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Operations Which Preserve Path-Width at Most Two

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The number of excluded minors for the class of graphs with path-width at most two is very large. To give a practical characterization of the obstructions, we introduce some operations which preserve path-width at most two. We give a list of ten graphs such that any graph with path-width more than two can be reduced – by taking minors and applying our operations – to one of the graphs on our list. We think that our operations and excluded substructures give a far more transparent description of the class of graphs with path-width at most two than Kinnersley and Langston's characterization by 110 excluded minors (see [4]).

1. History and introduction

Kuratowski's theorem says that a graph is planar if and only if it has no minor isomorphic to K_5 or $K_{3,3}$. This theorem gives a good characterization of planarity. There is a large collection of similar results in graph theory, called excluded minor theorems. Actually, any minor-closed class of graphs can be characterized by a list of so-called excluded minors. This fact is a consequence of the celebrated theorem of Robertson and Seymour [8], which states that the list of excluded minors is necessarily finite. Minor-closed families of graphs naturally arise in graph theory: these include outerplanar, series-parallel, planar, and linklessly embeddable graphs. Any minor-monotone graph parameter (*e.g.*, the genus, the tree-width, the path-width, or the Colin de Verdière number of the graph) defines a sequence of minor-closed families by limiting from above the value of the parameter by any natural number t.

As we increase the value of t, the class grows and usually the number of excluded minors increases rapidly with t. So the determination of the excluded minors becomes increasingly

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difficult. The class of graphs with tree-width at most three requires four excluded minors [1]. D. P. Sanders, in his PhD thesis, determined more than 75 minimal forbidden minors for tree-width at most four, but the list was still incomplete. The classification of planar graphs requires two excluded minors. However, nobody has undertaken the task of explicitly listing the excluded minors for toroidal graphs. The class of graphs with path-width at most one requires two excluded minors. N. G. Kinnersley, in her PhD thesis, determined the 110 minimal forbidden minors for path-width at most two. It is known [10] that the number of excluded tree minors for the class of graphs with path-width at most t grows superexponentially.

It seems that even a very small limit on natural minor-monotone graph parameters creates classes with extreme complexity. However, the classes considered are important, since many NP-complete optimization problems can be solved efficiently for their members. Their recognition and representation problems are important.

But there is hope. Our main goal is to demonstrate that characterizations by excluded minors are very often inappropriate for certain classes. We introduce new operations, particular for the class of graphs with path-width at most two. Based on these operations, we can significantly reduce the number of excluded structures.

The novelty of our approach is that it gives a much better insight into the considered class than an overwhelming list of more than 100 graphs. Further, in [4], Kinnersley and Langston wrote the following about finding the excluded minors for path-width two: 'To assist in this heroic undertaking, massive computational power was used to verify that each obstruction represents a circuit that has no three-track layout, and to check that each proper minor of each obstruction represents a circuit that does have a three-track layout.' Checking by computer proves the correctness of the result, but it does not give a good understanding of the considered theorem. In contrast, we do not need computers to handle huge lists: our operations do the job for us. In Kinnersley and Langston's list, many graphs look alike, but the computer handles each of them independently. Our operations exhibit them as simple variations of some base graphs.

2. Notation and definitions

We only consider finite simple graphs. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. A *path-decomposition* of a graph G is a pair (P, W), where P is a path and $W = (W_p : p \in V(P))$ is a family of subsets of $V(G)$, satisfying

(1) $\bigcup_{p \in V(P)} W_p = V(G)$, and every edge of G has both ends in some W_p, and
(2) if $p, p', p'' \in V(P)$, and p' lies on the path from p to p'', then $W_p \cap W_{p''} \subseteq W_{p'}$.

The *width* of a path-decomposition is $\max(|W_p| - 1 : p \in V(P))$, and the *path-width* of G ($\mathrm{pw}(G)$ in notation) is the minimum width of all path-decompositions of G.

There are several alternative definitions of path-width (see [6], [7], [11]). For us, one is especially important: an *interval representation of a graph* G (or simply representation of G) is a function ϱ that assigns closed intervals of a line l to the vertices of G such that adjacent vertices correspond to intersecting intervals. We say that a point P of l is

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covered m times in a representation if and only if P is an element of exactly m intervals assigned to vertices. The width of the representation is the maximal m such that there exists a point which is covered m times. The path-width of the graph G is one less than the width of its minimal width representation. A minimal width representation is also called *optimal*.

We will use the above description as the definition of path-width. For its equivalence to the original definition by Robertson and Seymour, see, for instance, [5]. To get acquainted with the above notion we prove that the introduced parameter is minor-monotone. Let G be a graph with an optimal representation ϱ. Let G' be a graph that is obtained from G by deleting an edge from it. Then ϱ is a representation of G' too, proving that $\mathrm{pw}(G') \leq \mathrm{pw}(G)$. Let G'' be a graph that is obtained from G by contracting one of its edges, uv. Let x be the 'new' vertex representing the identified vertices u and v. Let ϱ' be a function on $V(G'') = V(G) \setminus \{u,v\} \cup \{x\}$ that is identical to ϱ on $V(G) \setminus \{u,v\}$ and has the value $\varrho(u) \cup \varrho(v)$ on x. Then ϱ' is a representation of G'' with width at most the width of ϱ. This proves that $\mathrm{pw}(G'') \leq \mathrm{pw}(G)$.

