum_{j \in \omega} m_j^1, \sum_{j \in \omega} m_j^2 \right) = (\vec{m}^1 \cdot \omega, \vec{m}^2 \cdot \omega).$$

Notice the $\vec{0}$-component and ω-component of $W_0^{[N_2]} \cap S_2^{-1}(V_0^{[N_2]} \cap S_1^{-1} V_0^{[N_2]})$ are W_0 and $W_0 \cap T^{-b} V_0 \cap T^{-a-b} V_0 =: W_\omega$, respectively, it follows from condition (M) that

$$\mu(W_\omega) = \mu_x^{[1]}(W_0 \times W_\omega) \geqslant \mu_x^{[N_2]}(W_0^{[N_2]} \cap S_2^{-1}(V_0^{[N_2]} \cap S_1^{-1} V_0^{[N_2]})) > 0.$$

As μ is ergodic, there exists $c \in \mathbb{N}$ such that

$$\mu(W_\omega \cap T^{-c}(V_0 \cap T^{-a} V_0 \cap T^{-b} V_0 \cap T^{-a-b} V_0)) > 0.$$

Now, let

$$W = W_0 \cap T^{-b} V_0 \cap T^{-a-b} V_0 \cap T^{-c} V_0 \cap T^{-c-a} V_0 \cap T^{-c-b} V_0 \cap T^{-c-a-b} V_0.$$

Then, $W \subseteq W_0 \subseteq V_0$ and $T^n W \subseteq V_0$ for any $n \in \mathrm{IP}(a, b, c) \setminus \{a\}$.

Recall that $\vec{m}^1 \in \mathrm{IP}(\vec{n}_1^1, \dots, \vec{n}_{N_1}^1)$, so there exist $1 \leqslant j_1 < \cdots < j_l \leqslant N_1$ such that $\vec{m}^1 = \vec{n}_{j_1}^1 + \cdots + \vec{n}_{j_l}^1$, which implies

$$a = \vec{m}^1 \cdot \omega = \sum_{s=1}^{l} \sum_{t \in \omega} n_{(j_s-1)N_2+t}^1 \in \mathrm{IP}(n_1^1, \dots, n_{m_1}^1) = R.$$

Then, we have $T^a W \subseteq T^a W_0 \subseteq V_1$.

This shows the case $d = 3$.

The general case. We first define inductively positive integers $m_d, \dots, m_1$ and $N_{d-1}, \dots, N_1$ as follows. Set $m_d = 1$. Assume that we have already defined $m_d, \dots, m_{i+1}$ and $N_{d-1}, \dots, N_{i+1}$. Let us consider the system $(X^{[m_{i+1}]}, \mu_x^{[m_{i+1}]}, \mathcal{F}^{[m_{i+1}]})$. As $\mu_x^{[m_{i+1}]}(V_0^{[m_{i+1}]}) > 0$ by condition (P), it follows from Corollary 2.16

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that there exists $N_i \in \mathbb{N}$ such that for any $\mathbf{n}_1, \dots, \mathbf{n}_{N_i} \in (\mathbb{N}^{m_{i+1}})^i$, one can find some $\mathbf{n} = (\vec{n}_1, \dots, \vec{n}_i) \in \mathrm{IP}(\mathbf{n}_1, \dots, \mathbf{n}_{N_i})$ such that $S_j = (T^{\vec{n}_j \cdot \epsilon})_{\epsilon \in \mathcal{V}_{m_{i+1}}}$, $j \in [i]$ and

$$\mu_x^{[m_{i+1}]}\Big(\bigcap_{\alpha \subseteq [i]} \Big(\prod_{j \in \alpha} S_j^{-1}\Big) V^{[m_{i+1}]}\Big) > 0.$$

Set $m_i = m_{i+1} N_i$. This finishes the inductive definition.

By the claim above, there is an IP_{m_1}-set R and $W_0 \subseteq V_0$ with $\mu(W_0) > 0$ such that

$$T^n W_0 \subseteq V_1 \quad \text{for all } n \in R, \tag{3.4}$$

$$\mu_x^{[d]}(W_0^{[d]}) > 0 \quad \text{for all } d \in \mathbb{N}. \tag{3.5}$$

For $i \in \{2, \dots, d\}$, inductively, we will construct $\vec{n}_1^i, \dots, \vec{n}_i^i \in \mathbb{N}^{m_i}$ and $W_\epsilon^i \subseteq W_0$, $\epsilon \in \mathcal{V}_{m_i}$ such that:

(A_i) $\vec{n}_1^i \cdot \epsilon \in R$ for any $\epsilon \in \mathcal{V}_{m_i}^*$;

(B_i) $\mu_x^{[m_i]}(\prod_{\epsilon \in \mathcal{V}_{m_i}} W_\epsilon^i) > 0$;

(C_i) $(\prod_{j \in \alpha} T^{\vec{n}_j^i \cdot \epsilon}) W_\epsilon^i \subseteq V_0$ for any $\epsilon \in \mathcal{V}_{m_i}$ and any $\alpha \subseteq [i]$, $\alpha \neq \{1\}$.

Assume this has been achieved, then there exist $n_1^d, \dots, n_d^d \in \mathbb{N}^{m_d} = \mathbb{N}$ and $W_0^d, W_1^d \subseteq W_0$ such that:

- $n_1^d \in R$;
- $\mu(W_1^d) = \mu_x^{[1]}(W_0^d \times W_1^d) > 0$;
- $(\prod_{j \in \alpha} T^{n_j^d}) W_1^d \subseteq V_0$ for any $\alpha \subseteq [i]$, $\alpha \neq \{1\}$.

Set $W = W_1^d$ and $n_i = n_i^d$ for $i \in [d]$. Then, we have $\mu(W) > 0$ and:

- $T^{n_1} W = T^{n_1^d} W_1^d \subseteq T^{n_1^d} W_0 \subseteq V_1$ by (3.4);
- $T^{\sum_{j \in \alpha} n_j} W = (\prod_{j \in \alpha} T^{n_j^d}) W_1^d \subseteq V_0$ for any $\alpha \subseteq [i]$, $\alpha \neq \{1\}$,

which shows the proposition.

We now return to the inductive construction.

