CYCLIC ARTINIAN MODULES WITHOUT A COMPOSITION SERIES

P. M. COHN

To the memory of Brian Hartley

1. *Introduction*

Let R be a ring (always understood to be associative with a unit element 1). It is well known that an R-module is Noetherian if and only if all its submodules are finitely generated and that it has a finite composition series if and only if it is Noetherian and Artinian. This raises the question whether every finitely generated Artinian module is Noetherian; here it is enough to consider cyclic Artinian modules, by an induction on the length. This question has been answered (negatively) by Brian Hartley [**5**], who gives a construction of an Artinian uniserial module of uncountable composition-length over the group algebra of a free group of countable rank. If we are just interested in finding cyclic modules that are Artinian but not Noetherian, there is a very simple construction based on the fact that over a free algebra every countably generated Artinian module can be embedded in a cyclic module which is again Artinian. This is described in §2 below.

Of course it is clear on general grounds that if examples of such modules exist, they can be found over free algebras; our object here is to describe a class of rings that leads naturally to such examples. We shall find that cyclic Artinian but not Noetherian modules exist over every 2-fir which is a local ring and has right but not left ACC_1. Examples of such rings can be found in [**1**, 2.10], or also in [**4**]. Some simple examples are given in §3 below.

The ring construction is necessarily asymmetric since over a commutative ring every cyclic Artinian module is Noetherian. For, any such module over a commutative ring A has the form A/I for an ideal I, so on replacing A by A/I we may take our module to be A itself. If this is Artinian, it must be Noetherian, as is well known, see, for example, [**3**, Theorem 9.9.9, p. 348].

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2. *The embedding theorem for countably generated modules*

Let $R = k\langle x_1, x_2, \ldots\rangle$ be the free k-algebra over a field k on a countable free generating set $x_1, x_2, \ldots$. The ground field k itself will be regarded as an R-module by letting each x_i act as zero. We shall find that every countably generated Artinian R-module can be embedded in a cyclic R-module which is again Artinian. More precisely, we have the following.

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PROPOSITION 1. *Let R be the free k-algebra of countable rank, as above, and let M be a countably generated right R-module. Then there is a cyclic R-module P with a submodule M' isomorphic to M such that $P/M' \cong k$. Moreover, if M is Artinian, then so is P.*

Proof. Let $v_1, v_2, \ldots$ be a countable generating set of M; we adjoin an element u to M and define

$$ux_i = v_i \quad \text{for } i = 1, 2, \ldots. \tag{1}$$

This is possible, because the x_i are free generators of R. If the module generated by $u, v_1, v_2, \ldots$ is denoted by P, then it is clear from (1) that P is generated by u alone and so is cyclic. Moreover, every element of P can be uniquely expressed as $\alpha u + x$, where $\alpha \in k, x \in M$, so $P/M \cong k$, where k is regarded as the R-module R/R_1 with R_1 the ideal generated by all the x_i (the augmentation ideal). In particular, P/M is Artinian, hence if M is Artinian, then so is P.

If the module M of Proposition 1 is not finitely generated, then it cannot be Noetherian, since the ascending sequence of submodules $\langle v_1\rangle, \langle v_1, v_2\rangle, \ldots$ does not break off. For if equality were to hold from the nth term onwards, then $\langle v_1, \ldots, v_n\rangle = M$ and M would be finitely generated. So when M is not finitely generated, P is cyclic Artinian but not Noetherian. Such a module M is easily formed by putting

$$v_i x_j = \begin{cases} v_{i-1} & \text{for } i > 1, \\ 0 & \text{for } i = 1. \end{cases}$$

for all x_j.