A *separation* of a graph G is a triple $(A, B, (v_1, v_2, \ldots, v_k))$, where A, B are subgraphs of G with $A \cup B = G$, $E(A) \cap E(B) = \emptyset$, and $V(A) \cap V(B) = \{v_1, v_2, \ldots, v_k\}$. We think of a separation as representing G as a result of some gluing: we can obtain G from A by gluing to it a copy of B by identifying the corresponding vertices.

Let a *structure* S be a pair $(G(S), (u_1, u_2, \ldots, u_j))$, where $G(S)$ is a graph, and $u_1, u_2, \ldots, u_j$ are distinct vertices of $G(S)$, called the *vertices of attachment* of S. A graph G has a structure S if G has a separation $(A, B, (v_1, v_2, \ldots, v_k))$, where $(B, (v_1, v_2, \ldots, v_k))$ is isomorphic to S.

A *reduction* R is a pair of structures, S_R and T_R, with the same sequence of vertices of attachment and $|V(S_R)| > |V(T_R)|$. Let G be a graph which has the structure S_R. Then we say that the reduction R can be performed on G. The result of the reduction is a graph H that can be obtained from a proper separation of G by replacing the side isomorphic to S_R by T_R.

We need some relaxations of the formal definition above. A *variation* R is a pair of structures, S_R and T_R, with the same sequence of vertices of attachment and $|V(S_R)| = |V(T_R)|$. Sometimes we allow the performance of a reduction or a variation only when the corresponding separation satisfies some additional requirements (for example, some degree conditions on the vertices of attachment, or containment of a cycle on the side of the separation that is not changed during the reduction or variation). We can call them conditional reduction and conditional variation. From now on we use the notion of *operation* instead of the words reduction, variation, conditional reduction, or conditional variation.

For an operation R, define the following partial order: $H \leq_R G$ if there is a sequence of graphs $H = G_1, G_2, \ldots, G_k = G$ such that, for every $i < k$, G_i is obtained from G_{i+1} by performing R. Naturally, we can define $\leq_{\mathscr{R}}$ if we have a set $\mathscr{R}$ of operations by $H \leq_{\mathscr{R}} G$ if $G_i \leq_R G_{i+1}$ for some $R \in \mathscr{R}$ for each i. If A is a class of graphs, then an operation R is *A-monotone* if $G \in A$ implies $H \in A$ for all graphs G, H satisfying $H \leq_R G$. An operation R is *A-safe* if, for all graphs G, H satisfying $H \leq_R G$, $G \in A$ if and only if $H \in A$.

By saying a graph *has* or *contains* an operation R, we mean that the operation can be performed on it. A graph G is reducible if G has a reduction. $R(G)$ abbreviates the result of a performance of R on G.

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3. PW2-safe operations

Let PW2 be the set of all simple graphs with path-width at most two. We will discuss PW2-monotone and PW2-safe operations. Some of them are well known (for example, taking minors), some are implicit in other works (compare Lemmas 3.2, 3.3 and 3.4 with [4]), and some are new. For the sake of completeness we list all of them with proof. Our proofs are different from the previous results (and perhaps simpler), and provide a unified approach to these operations.

First we start with some simple observations on representations.

Let $\mathscr{I}$ denote the set of all closed intervals of the real line. Let $\phi : V(G) \to \mathscr{I}$ be a representation of G. Let $\phi(v) = [l(v), r(v)]$, *i.e.*, $l(v), r(v)$ simply denotes the left and right endpoints, respectively, of the closed interval $\phi(v)$ representing the vertex v. If I is a closed interval, then $l(I)$ and $r(I)$ denote the left and right endpoints, respectively, of I. The real line introduces a 'new language'. We can refer to the left side of a point or interval on our line. We can write $P < Q$ or $P < I$ if and only if P is on the left-hand side of point Q or interval I (*i.e.*, $P < l(I)$). Similarly, for two intervals, $I_1 < I_2$ means that $r(I_1) < l(I_2)$. $P + \varepsilon$ is the point on the right of P, at distance ε from it. Q is between P and R if and only if $P < Q < R$ or $P > Q > R$. $[P]$ is the interval that contains exactly the point P.

Observation 3.1.