Step 1. Let $n_1^1, \dots, n_{m_1}^1 \in \mathbb{N}$ such that $R = \mathrm{IP}(n_1^1, \dots, n_{m_1}^1)$. Since $m_1 = m_2 N_1$, we divide $\{n_1^1, \dots, n_{m_1}^1\}$ into N_1 vectors of dimension m_2 as follows: set $\vec{r}_k = (n_{(k-1)m_2+1}^1, \dots, n_{km_2}^1)$ for $k \in [N_1]$. Let us consider the system $(X^{[m_2]}, \mu_x^{[m_2]}, \mathcal{F}^{[m_2]})$. Notice that $\mu_x^{[m_2]}(V_0^{[m_2]}) > 0$ by condition (P), thus, by the construction of N_1, there exists a non-empty subset β of $[N_1]$ such that

$$\vec{n}_1^2 = \sum_{k \in \beta} \vec{r}_k, \quad S_1^2 = (T^{-\vec{n}_1^2 \cdot \epsilon})_{\epsilon \in \mathcal{V}_{m_2}},$$

and

$$\mu_x^{[m_2]}(V_0^{[m_2]} \cap S_1^2 V_0^{[m_2]}) > 0.$$

As $\mu_x^{[m_2]}$ is $\mathcal{F}^{[m_2]}$-ergodic by condition (E) and $\mu_x^{[m_2]}(W_0^{[m_2]}) > 0$, there exists $\vec{n}_2^2 \in \mathbb{N}^{m_2}$ such that $S_2^2 = (T^{-\vec{n}_2^2 \cdot \epsilon})_{\epsilon \in \mathcal{V}_{m_2}}$ and

$$\mu_x^{[m_2]}(W_0^{[m_2]} \cap S_2^2(V_0^{[m_2]} \cap S_1^2 V_0^{[m_2]})) > 0.$$

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Set $W_\epsilon^2 = W_0 \cap T^{-\vec{n}_2^2\cdot\epsilon} V_0 \cap T^{-(\vec{n}_2^2+\vec{n}_1^2)\cdot\epsilon} V_0 \subseteq W_0$ for $\epsilon \in \mathcal{V}_{m_2}$. Then, we have:

- for any $\epsilon \in \mathcal{V}_{m_2}^*$,

$$\vec{n}_1^2 \cdot \epsilon = \sum_{k\in\beta} \vec{r}_k \cdot \epsilon = \sum_{k\in\beta} \sum_{j=1}^{m_2} n^1_{(k-1)m_2+j} \cdot \epsilon_j \in R;$$

- $\mu_x^{[m_2]}(\prod_{\epsilon\in\mathcal{V}_{m_2}} W_\epsilon^2) = \mu_x^{[m_2]}(W_0^{[m_2]} \cap S_2^2(V_0^{[m_2]} \cap S_1^2 V_0^{[m_2]})) > 0$;
- $W_\epsilon^2 \subseteq V_0 \cap T^{-\vec{n}_2^2\cdot\epsilon} V_0 \cap T^{-(\vec{n}_2^2+\vec{n}_1^2)\cdot\epsilon} V_0$ for any $\epsilon \in \mathcal{V}_{m_2}$.

Step i. Let $i \geqslant 2$ be an integer and assume that we have already chosen $\vec{n}_1^i, \ldots, \vec{n}_i^i \in \mathbb{N}^{m_i}$ and $W_\epsilon^i \subseteq W_0, \epsilon \in \mathcal{V}_{m_i}$ such that:

(A_i) $\vec{n}_1^i \cdot \epsilon \in R$ for any $\epsilon \in \mathcal{V}_{m_i}^*$;

(B_i) $\mu_x^{[m_i]}(\prod_{\epsilon\in\mathcal{V}_{m_i}} W_\epsilon^i) > 0$;

(C_i) $(\prod_{j\in\alpha} T^{\vec{n}_j^i\cdot\epsilon}) W_\epsilon^i \subseteq V$ for any $\epsilon \in \mathcal{V}_{m_i}$ and any $\alpha \subseteq [i], \alpha \neq \{1\}$.

For $j \in [i]$, set $\vec{n}_j^i = (n^i_{j,1}, \ldots, n^i_{j,m_i})$. As $m_i = m_{i+1}N_i$, we can divide $\{n^i_{j,1}, \ldots, n^i_{j,m_i}\}$ into N_i vectors of dimension m_{i+1} as follows: set $\vec{r}_{j,k} = (n^i_{j,(k-1)m_{i+1}+1}, \ldots, n^i_{j,km_{i+1}})$ for $k \in [N_i]$. Set $\mathbf{n}_k = (\vec{r}_{1,k}, \ldots, \vec{r}_{i,k}) \in (\mathbb{Z}^{m_i})^i$ for $k \in [N_i]$.

Let us consider the system $(X^{[m_{i+1}]}, \mu_x^{[m_{i+1}]}, \mathcal{F}^{[m_{i+1}]})$. Notice that $\mu_x^{[m_{i+1}]}(V_0^{[m_{i+1}]}) > 0$ by condition (P), thus, by the construction of N_i, there exist $1 \leqslant k_1 < \cdots < k_l \leqslant N_i$ such that for $j \in [i]$,

$$\vec{n}_j^{i+1} = \vec{r}_{j,k_1} + \cdots + \vec{r}_{j,k_l}, \quad S_j^{i+1} = (T^{-\vec{n}_j^{i+1}\cdot\epsilon})_{\epsilon\in\mathcal{V}_{m_{i+1}}},$$

and

$$\mu_x^{[m_{i+1}]}\Bigg(\underbrace{\bigcap_{\alpha\subseteq[i]}\Big(\prod_{j\in\alpha} S_j^{i+1}\Big)V_0^{[m_{i+1}]}}_{=:\mathbf{V}_1}\Bigg) > 0.$$

For $s \in [m_{i+1}]$, set $\omega_s = \{(k_1-1)m_{i+1}+s, \ldots, (k_l-1)m_{i+1}+s\}$. Then, $\omega_1, \ldots, \omega_{m_{i+1}}$ are non-empty pairwise disjoint subsets of $[m_i]$. Let $p : \mathcal{V}_{m_{i+1}} \to \mathcal{V}_{m_i}$ be the map such that $p(\emptyset) = \emptyset$ and $p(\eta) = \bigcup_{s\in\eta} \omega_s$ for $\eta \in \mathcal{V}_{m_{i+1}}^*$. It follows from condition (M) that

$$\mu_x^{[m_{i+1}]}\Bigg(\underbrace{\prod_{\eta\in\mathcal{V}_{m_{i+1}}} W^i_{p(\eta)}}_{=:\mathbf{V}_2}\Bigg) \geqslant \mu_x^{[m_i]}\Bigg(\prod_{\epsilon\in\mathcal{V}_{m_i}} W_\epsilon^i\Bigg) > 0.$$

Moreover, for $j \in [i]$ and $\eta \in \mathcal{V}_{m_{i+1}}^*$, we have

$$\begin{aligned}\vec{n}_j^{i+1} \cdot \eta = \sum_{t=1}^{l} \vec{r}_{j,k_t} \cdot \eta &= \sum_{t=1}^{l} \sum_{s\in\eta} n_{j,(k_t-1)m_{i+1}+s} \\ &= \sum_{s\in\eta} \vec{n}_j^i \cdot \omega_s = \vec{n}_j^i \cdot p(\eta). \end{aligned} \tag{3.6}$$

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Recall that $\mu_x^{[m_{i+1}]}$ is $\mathcal{F}^{[m_{i+1}]}$-ergodic by condition (E), so there exists $\vec{n}_{i+1}^{\,i+1} \in \mathbb{N}^{m_{i+1}}$ such that $S_{i+1}^{i+1} = (T^{-\vec{n}_{i+1}^{\,i+1}\cdot\eta})_{\eta\in\mathcal{V}_{m_{i+1}}}$ and

$$\mu_x^{[m_{i+1}]}(\mathbf{V}_1 \cap S_{i+1}^{i+1}\mathbf{V}_2) > 0. \tag{3.7}$$