Clearly M is the union of the chain of submodules $M_n = \langle v_1, \ldots, v_n\rangle = \langle v_n\rangle$, but $v_{n+1} \notin M_n$; hence M is not finitely generated and so is not Noetherian. To show that M is Artinian, take a non-zero submodule N of M, say N contains $f = \alpha_1 v_1 + \cdots + \alpha_n v_n$ (for $\alpha_i \in k$), where $\alpha_n \neq 0$; on multiplying by α_n^{-1} we may suppose that $\alpha_n = 1$. By acting with x_1^{n-1} we find that $v_1 \in N$. Next we act with x_1^{n-2} and subtract $\alpha_{n-1} v_1$; this shows that $v_2 \in N$ and continuing in this way we see that N contains M_n, and M_n is an n-dimensional k-space with basis $v_1, \ldots, v_n$. Now, either N contains elements involving v_n for arbitrarily large n; then the above reasoning shows that $N = M$. Or there exists n such that the elements of N only involve v_i for $i \leqslant n$ and so $N \subseteq M_n$ for some n; then N as submodule of M_n is finite-dimensional and so has a composition series. It follows that M is Artinian, as claimed.

A similar construction can be carried out for modules over a free algebra of rank 2. For if $R = k\langle x, y\rangle$, we can define M as the R-module generated by $v_1, v_2, \ldots$ with defining relations

$$v_i xy = \begin{cases} v_{i-1} & \text{for } i > 1, \\ 0 & \text{for } i = 1. \end{cases}$$

As before we can verify that M is Artinian but not finitely generated and so not Noetherian. Now we form P by adjoining an element u with defining relations $ux = 0$, $uy^n x = v_n$ for $n = 1, 2, \ldots$. Then $P/M \cong k$ as before, and we have a statement as in Proposition 1.

This is probably the simplest construction (it is contained as a special case in Hartley's construction), but it tells us nothing about the properties of the ring that are needed, since every module can be regarded as a module over a free ring. We shall therefore in the next section take a closer look at the type of ring involved.

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3. *The case of* 2*-firs*

We recall that a 2*-fir* is an integral domain in which any two elements with a common non-zero right multiple generate a right ideal which is principal [**1**, p. 67]; this notion is left-right symmetric even though the definition is not. We also recall that *right* ACC_1 in a ring (or monoid) is the ascending chain condition for principal right ideals; similarly for DCC_1, the descending chain condition, and corresponding conditions can be defined on the left. By the *restricted* DCC we understand the DCC on the set of right (respectively, left) ideals containing a fixed non-zero element; similarly for the restricted DCC_1. For any ring R, the set of all non-zero elements of R is denoted by $R^\times$.

If R is an integral domain and $a \in R^\times$, then any series of factorizations

$$a = a_1 p_1 = a_2 p_2 p_1 = \cdots = a_n p_n \cdots p_1 = \ldots,$$

where the p_i are non-units, gives rise to a strictly ascending chain of principal right ideals $aR \subset a_1 R \subset a_2 R \ldots$, and so it must break off whenever R satisfies right ACC_1. It also yields a descending chain of principal left ideals $R \supset Rp_1 \supset Rp_2 p_1 \supset \cdots \supset Ra$, and this shows that in an integral domain, the restricted left DCC_1 implies the right ACC_1 and it is easy to see that this is in fact an equivalence. This is just an instance of the factorial duality described in [**1**, p. 166].

For the next lemma we recall a definition and a result characterizing 2-firs that are local rings. An integral domain R is called *rigid* if the left divisors of any element are totally ordered by divisibility, that is, $ab' = ba' \neq 0$ implies that $a = bc$ or $b = ac$ for some $c \in R$. It is not hard to prove that an integral domain is rigid if and only if it is a 2-fir and a local ring (that is, all non-units form an ideal), see [**1**, Theorem 3.4.7, p. 172]. We also recall that an element c of R is called *left large* if Rc is a large left ideal, that is, it meets every non-zero left ideal of R.

LEMMA 2. *Let R be a* 2*-fir and a local ring. If R satisfies the right ACC_1 then it satisfies the left DCC, for any chain containing a fixed left large element.*

Proof. By the result just quoted R is rigid; thus the principal left ideals containing a given non-zero left ideal form a chain. Further, given any non-zero elements a, b of R, if a is left large, then $Ra \cap Rb \supseteq Rc \neq 0$, so either $Rb \supseteq Ra$ or $Ra \supseteq Rb$. Now consider a strictly descending chain of left ideals of R containing a left large element a of R: $I_1 \supset I_2 \supset \cdots \supset Ra$. Take $b_i \in I_i \setminus I_{i+1}$; then $I_i \supseteq Rb_i$ and for any $c \in I_{i+1} \setminus Ra$, we have that Rb_i and Rc contain Ra and $b_i \notin Rc$, hence $c \in Rb_i$ by rigidity, so $Rb_i \supset I_{i+1}$ and we obtain the chain $I_i \supseteq Rb_1 \supset I_2 \supseteq Rb_2 \supset \cdots \supset Ra$.