(i) *K_n has essentially one representation: the representing intervals are n intervals with non-empty intersection. Hence the minimal width of the representations is n.*

(ii) *If G is connected, then, for any representation, the union of the representing intervals is an interval (i.e., connected).*

(iii) *Assume there exists a representation of a graph G. If there is a point p on the real line, such that $r(x) = p$ (resp. $l(x) = p$) for some vertex x, and the width at p is k, then, for appropriately chosen small positive ε, the width at any point of the open interval $(p, p + \varepsilon)$ (resp. $(p - \varepsilon, p)$), is at most $k - 1$.*

The importance of (iii) is that it gives us some 'space' to deform a representation without increasing the width.

In a series of lemmas we prove the main features of the operations. Unfortunately the formal proofs are hard to read. We advise the reader to have paper and pencil in hand, so every detail can be drawn and visualized.

Operation 1. O_1 = deletion of an edge or isolated vertex. It preserves path-width at most two, so it is a PW2-monotone reduction.

Operation 2. O_2 = contraction of an edge. It preserves path-width at most two, so it is a PW2-monotone reduction.

Operation 3. Let O_3 be the variation defined as follows. Let $S_3 = xaby$, a path of length three. The attachments are x and y (the two endvertices of the path). Let T_3 be a star

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with three branches (a claw) with vertices (x, a', b', y), where $\deg(a') = \deg(x) = \deg(y) = 1$ and $\deg(b') = 3$. The attachments are x and y (two leaves of the star). Then $O_3(G) = (G - S_3) \cup T_3$. (See Figure 1, where the attachments are drawn as full circles.)

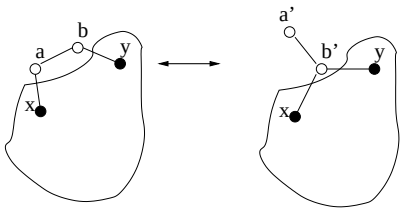


Figure 1 O_3 variation

Lemma 3.2. *O_3 is a* PW2*-safe variation, i.e.,* $\mathrm{pw}(G) \leq 2$ *if and only if* $\mathrm{pw}(O_3(G)) \leq 2$.

Proof. Let ϱ be an optimal representation of G. Then let ϱ' be the following representation of $O_3(G)$. It is identical to ϱ on $V(O_3(G)) \setminus \{a', b'\} = V(G) \setminus \{a, b\}$, $\varrho'(b') = \varrho(a) \cup \varrho(b)$ and $\varrho'(a') = \varrho(a) \cap \varrho(b)$. The width of ϱ' is at most the width of ϱ.

Now let ψ be an optimal representation of $O_3(G)$. Since the degree of a' is one, we can assume that $\psi(a')$ is an interval of one point: $l(a') = r(a') = P$. (This assumption is not vital, but it simplifies the picture behind the proof.)

The natural way to modify ψ (without increasing the width), obtaining a representation ψ' of G, is as follows. Define $\psi'(a) = [l(b'), P]$, $\psi'(b) = [P, r(b')]$ and leave the intervals assigned by ψ unchanged in other cases (or do the same but exchange the image of a and b). If none of these modifications works, then $\psi(x)$ and $\psi(y)$ are on the same (open) side of P. We can assume that $P < \psi(x), \psi(y)$, and $l(x) \leq l(y)$. $\psi(a') \cap \psi(b') \neq \emptyset$ and $\psi(y) \cap \psi(b') \neq \emptyset$; hence $P, l(y) \in \psi(b')$, which implies $[P, l(y)] \subset \psi(b')$, and $l(y)$ has width at least three. By Observation 3.13.1, ψ remains optimal by setting $\psi(a') = [l(y) - \varepsilon]$. This variation of ψ brings us to the case when the first (naive) approach works. This proves the claim. □

Operation 4. Instead of a formal description, let Figure 2 define operation O_4. (Do not forget that the full circles denote the attachments.)

Lemma 3.3. *O_4 is a* PW2*-safe reduction, i.e.,* $\mathrm{pw}(G) \leq 2$ *if and only if* $\mathrm{pw}(O_4(G)) \leq 2$.

Proof. Again, name the vertices participating in the reduction as shown in Figure 2 $(V(O_4(G)) = V(G) \setminus \{a_1, a_2, a_3, b_1, b_2, b_3\} \cup \{a, b\})$.

First let ψ be an optimal representation of $O_4(G)$. We can assume that $l(a) = r(a) = l(b) = r(b) = P \in \psi(x)$, that is, $\psi(a) = \psi(b)$, a single point P. Then a small neighbourhood of P is such that only $\psi(x), \psi(a)$ and $\psi(b)$ covers it. We can remove $\psi(a)$ and $\psi(b)$ and define $\psi'(a_1) = \psi'(b_1) = P - \varepsilon$, $\psi'(a_2) = \psi'(b_2) = P$, $\psi'(a_3) = \psi'(b_3) = P + \varepsilon$. For any other vertex of G, its representing interval will be the same as in ψ. In this way we have obtained a representation ψ' of G of width at most three.