For $\eta \in \mathcal{V}_{m_{i+1}}$, set

$$W_\eta^{i+1} = W_{p(\eta)}^i \cap T^{-\vec{n}_{i+1}^{\,i+1}\cdot\eta}\Big(\bigcap_{\alpha\subseteq[i]}\Big(\prod_{j\in\alpha} T^{-\vec{n}_j^{\,i+1}\cdot\eta}\Big)V_0\Big).$$

Then, we have

$$\prod_{\eta\in\mathcal{V}_{m_{i+1}}} W_\eta^{i+1} = \mathbf{V}_1 \cap S_{i+1}^{i+1}\mathbf{V}_2, \tag{3.8}$$

$$T^{\vec{n}_{i+1}^{\,i+1}\cdot\eta}\Big(\prod_{j\in\alpha} T^{\vec{n}_j^{\,i+1}\cdot\eta}\Big)W_\eta^{i+1} \subseteq V_0 \quad \text{for all } \alpha \subseteq [i], \tag{3.9}$$

$$W_\eta^{i+1} \subseteq W_{p(\eta)}^i. \tag{3.10}$$

We next show that $\vec{n}_1^{\,i+1}, \ldots, \vec{n}_{i+1}^{\,i+1}$ and sets W_η^{i+1}, $\eta \in \mathcal{V}_{m_{i+1}}$ satisfy $(A)_{i+1}$, $(B)_{i+1}$, $(C)_{i+1}$.

By (3.6) and $(A)_i$ for any $\eta \in \mathcal{V}_{m_{i+1}}^*$, we have

$$\vec{n}_1^{\,i+1}\cdot\eta = \vec{n}_1^{\,i}\cdot p(\eta) \in R.$$

By (3.7) and (3.8), we have

$$\mu_x^{[m_{i+1}]}\Big(\prod_{\eta\in\mathcal{V}_{m_{i+1}}} W_\eta^{i+1}\Big) = \mu_x^{[m_{i+1}]}(\mathbf{V}_1 \cap S_{i+1}^{i+1}\mathbf{V}_2) > 0.$$

It remains to show $(C)_{i+1}$. Let $\alpha \subseteq [i+1]$ with $\alpha \neq \{1\}$.

When $i+1 \in \alpha$, set $\alpha' = \alpha\backslash\{i+1\}$. Then, by (3.9), we have

$$\Big(\prod_{j\in\alpha} T^{\vec{n}_j^{\,i+1}\cdot\eta}\Big)W_\eta^{i+1} = T^{\vec{n}_{i+1}^{\,i+1}\cdot\eta}\Big(\prod_{j\in\alpha'} T^{\vec{n}_j^{\,i+1}\cdot\eta}\Big)W_\eta^{i+1} \subseteq V_0.$$

When $i+1 \notin \alpha$, we can view α as a subset of $[i]$. Hence, by $(C)_i$, (3.6), and (3.10), we have

$$\Big(\prod_{j\in\alpha} T^{\vec{n}_j^{\,i+1}\cdot\eta}\Big)W_\eta^{i+1} = \Big(\prod_{j\in\alpha} T^{\vec{n}_j^{\,i}\cdot p(\eta)}\Big)W_\eta^{i+1} \subseteq \Big(\prod_{j\in\alpha} T^{\vec{n}_j^{\,i}\cdot p(\eta)}\Big)W_{p(\eta)}^i \subseteq V_0.$$

We finish the construction by induction and, hence, the proof is complete. □

3.2.2. *Step 2:* $|\kappa| = 1$.

PROPOSITION 3.4. *π has the property $F_d(\kappa)$ for any $d \in \mathbb{N}$ and any $\kappa \subseteq \mathcal{V}_d^*$ with $|\kappa| = 1$.*

Proof. Fix $d \in \mathbb{N}$ and let $\omega \in \mathcal{V}_d^*$ with $\kappa = \{\omega\}$. Assume that V is a measurable subset of X with $\mu(V) > 0$ and V_1 is an open subset of X such that $\pi(V) \subseteq \pi(V_1)$. It follows from

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Proposition 3.3 that there exists $W_0 \subseteq V$ with $\mu(W_0) > 0$ and $(m_1, \dots, m_d) \in \mathbb{Z}^d$ such that:

- $T^{m_1} W_0 \subseteq V_1$;
- $T^{\sum_{i\in\epsilon} m_i} W_0 \subseteq V$ for all $\epsilon \subseteq [d]$, $\epsilon \neq \{1\}$.

Set $a = \min \omega$. Let $\vec{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ such that

$$n_i = \begin{cases} m_a, & i = 1, \\ m_1, & i = a, \\ -m_i, & i \in \omega\backslash\{a\}, \\ m_i & \text{otherwise}, \end{cases}$$

and let $W = T^{\sum_{i\in\omega\backslash\{a\}} m_i} W_0$. Then, $W \subseteq V$ and $\mu(W) > 0$.

Set $\omega' = \omega\backslash\{a\}$. For $\epsilon \in \mathcal{V}_d$, we have

$$\begin{aligned} \sum_{i\in\epsilon} n_i + \sum_{i\in\omega'} m_i &= \sum_{i\in\epsilon\backslash\omega'} n_i + \sum_{i\in\epsilon\cap\omega'} n_i + \sum_{i\in\epsilon\cap\omega'} m_i + \sum_{i\in\omega'\backslash\epsilon} m_i \\ &= \sum_{i\in\epsilon\backslash\omega'} n_i + \sum_{i\in\omega'\backslash\epsilon} m_i, \end{aligned}$$

which implies that

$$\sum_{i\in\omega} n_i + \sum_{i\in\omega'} m_i = n_a = m_1$$

and

$$\left\{ \sum_{i\in\epsilon} n_i + \sum_{i\in\omega'} m_i : \epsilon \in \mathcal{V}_d \right\} = \{\vec{m} \cdot \epsilon : \epsilon \in \mathcal{V}_d\}.$$

Thus, we have:

- $T^{\vec{n}\cdot\omega} W = T^{\sum_{i\in\omega} n_i + \sum_{i\in\omega'} m_i} W_0 = T^{m_1} W_0 \subseteq V_1$;
- $T^{\vec{n}\cdot\epsilon} W = T^{\sum_{i\in\epsilon} n_i + \sum_{i\in\omega'} m_i} W_0 \subseteq V$ for all $\epsilon \in \mathcal{V}_d\backslash\{\omega\}$.

This shows that π has the property $F_d(\kappa)$ and, hence, finishes the proof. □

3.2.3. *Step 3: $\kappa = \mathcal{V}_d^*$.* We will now employ induction on the number of elements in κ together with Propositions 3.3 and 3.4 to show Proposition 3.2. Before the induction, we show a special case when $d = 2$ and $\kappa = \{\{1\}, \{2\}\}$ as an example.

