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The S-unit equation over function fields

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In the study of integral solutions to Diophantine equations, many problems can be reduced to that of solving the equation

$$u+v=1$$

in S-units of the given ring. To accomplish this over number fields, the only known effective method is to use Baker's deep results on linear forms in logarithms, which yield relatively weak upper bounds. For function fields, R. C. Mason [2] has recently given a remarkably strong effective upper bound. In this note we give an independent proof of Mason's bound, relying only on elementary algebraic geometry, principally the Riemann–Hurwitz formula.

We start with some notation:

k	an algebraically closed field of characteristic p (possibly $p=0$),
C/k	a smooth projective curve of genus g,
K	the function field of C,
S	a finite set of points of $C(k)$,
$\mathcal{O}_S$	the ring of S-integers in K (i.e. functions whose poles lie in S),
$\deg(f)$	for $f\in K$, the degree of the map $f\colon C\to\mathbb{P}^1$,
$\deg_s$	separable degree.

THEOREM. *Let u and v be S-units satisfying*

$$u+v=1.$$

Then either $u, v\in k^*$ or

$$\deg_s(u)\leqslant 2g-2+|S|.$$

Proof. Assume that u and v are not elements of k.

Let n be an integer prime to p (arbitrary if $p=0$). Let

$$x=u^{1/n},\quad y=v^{1/n}\quad\text{and}\quad K'=K(x,y).$$

Then K' is the function field of some non-singular projective curve C'/k of genus g', and we have the map

$$\phi\colon C'\to C$$

corresponding to the inclusion of fields $K\subset K'$. Note that the degree of ϕ divides n^2, so ϕ is separable and never wildly ramified. Further, since u and v are S-units, the only points of $C(k)$ at which ϕ may ramify are points of S. Hence the Hurwitz genus formula ([1], IV, cor. 2·4) yields

$$\begin{aligned}2g'-2&=(\deg\phi)(2g-2)+\sum_{p\in S}[e_p(\phi)-1]\\&\leqslant(\deg\phi)(2g-2+|S|).\end{aligned}\tag{1}$$

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On the other hand, there is a natural map from C' to the Fermat curve

$$F_n\colon X^n + Y^n = 1$$

given by

$$\psi\colon C' \to F_n,$$
$$P' \to (x(P'), y(P')).$$

If ψ is separable, then we may again apply the Hurwitz formula to obtain

$$2g' - 2 = (\deg \psi)(2g(F_n) - 2) + \deg(\text{ramification divisor}),$$

and so

$$2g' - 2 \geqslant (\deg_s \psi)(n^2 - 3n) \tag{2}$$

since F_n has genus $(n-1)(n-2)/2$. However, when ψ is inseparable, (2) is still valid, because ψ may be factored into a purely inseparable map followed by a separable map, and for the former the genus does not change ([1], IV, Prop. 2·5).

Finally, let λ be the separable map of degree n^2 given by

$$\lambda\colon F_n \to \mathbb{P}^1,$$
$$(X, Y) \to X^n.$$

Then we have a commutative diagram

$$\begin{array}{ccc} C' & \xrightarrow{\phi} & C \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle u} \\ F_n & \xrightarrow{\lambda} & \mathbb{P}^1. \end{array}$$

Since ϕ and λ are separable, taking separable degrees gives

$$n^2(\deg_s \psi) = (\deg_s u)(\deg \phi). \tag{3}$$

Combining (1), (2), and (3) yields

$$(1 - 3/n)(\deg_s u) \leqslant 2g - 2 + |S|.$$

Letting $n \to \infty$ (through integers prime to p if $p > 0$) gives the desired result.

Remark. It is worth mentioning that, for $g = 0$, $C = \mathbb{P}^1$, $K = k(T)$, one can easily give examples for which the upper bound of the theorem is attained and S is large. Thus let n be prime to p (if $p > 0$), $\{\delta_1, \ldots, \delta_n\}$ the set of nth roots of unity, and

$$S = \{0, \infty, \delta_1, \ldots, \delta_n\}.$$

Then T^n and $1 - T^n$ are S-units, and

$$\deg(T^n) = n = 2g - 2 + |S|.$$

It would be of interest to know if there are similar examples for curves of higher genus.

REFERENCES

[1] R. HARTSHORNE. *Algebraic Geometry* (Springer-Verlag, 1977).
[2] R. C. MASON. The hyperelliptic equation over function fields. *Math. Proc. Cambridge Philos. Soc.* **93** (1983), 219–230.

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