By the factorial duality the chain Rb_i must break off, hence the chain of the I_i also breaks off.

This lemma will lead to a construction of examples using right principal Bezout domains, but we have to keep in mind that the rings used cannot also be left principal (for then all non-free cyclic modules would have finite length), nor can they be local, as is shown by the next result.

THEOREM 3. *Let R be a Bezout domain which is right principal and a local ring. Then R is also left principal.*

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Proof. Since R is local, the principal right ideals form a chain, as do the principal left ideals. Thus R satisfies right ACC_1; if left ACC_1 also holds, R is left principal, so we may assume that left ACC_1 fails to hold. We first show that for any $a, b \in R^\times$, if b is a non-unit, then

$$ab = ca \quad \text{for some } c \in R, \text{ not zero or a unit.} \tag{2}$$

For we have $Ra \subseteq Rab$ or $Rab \subset Ra$. If the former holds, then

$$a = cab = c^2ab^2 = \cdots,$$

hence $aR \subset caR \subset c^2aR \subset \ldots$, with proper inclusions because b is a non-unit. But this contradicts right ACC_1, hence $Rab \subset Ra$, and so $ab = ca$ with a non-unit c, as claimed.

To complete the proof of the theorem, we remark that since the unique maximal ideal is principal as right ideal, pR say, we have, for any $a \in R$, by (2), $ap = p'a = pa'$ for some $a' \in R$. We consider two cases. Suppose first that $\bigcap p^nR = 0$; then every element of $R^\times$ has the form p^nu, where $n \geqslant 0$ and u is a unit, so we can define a valuation v on R by putting $v(0) = \infty, v(p^nu) = n$. Given $a, b \in R^\times$, say $a = p^nu, b = p^mv$, where u, v are units, we have $up^m = pu'p^{m-1} = \cdots = p^mu^*$ for some $u^* \in R$, and it follows that $v(ab) \geqslant v(a) + v(b)$. Further it is clear that $v(c) = 0$ if and only if c is a unit, so $v(c)$ must be a positive integer for any non-unit $c \neq 0$. Now given any $Rb \supset Ra \neq 0$, say $a = cb$, where c is a non-unit, we have

$$v(a) = v(cb) \geqslant v(c) + v(b) > v(b),$$

hence in any strictly ascending chain $Ra_1 \subset Ra_2 \subset \ldots$ the values are strictly decreasing positive integers, so the chain must break off.

There remains the case when $\bigcap p^nR \neq 0$; then $\bigcap p^nR = xR$ for some $x \in R$. We now have $x = px_1 = p^2x_2 = \cdots$, where all the x_i are non-units. By (2) it follows that $p^nx_n = y_np^n$ for a non-unit y_n. Thus $x = y_np^n$ for $n = 1, 2, \ldots$, and so we have $y_1R \subset y_2R \subset \cdots$, which again contradicts right ACC_1.

In order to obtain our example we shall need the fact that in a right principal Bezout domain R every element c, which is not zero or a unit, has an atomic left factor, where an *atomic element* or *atom* is a non-unit which cannot be written as a product of two non-units. To see this result, we note that cR is a proper right ideal, contained in a maximal proper right ideal pR and p is the required atomic left factor of c.

THEOREM 4. *Let R be a Bezout domain which is right principal such that, for every element not zero or a unit, the number of possible atomic left factors is finite* (*up to associates*). *Then any set of left ideals of R containing a fixed non-zero element satisfies* DCC.