A special case. $d = 2$ and $\kappa = \{\{1\}, \{2\}\}$. Assume that V is a measurable subset of X with $\mu(V) > 0$ and V_{10}, V_{01} are open subsets of X such that $\pi(V) \subseteq \pi(V_{10}) \cap \pi(V_{01})$. We next show that there is $(n_1, n_2) \in \mathbb{Z}^2$ such that

$$\mu(V \cap T^{-n_1} V_{10} \cap T^{-n_2} V_{01} \cap T^{-n_1-n_2} V) > 0.$$

By Proposition 3.4, π has the property $F_2(\{\{1, 2\}\})$. As $\pi(V) \subseteq \pi(V_{01})$, there is $(a_1, a_2) \in \mathbb{Z}^2$ such that

$$\mu(V \cap T^{-a_1} V \cap T^{-a_2} V \cap T^{-a_1-a_2} V_{01}) > 0.$$

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Set $W_1 = V \cap T^{-a_1}V \cap T^{-a_2}V \cap T^{-a_1-a_2}V_{01}$. Then, $\mu(W_1) > 0$. Again by Proposition 3.4, π has the property $F_2(\{\{1\}\})$. As $\pi(W_1) \subseteq \pi(V) \subseteq \pi(V_{10})$, there is $(b_1, b_2) \in \mathbb{Z}^2$ such that

$$\mu(W_1 \cap T^{-b_1}V_{10} \cap T^{-b_2}W_1 \cap T^{-b_1-b_2}W_1) > 0.$$

Set $n_1 = b_1 - a_1, n_2 = b_2 + a_2, W_2 = W_1 \cap T^{-b_1}V_{10} \cap T^{-b_2}W_1 \cap T^{-b_1-b_2}W_1$ and $W = T^{a_1}W_2$. Then, $\mu(W) > 0$:

- $W = T^{a_1}W_2 \subseteq T^{a_1}W_1 \subseteq V$;
- $T^{n_1}W = T^{b_1-a_1}(T^{a_1}W_2) = T^{b_1}W_2 \subseteq V_{10}$;
- $T^{n_2}W = T^{b_2+a_2}(T^{a_1}W_2) = T^{a_1+a_2}(T^{b_2}W_2) \subseteq T^{a_1+a_2}W_1 \subseteq V_{01}$;
- $T^{n_1+n_2}W = T^{b_1-a_1+b_2+a_2}(T^{a_1}W_2) = T^{a_2}(T^{b_1+b_2}W_2) \subseteq T^{a_2}W_1 \subseteq V$,

and thus,

$$\mu(V \cap T^{-n_1}V_{10} \cap T^{-n_2}V_{01} \cap T^{-n_1-n_2}V) \geqslant \mu(W) > 0.$$

This shows that π has the property $F_2(\{\{1\}, \{2\}\})$.

Now, we can give a proof of Proposition 3.2.

Proof of Proposition 3.2. Fix $d \in \mathbb{N}$ and let $\kappa \subseteq \mathcal{V}_d^*$. When $|\kappa| = 1$, it follows from Proposition 3.4. Let $i \geqslant 2$ be an integer and assume that we have already shown that π has the property $F_d(\kappa)$ for any $\kappa \subseteq \mathcal{V}_d^*$ with $|\kappa| = i - 1$.

Now, fix $\kappa \subseteq \mathcal{V}_d^*$ with $|\kappa| = i$. Assume that V is a measurable subset of X with $\mu(V) > 0$ and $V_\epsilon, \epsilon \in \kappa$ are open subsets X such that $\pi(V) \subseteq \bigcap_{\epsilon\in\kappa} \pi(V_\epsilon)$.

Choose $\omega \in \kappa$ and let Φ be a permutation of $\mathcal{V}_d$ such that for any $\epsilon \in \mathcal{V}_d$,

$$\Phi(\epsilon)_i = \begin{cases} 1 - \epsilon_i & \text{if } \omega_i = 1, \\ \epsilon_i & \text{if } \omega_i = 0. \end{cases}$$

Then, we have $\Phi(\vec{0}) = \omega$, $\Phi(\omega) = \vec{0}$ and $\Phi(\Phi(\epsilon)) = \epsilon$ for any $\epsilon \in \mathcal{V}_d$. Set

$$\kappa' = \{\Phi(\epsilon) : \epsilon \in \kappa \backslash \{\omega\}\}.$$

Then, $|\kappa'| = i - 1$. Now, let $U_{\Phi(\epsilon)} = V_\epsilon$ for $\epsilon \in \kappa\backslash\{\omega\}$ and let $U = V$. Then, we have

$$\pi(U) = \pi(V) \subseteq \bigcap_{\epsilon\in\kappa} \pi(V_\epsilon) \subseteq \bigcap_{\epsilon\in\kappa\backslash\{\omega\}} \pi(V_\epsilon) = \bigcap_{\epsilon\in\kappa'} \pi(U_\epsilon).$$

By our inductive assumption, π has the property $F_d(\kappa')$. Thus, there is $\vec{a} \in \mathbb{Z}^d$ such that

$$\mu\left(\bigcap_{\epsilon\in\mathcal{V}_d} T^{-\vec{a}\cdot\epsilon}U_\epsilon\right) > 0,$$

where $U_\epsilon = U$ for $\epsilon \in \mathcal{V}_d\backslash\kappa'$. Set $W = \bigcap_{\epsilon\in\mathcal{V}_d} T^{-\vec{a}\cdot\epsilon}U_\epsilon$. Then, $\mu(W) > 0$ and

$$T^{\vec{a}\cdot\Phi(\epsilon)}W \subseteq U_{\Phi(\epsilon)} = V_\epsilon \quad \text{for all } \epsilon \in \kappa\backslash\{\omega\}, \tag{3.11}$$

$$T^{\vec{a}\cdot\Phi(\epsilon)}W \subseteq U_{\Phi(\epsilon)} = U = V \quad \text{for all } \epsilon \in (\mathcal{V}_d\backslash\kappa) \cup \{\omega\}. \tag{3.12}$$

In particular, $W = T^{\vec{a}\cdot\Phi(\omega)}W \subseteq U_{\Phi(\omega)} = U = V$ as $\Phi(\omega) = \vec{0}$ and $U_{\Phi(\omega)} = U$. Again, by our inductive assumption, π has the property $F_d(\{\omega\})$. As

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$\pi(W) \subseteq \pi(U) = \pi(V) \subseteq \pi(V_\omega)$, there is $\vec{b} \in \mathbb{Z}^d$ such that

$$\mu\Big(\bigcap_{\epsilon\in\mathcal{V}_d} T^{-\vec{b}\cdot\epsilon} W_\epsilon\Big) > 0,$$

where $W_\omega = V_\omega$ and $W_\epsilon = W$ for $\epsilon \in \mathcal{V}_d\backslash\{\omega\}$. Set $G = \bigcap_{\epsilon\in\mathcal{V}_d} T^{-\vec{b}\cdot\epsilon} W_\epsilon$. Then, $\mu(G) > 0$ and

$$T^{\vec{b}\cdot\omega} G \subseteq W_\omega = V_\omega, \tag{3.13}$$

$$T^{\vec{b}\cdot\epsilon} G \subseteq W_\epsilon = W \quad \text{for all } \epsilon \in \mathcal{V}_d\backslash\{\omega\}. \tag{3.14}$$