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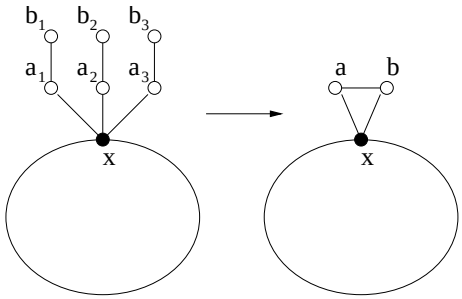


Figure 2 O_4 reduction

Now let ϱ be an optimal representation of G. If one of $\varrho(a_1) \cap \varrho(b_1)$, $\varrho(a_2) \cap \varrho(b_2)$ and $\varrho(a_3) \cap \varrho(b_3)$ intersects $\varrho(x)$, then there will be a point P of $\varrho(x)$ that is covered three times by the intervals playing roles in the reduction. This point can represent a and b. Leaving the representing intervals of other vertices of $O_4(G)$ (*i.e.*, elements of $V(O_4(G)) \setminus \{a, b, c\} = V(G) \setminus \{a_1, a_2, a_3, b_1, b_2, b_3, x\}$) as in ϱ, we obtain a desirable representation.

If none of $\varrho(a_1) \cap \varrho(b_1)$, $\varrho(a_2) \cap \varrho(b_2)$ and $\varrho(a_3) \cap \varrho(b_3)$ intersects $\varrho(x)$, then at least two of the intersections ($\varrho(a_1) \cap \varrho(b_1)$ and $\varrho(a_2) \cap \varrho(b_2)$, say) will be located on the same side (the left-hand side, say) of $\varrho(x)$. This implies that $l(x)$ will be covered by $\varrho(x)$, $\varrho(a_1)$ and $\varrho(a_2)$, which means that $l(x)$ can play the role of P, and the previous argument works. This proves the claim. □

Operation 5. Instead of a formal description, let Figure 3 define operation O_5. We stress that there is one important condition: we can perform O_5 only when the degree of x is at least three.

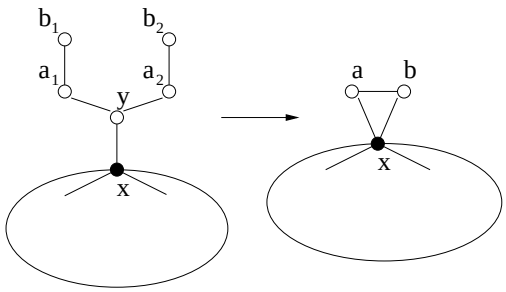


Figure 3 O_5 reduction

Lemma 3.4. *O_5 is a* PW2*-safe reduction, i.e.,* $\mathrm{pw}(G) \leq 2$ *if and only if* $\mathrm{pw}(O_5(G)) \leq 2$.

Proof. Again, name the vertices participating in the reduction as shown in Figure 3 ($V(O_5(G)) = V(G) \setminus \{a_1, a_2, b_1, b_2, y\} \cup \{a, b\}$).

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An optimal representation of $O_5(G)$ can be transformed into a width-three representation of G, as in Lemma 3.3.

Now we take an optimal representation ϱ of G. As in the previous proof, the only problem occurs if neither $I_1 = \varrho(a_1) \cap \varrho(b_1)$ nor $I_2 = \varrho(a_2) \cap \varrho(b_2)$ intersects $\varrho(x)$. Hence I_1, I_2 and $\varrho(x)$ are pairwise disjoint. We can also assume that $\widehat{I_1} = \varrho(a_1) \cup \varrho(b_1)$ and $\widehat{I_2} = \varrho(a_2) \cup \varrho(b_2)$ do not intersect (otherwise ϱ would also represent $G + e$ – where e is an edge between the sets $\{a_1, b_1\}$ and $\{a_2, b_2\}$ – which clearly contains $O_5(G)$ as a minor). We consider two cases.

Case 1: I_1 and I_2 are on the same side of $\varrho(x)$, say $I_1 < I_2 < \varrho(x)$. Then I_2 separates $\widehat{I_1}$ from $\varrho(x)$, and hence $\varrho(y)$ must contain I_2. Let P be any point of I_2. We can define ϱ', a representation of $O_5(G)$ (without increasing the width of ϱ), by $\varrho'(a) = \varrho'(b) = [P]$, $\varrho'(x) = \varrho(x) \cup \varrho(y)$, and all the other vertices have their representing intervals as in ϱ.