Proof. Consider a chain of left ideals with non-zero intersection

$$I_1 \supset I_2 \supset \cdots, \tag{3}$$

$$\bigcap I_n = I \supseteq Ra \neq 0. \tag{4}$$

Choose $a_1' \in I_1 \setminus I_2$ and let $Ra_1' + Ra = Ra_1$; then $a_1 \in I_1 \setminus I_2$ and $a = c_1a_1$ for some $c_1 \in R$. Suppose first that $Ra_1 \cap I_r \not\subseteq I_{r+1}$ for infinitely many r; by going over to a subchain of (3) we can ensure that this holds for all r. By induction assume that

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$a_i \in I_i \backslash I_{i+1}$ with $Ra_i \supset Ra$ for $i < n$; if $Ra_{n-1} \cap I_r \not\subseteq I_{r+1}$ for infinitely many $r \geqslant n$, then we can find $a_n \in Ra_{n-1} \cap (I_n \backslash I_{n+1})$ with $Ra_n \supseteq Ra$ and continue in this way. Thus $a_n = b_{n-1} a_{n-1}$, where b_{n-1} is a non-unit, because otherwise $Ra_n = Ra_{n-1}$, but $a_{n-1} \notin I_n, a_n \in I_n$. So the b_i are non-units and if

$$a = c_n a_n = c_n b_{n-1} a_{n-1} = \cdots = c_n b_{n-1} b_{n-2} \cdots b_1 a_1,$$

then $c_1 = c_2 b_1 = c_3 b_2 b_1 = \ldots$, hence $c_1 R \subset c_2 R \subset \ldots$, which is a contradiction.

It follows that there is an $r \geqslant 1$ such that $Ra_r \cap I_n \subseteq I_{n+1}$ for all but finitely many n, say for $n \geqslant n_0$. Omitting the first n_0 of the I_i and renumbering the rest, we have $Ra_1 \cap I_n \subseteq I_{n+1}$ for all n, so $Ra_1 \cap I_n \subseteq \bigcap I_{n+r} = I$ and so $Ra_1 \cap I_n = I$ for all large n. Further we have

$$I_n / I \cong I_n / (Ra_1 \cap I_n) \cong (Ra_1 + I_n) / Ra_1,$$

and so, on replacing I_n by $Ra_1 + I_n$, we obtain a chain as in (3) with an intersection which is principal:

$$I_1 \supset I_2 \supset \ldots, \quad \text{and} \quad \bigcap I_n = Ra.$$

By hypothesis a has only finitely many atomic left factors, say $a = p_i a_i$, where p_i for $i = 1, \ldots, r$ are non-associated atoms. For each i there is an $n(i)$ such that $Ra_i \not\subseteq I_{n(i)}$. Let $m = \max\{n(1), \ldots, n(r)\}$; then $Ra_i \not\subseteq I_m$ for $i = 1, \ldots, r$. Next take $c \in I_m$ such that $a \in Rc$; then $a = b'c$ for some b', which for suitable choice of c will be a non-unit. If p is an atomic left factor of b', we have $b' = pb$, hence $a = pbc$, so p must be associated to one of the p_i, say $p = p_1$, by adjusting unit factors. This means that $a_1 = bc$ and so $Ra_1 \subseteq Rc \subseteq I_m$, which is a contradiction, and it proves that the chain (3) breaks off.

To get an example of the required form, let A be the k-algebra (over a commutative field k) generated by elements y, z_n (with $n \in \mathbb{Z}$) with defining relations $z_{n-1} = yz_n$. From the results of [**1**, p. 142], it follows that A is a right fir, but it is not a left fir, as is shown by the ascending chain of principal left ideals $Az_0 \subset Az_1 \subset \cdots$. If we take the coproduct of A with the free algebra $k\langle X \rangle$ on an infinite set X and then localize at the set of all full matrices totally prime to y, z_n (with $n \in \mathbb{Z}$), we obtain a simple Bezout domain R which is right but not left principal, because z_0 has the factorizations $z_0 = y^n z_n$. Briefly, the effect of the localization is to make any (non-zero) non-unit invertible unless it has a factor similar to y or z_n (see [**2**, **4**] for details). We remark that R is a ring with a single atom y. Thus R is a ring such that R/Rz_0 is cyclic Artinian but not Noetherian, as the chain $Rz_0 \subset Rz_1 \subset \cdots$ shows.

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Department of Mathematics
University College London
Gower Street
London WC1E 6BT

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