Let $\vec{c} = (c_1, \ldots, c_d) \in \mathbb{Z}^d$ such that

$$c_i = \begin{cases} -a_i & \text{if } \omega_i = 1, \\ a_i & \text{if } \omega_i = 0. \end{cases}$$

Set $\vec{n} = \vec{b} + \vec{c}$ and $W = T^{\vec{a}\cdot\omega} G$. Then, $\mu(W) = \mu(G) > 0$. We next show

$$T^{\vec{n}\cdot\epsilon} W \subseteq \begin{cases} V_\epsilon & \text{if } \epsilon \in \kappa, \\ V & \text{if } \epsilon \in \mathcal{V}_d\backslash\kappa. \end{cases}$$

Notice that for $\epsilon \in \mathcal{V}_d$,

$$\begin{aligned} &(\vec{b} + \vec{c}) \cdot \epsilon + \vec{a} \cdot \omega \\ &\quad = \vec{b} \cdot \epsilon + \sum_{i\in[d]\backslash\omega} a_i\epsilon_i - \sum_{i\in\omega} a_i\epsilon_i + \sum_{i\in\omega} a_i \\ &\quad = \vec{b} \cdot \epsilon + \sum_{i\in[d]\backslash\omega} a_i\epsilon_i + \sum_{i\in\omega} a_i(1 - \epsilon_i) \\ &\quad = \vec{b} \cdot \epsilon + \vec{a} \cdot \Phi(\epsilon), \end{aligned}$$

thus, we have

$$T^{\vec{n}\cdot\epsilon} W = T^{(\vec{b}+\vec{c})\cdot\epsilon}(T^{\vec{a}\cdot\omega} G) = T^{\vec{a}\cdot\Phi(\epsilon)}(T^{\vec{b}\cdot\epsilon} G). \tag{3.15}$$

When $\epsilon = \omega$, as $\Phi(\omega) = \vec{0}$, we have

$$T^{\vec{n}\cdot\omega} W \overset{(3.15)}{=} T^{\vec{a}\cdot\Phi(\omega)}(T^{\vec{b}\cdot\omega} G) = T^{\vec{b}\cdot\omega} G \overset{(3.13)}{\subseteq} V_\omega.$$

When $\epsilon \in \kappa\backslash\{\omega\}$, we have

$$T^{\vec{n}\cdot\epsilon} W \overset{(3.15)}{=} T^{\vec{a}\cdot\Phi(\epsilon)}(T^{\vec{b}\cdot\epsilon} G) \overset{(3.14)}{\subseteq} T^{\vec{a}\cdot\Phi(\epsilon)} W \overset{(3.11)}{\subseteq} V_\epsilon.$$

When $\epsilon \in \mathcal{V}_d\backslash\kappa$, we have

$$T^{\vec{n}\cdot\epsilon} W \overset{(3.15)}{=} T^{\vec{a}\cdot\Phi(\epsilon)}(T^{\vec{b}\cdot\epsilon} G) \overset{(3.14)}{\subseteq} T^{\vec{a}\cdot\Phi(\epsilon)} W \overset{(3.12)}{\subseteq} V.$$

This completes the induction procedure and, hence, the proof of the proposition. □

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4. *Proof of Theorem 1.5*

To end this paper, using Theorem 1.4, we show Theorem 1.5. Set $\mathbb{N}^* = \{\infty\} \cup \{1, 2, \ldots\}$. The proof of the following lemma is derived from [**19**, Proposition 3.7].

LEMMA 4.1. *Let $\pi : (X, T) \to (Y, T)$ be a factor map of systems. For $i \in \mathbb{N}^*$, set*

$$Y_i = \{y \in Y : |\pi^{-1}(y)| = i\}.$$

Then, Y_i is Borel measurable for every $i \in \mathbb{N}^*$.

Proof. Let 2^X be the hyperspace of X with Hausdorff metric $\rho_{X,H}$ and consider the map

$$F : Y \to 2^X, \quad y \mapsto \pi^{-1}(y).$$

Then, F is upper semi-continuous and, hence, Borel measurable. Define

$$G : 2^X \to \mathbb{N}^*, \quad A \mapsto |A|.$$

Assume that $B \in 2^X$ is a finite set. Let

$$\epsilon_B = \min_{x,x' \in B,\, x \neq x'} \rho_X(x, x').$$

Then, $\epsilon_B > 0$ and for any $B' \in 2^X$ with $\rho_{X,H}(B, B') < \epsilon_B/2$, it follows that $|B'| \geqslant |B|$.

Now, let $A_0 \in 2^X$. If $|A_0| < \infty$, then $|A| \geqslant |A_0|$ for any $A \in 2^X$ with $\rho_{X,H}(A_0, A) < \epsilon_{A_0}/2$. If $|A_0| = \infty$, then for any positive integer m, one can choose $A_m \subseteq A_0$ with $|A_m| = m$. Then, $|A| \geqslant m$ for any $A \in 2^X$ with $\rho_{X,H}(A_0, A) < \epsilon_{A_m}/2$. This implies $\lim_{A \to A_0} |A| = \infty$.

Equivalently, this can be expressed as

$$\liminf_{A \to A_0} G(A) \geqslant G(A_0).$$

That is, G is lower semi-continuous. It follows that $G \circ F : Y \to \mathbb{N}^*$ is Borel measurable. This implies that $Y_i = (G \circ F)^{-1}(i)$ is Borel measurable for each $i \in \mathbb{N}^*$. □

Let X be a standard Borel space and let $\mathcal{B}(X)$ be the set of Borel sets of X. A subset A of X is analytic if there exist a standard Borel space X', $A' \in \mathcal{B}(X')$ and a measurable map $f : X' \to X$ such that $A = f(A')$. Let μ be a Borel measure on X. A subset A is called a μ-measurable set if $A = B \triangle N$ for some $B \in \mathcal{B}(X)$ and μ-null set N. A subset $A \subseteq X$ is called universally measurable if it is μ-measurable for every Borel measure μ. Lusin's theorem says that every analytic set is universally measurable (see [**23**, Theorem 21.10] for example).

Now, we can prove Theorem 1.5.

Proof of Theorem 1.5. Let (X, T) be a minimal system without any non-trivial Ind_{fip}-$(k+1)$-tuples. Let $\pi : X \to X/\mathbf{RP}^{[\infty]} = Y$ be the factor map and suppose that $m \in M(Y, T)$ is the unique ergodic measure. For $i \in \mathbb{N}^*$, set $Y_i = \{y \in Y : |\pi^{-1}(y)| = i\}$. Clearly, Y_i is T-invariant and is measurable by Lemma 4.1 for every $i \in \mathbb{N}^*$. Since m is ergodic, there exists $j \in \mathbb{N}^*$ such that $m(Y_j) = 1$ and $m(Y_i) = 0$ for all $i \neq j$.