Case 2: $\varrho(x)$ separates I_1 and I_2, say $I_1 < \varrho(x) < I_2$. We are easily done if $\varrho(a_1)$ and $\varrho(a_2)$ intersect $\varrho(y)$ over $\varrho(x)$. Hence we can assume that $\varrho(x) \subset \varrho(y)$. There are at least two neighbours of x different from y, say n, N. Hence $\varrho(n) \not\supset \varrho(x)$. So one endpoint of $\varrho(n)$, say $l(n)$, is in $\varrho(x)$. Then we can take $\varrho'(x) = \varrho(x) \cup \varrho(y)$ and $\varrho'(a) = \varrho'(b) = [l(n) - \varepsilon]$, and leave all the other representing intervals as in ϱ. The constructed ϱ' is a representation of $O_5(G)$ having width at most three, if the width of ϱ did not exceed three. □

For a moment we stop the flow of lemmas to exhibit the strength of our operations. In [10], the minimal acyclic forbidden minors (*i.e.*, excluded trees) for path-width at most two were determined. The ten trees 'look very similar'. The last two operations can be used to reduce the excluded trees to one fundamental graph, D_3. In fact, we can say more. Ten other graphs (altogether 20 out of the 110 excluded minors) are reducible to D_3. Figure 4 shows an example.

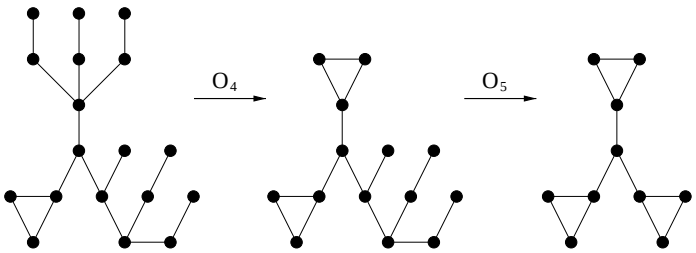


Figure 4 20 excluded minors are reducible to D_3

Operation 6. Instead of a formal description, let Figure 5 define operation O_6. We point out that this is a conditional operation, that is, we assume that the exchange of two structures can be done under a certain assumption on the unchanged part: x must be contained in a cycle with a chord (vw).

Lemma 3.5. *O_6 is a* PW2*-safe reduction, i.e.,* $\mathrm{pw}(G) \leq 2$ *if and only if* $\mathrm{pw}(O_6(G)) \leq 2$.

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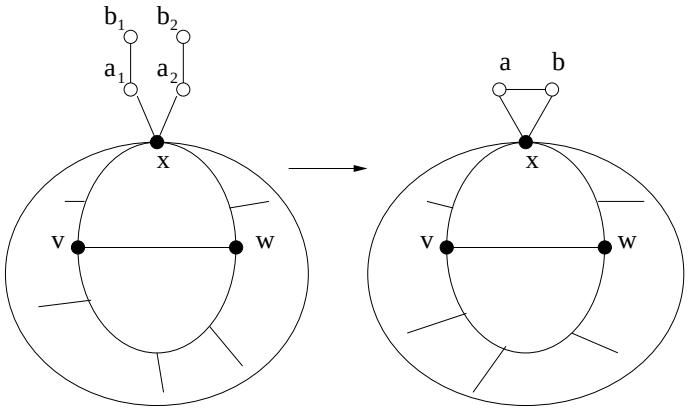


Figure 5 O_6 reduction

Proof. Again, name the vertices participating in the reduction as shown in Figure 5 ($V(O_6(G)) = V(G) \setminus \{a_1, a_2, b_1, b_2, \} \cup \{a, b\}$).

An optimal representation of $O_6(G)$ can be transformed into a width-three representation of G as in Lemma 3.3.

Now we take an optimal representation ϱ of G. As in the proof of Lemma 3.4, the only problem occurs if neither $\varrho(a_1) \cap \varrho(b_1)$ nor $\varrho(a_2) \cap \varrho(b_2)$ intersects $\varrho(x)$, and $\varrho(x)$ separates $\varrho(a_1) \cap \varrho(b_1)$ and $\varrho(a_2) \cap \varrho(b_2)$, say $\varrho(a_1) \cap \varrho(b_1) < \varrho(x) < \varrho(a_2) \cap \varrho(b_2)$. We are also done if $\varrho(a_1) \cap \varrho(a_2) \cap \varrho(x) \neq \emptyset$ (both $\varrho(a_1)$ and $\varrho(a_2)$ must intersect $\varrho(x)$). Hence we can assume that $l(x) \leq r(a_1) < l(a_2) \leq r(x)$. In particular, $l(x)$ and $r(x)$ are covered twice by the intervals corresponding to the vertices of S_{O_6}, the structure exchanged during the application of O_6.

Now we consider the remaining part of G. It contains a cycle C going through x, and having a vw chord. We partition C into four arcs (one of them consists of only one vertex, x), according to Figure 6. For each arc we take the union of the intervals assigned to its vertices. Let $\varrho(x)$, I_1, I_2 and I_3 be the corresponding intervals ($I_i = \cup_{v \in A_i} \varrho(v)$). These four intervals cannot be pairwise intersecting (our representation has width at most three). The only possibility is that $\varrho(x)$ and I_3 are disjoint. Because of the symmetry of our argument so far, we can assume that $I_3 < \varrho(x)$. I_1 and I_2 intersect both I_3 and $\varrho(x)$; hence they cover $l(x)$. This contradicts the fact that $l(x)$ is covered by at most three intervals, and thus proves the claim. □

Operation 7. Instead of a formal description, let Figure 7 define operation O_7. There is one important condition: the operation can be performed only when the degrees of x and y are at least three.