CLAIM 1. *We have $j \leqslant k$.*

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Proof of Claim 1. Suppose for a contradiction that $j \geqslant k+1$. Let $\mu \in M(X,T)$ be an ergodic measure. It follows from Theorem 1.4 that there exists a subset Ω of X with $\mu(\Omega)=1$ such that $\{x\} \times (\pi^{-1}(\pi(x)))^{2^d-1} \subseteq \overline{\mathcal{F}^{[d]}}(x^{[d]})$ for any $x \in \Omega$ and any $d \in \mathbb{N}$. Notice that (Y,T) is uniquely ergodic, we have $\pi_*(\mu)=m$ and, thus, $\mu(\Omega \cap \pi^{-1}(Y_j))=1$. Now, let $x \in \Omega \cap \pi^{-1}(Y_j)$ and let $x_1,\dots,x_{k+1} \in \pi^{-1}(x)$ be distinct points. It follows that $\{x\} \times \{x_1,\dots,x_{k+1}\}^{2^{d+1}-1} \subseteq \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$ for any $d \geqslant 1$ and, thus, $\{x_1,\dots,x_{k+1}\}^{[d]} \subseteq \mathbf{Q}^{[d]}(X)$ for any $d \geqslant 1$, which implies $(x_1,\dots,x_{k+1})$ is an Ind_{fip}-$(k+1)$-tuple by Lemma 2.12, which is a contradiction. From this, we conclude that $j \leqslant k$. □

CLAIM 2. *The system (X,T) has at most j ergodic measures.*

Proof of Claim 2. Suppose for a contradiction that $\mu_1,\dots,\mu_{j+1}$ are distinct ergodic measures of (X,T). Let W_i be the set of all μ_i-generic points for $i \in [j+1]$. By Proposition 2.1, we have $\mu_i(W_i)=1$ for every $i \in [j+1]$. By Lusin's theorem, $\{\pi(W_i)\}_{i=1}^{j+1}$ are universally measurable as they are analytic sets.

Notice that $\pi_*(\mu_1)=\cdots=\pi_*(\mu_{j+1})=m$; hence, we have

$$m(\pi(W_1))=\cdots=m(\pi(W_{j+1}))=1.$$

Set $Y'=\bigcap_{i=1}^{j+1}\pi(W_i)$. It follows that $m(Y')=1$ and $|\pi^{-1}(y)| \geqslant j+1$ for any $y \in Y'$. Thus, $Y' \cap Y_j \neq \emptyset$, which is a contradiction by Claim 2.

This shows Claim 2. □

It follows from Proposition 2.14 that there exists $j' \in [j]$ such that $Y_{j'}$ contains a dense G_δ set, that is, π is an almost j'-to-one extension.

This finishes the proof. □

A. *Appendix. Proof of Lemma 2.17*

In this section, we give the proof of Lemma 2.17.

A.1. *Measure $\mu^{[d]}$.*

Definition A.1. [13, 14] Let $(X,\mathcal{X},\mu,T)$ be an ergodic MPS. Let $\mu^{[1]}=\mu\times\mu$. For $d \in \mathbb{N}$, let $\mathcal{I}_{T^{[d]}}$ be the $T^{[d]}$-invariant σ-algebra of $(X^{[d]},\mathcal{X}^{[d]},\mu^{[d]})$. Define $\mu^{[d+1]}$ to be the relative independent joining of two copies of $\mu^{[d]}$ over $\mathcal{I}_{T^{[d]}}$. That is, for $f_\epsilon \in L^\infty(\mu)$, $\epsilon \in \mathcal{V}_d$:

$$\int_{X^{[d+1]}} \prod_{\epsilon\in\mathcal{V}_d} f_\epsilon(x_\epsilon)\,d\mu^{[d+1]}(\mathbf{x}) = \int_{X^{[d]}} \mathbb{E}\Big(\prod_{\eta\in\mathcal{V}_d} f_{\eta\underline{0}}|\mathcal{I}_{T^{[d]}}\Big)(\mathbf{x})\,\mathbb{E}\Big(\prod_{\eta\in\mathcal{V}_d} f_{\eta\underline{1}}|\mathcal{I}_{T^{[d]}}\Big)(\mathbf{x})\,d\mu^{[d]}(\mathbf{x}).$$

PROPOSITION A.2. [15, Lemmas 5.1, 5.6] *Let (X,T) be a minimal system, $d \in \mathbb{N}$, and $\mu \in M(X,T)$ be an ergodic measure. Then:*

- *$\mu^{[d]}$ is a $\mathcal{G}^{[d]}$-invariant measure on $\mathbf{Q}^{[d]}(X)$;*
- *$\mu^{[d]}(V^{[d]}) \geqslant \mu(V)^{2^d}$ for any measurable subset V of X.*

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A.2. *Topological models.*

Definition A.3. Let $(X, \mathcal{X}, \mu, T)$ be an MPS. We say that the system $(\hat{X}, \hat{\mathcal{X}}, \hat{\mu}, \hat{T})$ is a *topological model* (or just a *model*) for $(X, \mathcal{X}, \mu, T)$ if $(\hat{X}, \hat{T})$ is a TDS and $\hat{\mu}$ is a $\hat{T}$-invariant probability measure on $\hat{\mathcal{X}}$, the Borel σ-algebra of $\hat{X}$, and the systems $(X, \mathcal{X}, \mu, T)$ and $(\hat{X}, \hat{\mathcal{X}}, \hat{\mu}, \hat{T})$ are measure theoretically isomorphic.

THEOREM A.4. [21, Theorem A] *Let $(X, \mathcal{X}, \mu, T)$ be an ergodic MPS and $d \in \mathbb{N}$. Then, it has a strictly ergodic (a TDS is strictly ergodic if it is minimal and has a unique ergodic measure) model $(\hat{X}, \hat{\mathcal{X}}, \hat{\mu}, \hat{T})$ such that:*

(1) $(\mathbf{Q}^{[d]}(\hat{X}), \mathcal{G}^{[d]})$ *is strictly ergodic;*
(2) $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ *is strictly ergodic for all $x \in \hat{X}$.*

A.3. *Measure disintegration.*

THEOREM A.5. [8, Theorem A.7] *Let X and Y be two standard Borel spaces, and $\pi : X \to Y$ a Borel map. Let $\mu \in M(X)$ be a Borel probability measure of X and $\nu = \pi_*(\mu)$ its image in $M(Y)$. Then, there is a Borel map $y \mapsto \mu_y$, from Y into $M(X)$ such that:*

- *for ν-almost every (a.e.) $y \in Y$, $\mu_y(\pi^{-1}(y)) = 1$;*
- *$\mu = \int_Y \mu_y \, d\nu(y)$.*

Moreover, such a map is unique in the following sense. If $y \mapsto \mu'_y$ is another such map, then $\mu_y = \mu'_y$ for ν-a.e. $y \in Y$. The collection $\{\nu_y\}_{y \in Y}$ is called a measure disintegration of μ.