Lemma 3.6. *O_7 is a* PW2*-safe reduction, i.e.,* $\mathrm{pw}(G) \leq 2$ *if and only if* $\mathrm{pw}(O_7(G)) \leq 2$.

Proof. Again, name the vertices participating in the reduction as shown in Figure 7 ($V(O_6(G)) = V(G) \setminus \{z, a, b\} \cup \{a', b'\}$).

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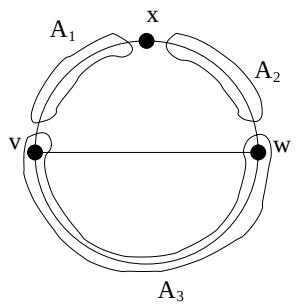


Figure 6 Four-arc partition

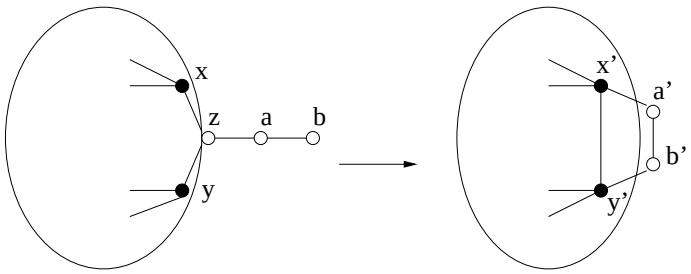


Figure 7 O_7 reduction

An optimal representation ψ of $O_6(G)$ can be transformed into a width-three representation of G as follows: $\psi(x) \cap \psi(b') \neq \emptyset$ and $\psi(y) \cap \psi(a') \neq \emptyset$ would mean that $\psi(x), \psi(y), \psi(a')$ and $\psi(b')$ are pairwise intersecting, which contradicts our assumption on the width of ψ. So we can assume that $\psi(x) \cap \psi(b') = \emptyset$; moreover, $\psi(x) < \psi(b')$. Hence $\psi(a')$ and $\psi(y)$ both contain $r(x)$ and $l(b')$. We can assume that $l(b') = r(b') = P$ is covered by $\psi(a')$, $\psi(b')$ and $\psi(y)$. So, by removing $\psi(a')$ and $\psi(b')$, we have 'space' around P to define a representation of G: $\psi'(z) = [l(a'), P]$, $\psi'(a) = [P - \varepsilon, P + \varepsilon]$, and $\psi'(b) = [P + \varepsilon]$.

Now we take an optimal representation ϱ of G. If $\varrho(x)$ and $\varrho(y)$ are intersecting, then ϱ is a representation of $(G - b) + xy$. O_3 can be applied on this graph to obtain $O_7(G)$ and we are done.

Henceforth we assume that that $\varrho(x)$ and $\varrho(y)$ are disjoint, and $\varrho(x) < \varrho(y)$. If $\varrho(b)$ intersects $\varrho(x)$ or $\varrho(y)$ (let us assume that $\varrho(b) \cap \varrho(x) \neq \emptyset$), then we can easily obtain a representation of $O_7(G)$ by $\varrho'(x) = \varrho(x)$, $\varrho'(a) = \varrho(b)$, $\varrho'(b) = \varrho(a)$, $\varrho'(y) = \varrho(y) \cup \varrho(z)$, and of course taking all other representing intervals from ϱ. So, from now on, $\varrho(b)$, $\varrho(x)$ and $\varrho(y)$ are pairwise intersecting. We consider two cases.

Case 1: $\varrho(b)$ is between $\varrho(x)$ and $\varrho(y)$; $\varrho(z)$ intersects both $\varrho(x)$ and $\varrho(y)$, and hence $\varrho(b) \subset [r(x), l(y)] \subset \varrho(z)$. Since $\varrho(a)$ intersects $\varrho(b)$, there is a point P that is covered by $\varrho(z)$, $\varrho(a)$ and $\varrho(b)$. Let $\varrho'(x) = [l(x), P]$, $\varrho'(y) = [P, r(x)]$, $\varrho'(a') = [P - \varepsilon, P + \varepsilon]$, $\varrho'(b') = [P + \varepsilon]$, and all other representing intervals are from ϱ. This is a desired representation of $O_7(G)$.

Case 2: We may assume that $\varrho(b)$ is the rightmost among $\varrho(x)$, $\varrho(y)$ and $\varrho(b)$. As before,

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we have $[r(x), l(y)] \subset \varrho(z)$. Now we use the fact that y has two neighbours different from z. Since $l(y)$ is already covered by $\varrho(z)$ and $\varrho(y)$, at least one of the neighbours of y (say n) is such that $l(n) \in \varrho(y) \setminus \varrho(z)$. For suitable small ε the interval $[l(n) - \varepsilon, l(n))$ is covered only by $\varrho(x) \cup \varrho(a) \cup \varrho(b)$ and $\varrho(y)$. This allows us to define ϱ' as in Case 1. □

Operation 8. Instead of a formal description, let Figure 8 define operation O_8.

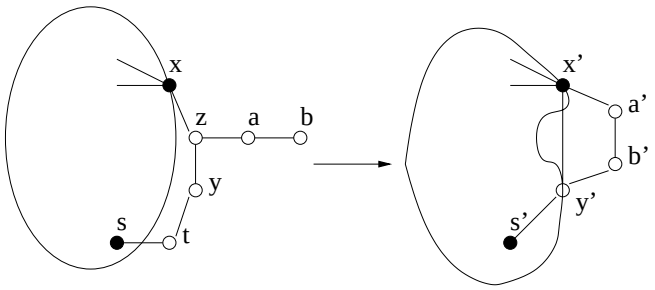


Figure 8 O_8 reduction

Lemma 3.7. *O_8 is a* PW2*-safe reduction, i.e.,* $\mathrm{pw}(G) \leq 2$ *if and only if* $\mathrm{pw}(O_8(G)) \leq 2$. *Again there is an assumption: we can perform O_8 only when the degree of x is at least three.*

Proof. O_8 can be simulated by an application of O_3 followed by an application of O_7 and an edge deletion. So one direction of our claim follows from Lemmas 3.2 and 3.6.

The other direction is also easy. If $O_8(G)$ is given by an optimal representation (for notation see Figure 8), we can add a single neighbour (with degree one) to y' and still have a graph with a representation of width two. Indeed, one of the three neighbours of y' has a representing interval with an extreme point inside the representing interval of y'. Just 'next to' this extreme point we have enough room to represent the new neighbour. After this we again need to refer the reader to Lemmas 3.6 and 3.2 to complete the proof. □

4. The characterization theorem

Many excluded minor theorems have the following structure: 'A graph G has property P if and only if G has no H-minor, where H is from the list L.' This type of characterization for the PW2-property was given in [4] (for an alternative solution see [2]). There are 110 excluded minors. We substitute this list with 10 minimal excluded graphs with respect to $\leq_{\mathscr{R}}$, where $\mathscr{R} = \{O_1, \ldots, O_8\}$. One major point of our paper is that the appropriate theorem is not the excluded minor theorem, but the one we present.

Theorem 4.1. *The following statements are equivalent:*

(i) *G has path-width at most two,*

(ii) *G is not reducible to any of the graphs listed in Figure 9.*

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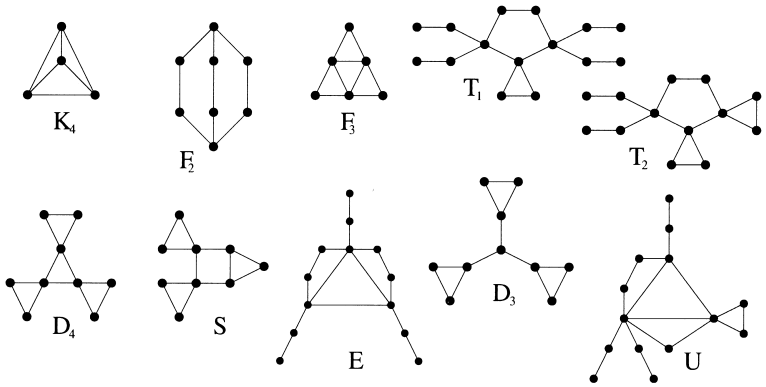


Figure 9 Non-reducible graphs

The (ii)⇒(i) implication can be proved based on [4]. If, like the authors, one checks that all the 110 excluded minors can be reduced to one of the graphs on our list, then it will be proved that any graph satisfying (ii) cannot have any minor from the Kinnersley and Langston list. Thus their theorem implies (i). The formal description of this argument would be too long, but we ask the reader to take a look at [4]'s list. Immediately one can see several reductions, and in most cases it is straightforward to see what should be done. In Figure 10 we give an example reducing a 'complicated' excluded minor to one of our graphs.

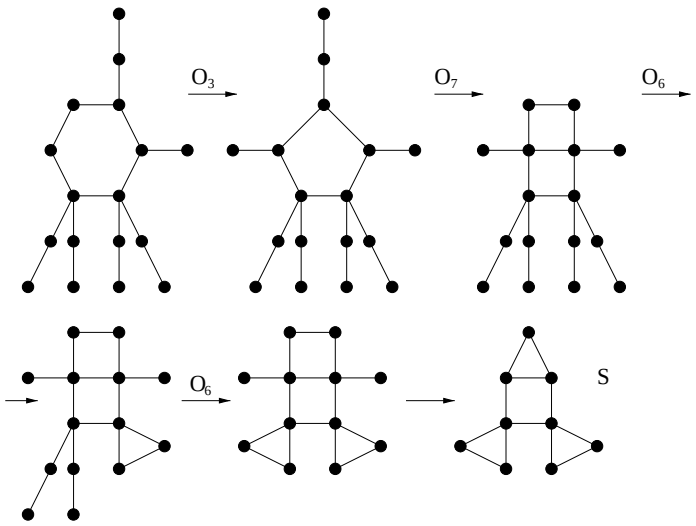


Figure 10 Reduction of an excluded minor

The (i)⇒(ii) implication follows easily if one sees that all graphs on our list have path-width more than two. This can be shown by brute force computation (as was done in [4] for the case of 110 graphs). In the next section we prove it by hand.