A.4. *Proof of Lemma 2.17.*

THEOREM A.6. *Let (X, T) be a minimal system and $\mu \in M(X, T)$ be an ergodic measure. Then, for every $d \in \mathbb{N}$, there exists a family $\{\mu_x^{[d]} = \delta_x \times \mu_x^{[d]_*}\}_{x \in X}$ of Borel probability measures on $X^{[d]}$ such that the following statements hold:*

(1) *for μ-a.e. $x \in X$, $\mu_x^{[d]}$ is $\mathcal{F}^{[d]}$-ergodic and $\mu_x^{[1]} = \delta_x \times \mu$;*
(2) *$\mu^{[d]} = \int_X \mu_x^{[d]} \, d\mu$;*
(3) *the map $X \to M(X^{[d]})$, $x \mapsto \mu_x^{[d]}$ is measurable;*
(4) *for any $k \in [d]$ and any morphism $p : \mathcal{V}_k \to \mathcal{V}_d$, one has $\bar{p}_*(\mu_x^{[d]}) = \mu_x^{[k]}$ for μ-a.e. $x \in X$.*

Proof. It follows from Theorem A.4 that there exists a strictly ergodic model $(\hat{X}, \hat{\mathcal{X}}, \hat{\mu}, \hat{T})$ such that:

- $(\mathbf{Q}^{[d]}(\hat{X}), \mathcal{G}^{[d]})$ is strictly ergodic;
- $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is strictly ergodic for all $x \in \hat{X}$.

By Proposition A.2, $\hat{\mu}^{[d]}$ is a $\mathcal{G}^{[d]}$-invariant measure on $\mathbf{Q}^{[d]}(\hat{X})$, which implies that $\hat{\mu}^{[d]}$ is the unique ergodic measure on $(\mathbf{Q}^{[d]}(\hat{X}), \mathcal{G}^{[d]})$. For $x \in \hat{X}$, let $\hat{\mu}_x^{[d]}$ be the unique ergodic measure on $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$. Notice that for any $\mathbf{x} \in \overline{\mathcal{F}^{[d]}}(x^{[d]})$, the $\vec{0}$-component of $\mathbf{x}$ is x. We write $\hat{\mu}_x^{[d]} = \delta_x \times \hat{\mu}_x^{[d]_*}$ for some Borel probability measure $\hat{\mu}_x^{[d]_*}$ on $\hat{X}_*^{[d]} = \hat{X}^{2^d - 1}$. In particular, when $d = 1$, we get that $(\overline{\mathcal{F}^{[1]}}(x^{[1]}), \mathcal{F}^{[1]}) = (\{x\} \times \hat{X}, \mathrm{id} \times \hat{T})$ and, thus, $\hat{\mu}_x^{[1]} = \delta_x \times \hat{\mu}$ for every $x \in \hat{X}$.

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CLAIM 1. *We have* $\hat{\mu}^{[d]} = \int_{\hat{X}} \hat{\mu}_x^{[d]}\, d\hat{\mu}$.

Proof of Claim 1. Recall that $(\mathbf{Q}^{[d]}(\hat{X}), \mathcal{G}^{[d]})$ is uniquely ergodic and

$$\left(\int_{\hat{X}} \hat{\mu}_x^{[d]}\, d\hat{\mu}\right)(\mathbf{Q}^{[d]}(\hat{X})) = \int_{\hat{X}} \hat{\mu}_x^{[d]}(\mathbf{Q}^{[d]}(\hat{X}))\, d\hat{\mu} = 1,$$

it suffices to show that $\int_{\hat{X}} \hat{\mu}_x^{[d]}\, d\hat{\mu}$ is $\mathcal{G}^{[d]}$-invariant. Notice that every $\hat{\mu}_x^{[d]}$ is $\mathcal{F}^{[d]}$-invariant and, thus, so is $\int_{\hat{X}} \hat{\mu}_x^{[d]}\, d\hat{\mu}$.

As $\hat{T}^{[d]}\hat{\mu}_x^{[d]}$ is an $\mathcal{F}^{[d]}$-ergodic measure on $\hat{T}^{[d]}\overline{\mathcal{F}^{[d]}}(x^{[d]}) = \overline{\mathcal{F}^{[d]}}((\hat{T}x)^{[d]})$ and notice $(\overline{\mathcal{F}^{[d]}}((\hat{T}x)^{[d]}), \mathcal{F}^{[d]})$ is uniquely ergodic, we deduce that $\hat{T}^{[d]}\hat{\mu}_x^{[d]} = \hat{\mu}_{\hat{T}x}^{[d]}$. This implies that

$$\hat{T}^{[d]}\hat{\mu}^{[d]} = \int_{\hat{X}} \hat{T}^{[d]}\hat{\mu}_x^{[d]}\, d\hat{\mu} = \int_{\hat{X}} \hat{\mu}_{\hat{T}x}^{[d]}\, d\hat{\mu} = \int_{\hat{X}} \hat{\mu}_x^{[d]}\, d\hat{\mu} = \hat{\mu}^{[d]}.$$

Since $\mathcal{G}^{[d]}$ is generated by $\mathcal{F}^{[d]}$ and $\hat{T}^{[d]}$, we have $\int_{\hat{X}} \hat{\mu}_x^{[d]}\, d\hat{\mu}$ is $\mathcal{G}^{[d]}$-invariant and, thus,

$$\hat{\mu}^{[d]} = \int_{\hat{X}} \hat{\mu}_x^{[d]}\, d\hat{\mu}.$$

This shows Claim 1. □

CLAIM 2. *The map* $\hat{X} \to M(\hat{X}^{[d]})$, $x \mapsto \hat{\mu}_x^{[d]}$ *is measurable.*

Proof of Claim 2. Let $\theta : \mathbf{Q}^{[d]}(\hat{X}) \to \hat{X}$ be the projection to the $\vec{0}$-coordinate. Then, $\theta_*(\hat{\mu}^{[d]})$ is a $\hat{T}$-invariant measure on X and, thus, $\theta_*(\hat{\mu}^{[d]}) = \hat{\mu}$ as $(\hat{X}, \hat{T})$ is uniquely ergodic.

Set $\mathbf{Q}_x^{[d]}(\hat{X}) = (\{x\} \times \hat{X}^{2^d-1}) \cap \mathbf{Q}^{[d]}(\hat{X})$ for $x \in \hat{X}$. We have $\overline{\mathcal{F}^{[d]}}(x^{[d]}) \subseteq \mathbf{Q}_x^{[d]}(\hat{X})$ and, thus, $\hat{\mu}_x^{[d]}(\mathbf{Q}_x^{[d]}(\hat{X})) = 1$.

By Claim 1 and Theorem A.5, we conclude that the map $\hat{X} \to M(\hat{X}^{[d]})$, $x \mapsto \hat{\mu}_x^{[d]}$ is measurable. This shows Claim 2. □

CLAIM 3. *For any* $x \in \hat{X}$, $k \in [d]$ *and morphism* $p : \mathcal{V}_k \to \mathcal{V}_d$, *one has* $\bar{p}_*(\hat{\mu}_x^{[d]}) = \hat{\mu}_x^{[k]}$.

Proof of Claim 3. Let $k \in [d]$, $x \in \hat{X}$ and let $p : \mathcal{V}_k \to \mathcal{V}_d$ be a morphism. Then, there exist pairwise disjoint non-empty subsets $\epsilon_1, \ldots, \epsilon_k$ of $[d]$ such that $p(\emptyset) = \emptyset$ and $p(\eta) = \bigcup_{i\in\eta} \epsilon_i$ for $\eta \in \mathcal{V}_k^*$. For any $\vec{n} \in \mathbb{Z}^d$, we have

$$\bar{p}((\hat{T}^{\vec{n}\cdot\epsilon}x)_{\epsilon\in\mathcal{V}_d}) = (\hat{T}^{\vec{n}\cdot(\sum_{i\in\eta}\epsilon_i)})_{\eta\in\mathcal{V}_k},$$

which implies that $\bar{p}(\overline{\mathcal{F}^{[d]}}(x^{[d]})) = \overline{\mathcal{F}^{[k]}}(x^{[k]})$. Hence, $\bar{p}_*(\hat{\mu}_x^{[d]})$ is an $\mathcal{F}^{[k]}$-invariant measure on $\overline{\mathcal{F}^{[k]}}(x^{[k]})$.

Observe that $(\overline{\mathcal{F}^{[k]}}(x^{[k]}), \mathcal{F}^{[k]})$ is also strictly ergodic, we deduce that $p_*(\hat{\mu}_x^{[d]}) = \hat{\mu}_x^{[k]}$.

This shows Claim 3. □

Let $\phi : X \to \hat{X}$ be the isomorphism. Then, $\phi^{[d]_*} : X_*^{[d]} \to \hat{X}_*^{[d]}$ is also an isomorphism. Define $\mu_x^{[d]_*} = \hat{\mu}_{\phi(x)}^{[d]_*} \circ \phi^{[d]_*}$ for $x \in X$. It is easy to check that $\{\mu_x^{[d]} = \delta_x \times \mu_x^{[d]_*}\}_{x\in X}$ is well defined and also satisfies statements (1)–(4).

This finishes the proof. □

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Proof of Lemma 2.17. Let $\{\mu_x^{[d]} = \delta_x \times \mu_x^{[d]_*}\}_{x\in X, d\in\mathbb{N}}$ be as in Theorem A.6. Then, there exists a subset Ω of X with full measure such that for any $x \in \Omega$ and any $d \in \mathbb{N}$:

- $\mu_x^{[d]}$ is $\mathcal{F}^{[d]}$-ergodic and $\mu_x^{[1]} = \delta_x \times \mu$;
- $\bar{p}_*(\mu_x^{[d]}) = \mu_x^{[k]}$ for any $k \in [d]$ and any morphism $p : \mathcal{V}_k \to \mathcal{V}_d$.

Moreover, we have:

- $\mu^{[d]} = \int_X \mu_x^{[d]}\, d\mu$;
- the map $X \to M(X^{[d]})$, $x \mapsto \mu_x^{[d]}$ is measurable.

CLAIM 1. *Let $k \in [d]$ and $p : \mathcal{V}_k \to \mathcal{V}_d$ be a morphism. Let $W_\epsilon, \epsilon \in \mathcal{V}_d$ be measurable subsets of X and $x \in \Omega$. Then,*

$$\mu_x^{[k]}\Big(\prod_{\eta\in\mathcal{V}_k} W_{p(\eta)}\Big) \geqslant \mu_x^{[d]}\Big(\prod_{\epsilon\in\mathcal{V}_d} W_\epsilon\Big).$$

Proof of Claim 1. Let

$$W'_\epsilon = \begin{cases} W_{p(\eta)} & \text{if } \epsilon = p(\eta) \text{ for some } \eta \in \mathcal{V}_k, \\ X & \text{otherwise.} \end{cases}$$

Since $x \in \Omega$, we have $\bar{p}_*(\mu_x^{[d]}) = \mu_x^{[k]}$ and, thus,

$$\mu_x^{[k]}\Big(\prod_{\eta\in\mathcal{V}_k} W_{p(\eta)}\Big) = \mu_x^{[d]}\Big(\prod_{\epsilon\in\mathcal{V}_d} W'_\epsilon\Big) \geqslant \mu_x^{[d]}\Big(\prod_{\epsilon\in\mathcal{V}_d} W_\epsilon\Big),$$

as $W_\epsilon \subseteq W'_\epsilon, \epsilon \in \mathcal{V}_d$. This shows Claim 1. □

CLAIM 2. *For any measurable subset A of X and $d \in \mathbb{N}$, there exists a measurable set $A' \subseteq A$ with $\mu(A') = \mu(A)$ such that $\mu_x^{[d]}(A^{[d]}) > 0$ for any $x \in A'$. (We remark this claim was also proved in* [**15**, Lemma 5.4].*)*

Proof of Claim 2. Let $B = \{x \in A : \mu_x^{[d]}(A^{[d]}) = 0\}$. It follows from Theorem A.6(3) that B is measurable. Then, we have

$$\begin{aligned} 0 &= \int_B \mu_x^{[d]}(A^{[d]})d\mu \\ &= \int_X \mu_x^{[d]}(B \times A^{2^d-1})\, d\mu \qquad (\text{as } \mu_x^{[d]} = \delta_x \times \mu_x^{[d]_*}) \\ &= \mu^{[d]}(B \times A^{2^d-1}) \qquad (\text{as } \mu^{[d]} = \int_X \mu_x^{[d]}\, d\mu) \\ &\geqslant \mu^{[d]}(B \times B^{2^d-1}) \qquad (\text{as } B \subseteq A) \\ &\geqslant \mu(B)^{2^d}, \qquad (\text{by Proposition A.2}), \end{aligned}$$

which implies $\mu(B) = 0$.

This shows the claim. □

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Let $\{V_i\}_{i\in\mathbb{N}}$ be a countable base for the topology of X. For every $i \in \mathbb{N}$, it follows from Claim 2 that there exists a measurable set $U_{d,i} \subseteq M \cap V_i$ with $\mu(U_{d,i}) = \mu(M \cap V_i)$ such that $\mu_x^{[d]}((M \cap V_i)^{[d]}) > 0$ for every $x \in U_{d,i}$.

Set $M_0 = \Omega \cap (\bigcap_{d\in\mathbb{N},i\in\mathbb{N}} U_{d,i} \cup (M\backslash V_i))$. Then, we have $\mu(M_0) = \mu(M)$. We next show that M_0 meets our requirement.

Let $x \in M_0$ and $d \in \mathbb{N}$. Then, x satisfies statements (1) and (2) as $x \in \Omega$. Let V be a neighborhood of x. Recall that $\{V_i\}_{i\in\mathbb{N}}$ is a base for the topology of X, we can choose V_i with $x \in V_i \subseteq V$, which implies that

$$\mu_x^{[d]}((M \cap V)^{[d]}) \geqslant \mu_x^{[d]}((M \cap V_i)^{[d]}) > 0.$$

This completes the proof. □

Acknowledgements. The authors would like to express their gratitude to Professors Wen Huang, Song Shao, Hui Xu and Xiangdong Ye for their insightful discussions and valuable comments. This research is supported by National Key R&D Program of China (Nos. 2024YFA1013601, 2024YFA1013600), USTC Research Funds of the Double First-Class Initiative (YD0010002009), and NNSF of China (12401243, 12426201, and 12471188).

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