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5. Path-width of the non-reducible graphs

The 110 excluded minors can be reduced to the ten fundamental graphs in Figure 9. These graphs are non-reducible in the sense that they do not contain any reduction. However, some of the ten graphs permit the variation O_3. In such cases, we just picked out one of the versions of the graph.

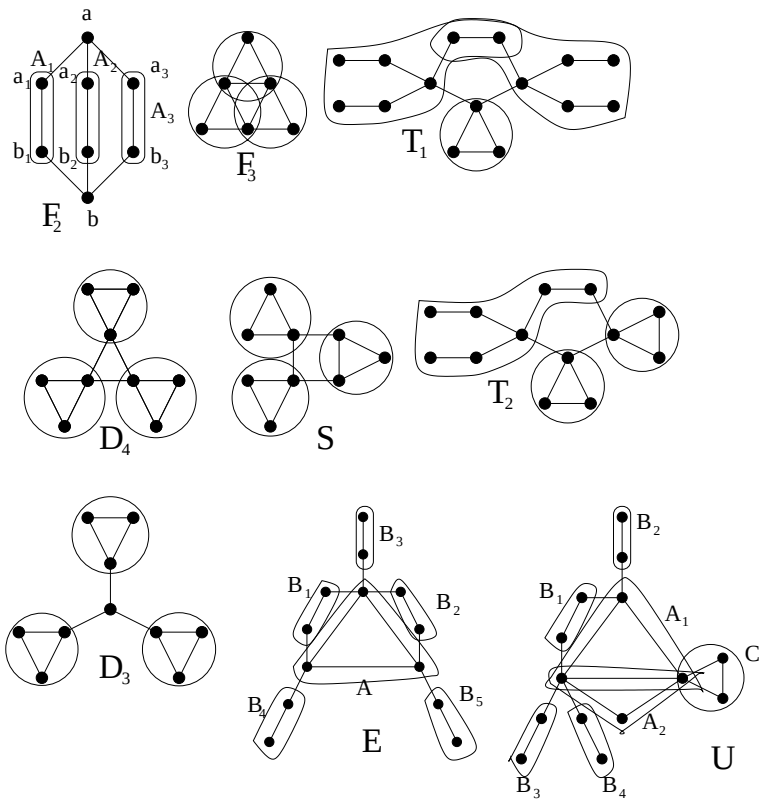


Figure 11 Non-reducible graphs

We prove that the ten non-reducible graphs have path-width at least three by proving that any representation of them has width at least four. As we mentioned earlier, this gives that all the 110 excluded minors of [4] have path-width at least three.

Lemma 5.1. *Any representation of K_4 has width at least four.*

Proof. Observation 3.1. □

Lemma 5.2. *Any representation of F_2 has width at least four.*

Proof. Let ϱ be any representation of F_2. Let A_1, A_2 and A_3 be the three two-element sets as in Figure 9. Let $I_i = \cap_{v \in A_i} \varrho(v)$. We can assume the I_is ($i = 1, 2, 3$) are pairwise

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disjoint (otherwise the width of ϱ is at least four). By symmetry assume that $I_1 < I_2 < I_3$. Then $\varrho(a_1) \cup \varrho(a) \cup \varrho(a_3)$ and $\varrho(b_1) \cup \varrho(b) \cup \varrho(b_3)$ covers I_2 (that is covered by $\varrho(a_2)$ and $\varrho(b_2)$ too). Hence the width of the representation is at least four in this case too. □

Lemma 5.3. *Let $A_i \subset V(G)$, $i = 1, 2, 3$ such that:*

(i) $|A_i \cap A_j| \leq 2$, $i \neq j \in \{1, 2, 3\}$,
(ii) *any representation of $G|_{A_i}$ has width at least three,*
(iii) $G - A_i$ $(i = 1, 2, 3)$ *is connected.*

Then any representation of G has width at least four.

Proof. Let ϱ be any representation of G. This contains representations of the graphs spanned by A_1 A_2 and A_3. By our assumption, there exist intervals I_1, I_2 and I_3 such that I_i $(i = 1, 2, 3)$ is covered by three intervals representing three vertices from A_i. Since two A_is have at most two common elements, a non-empty intersection among the I_is would mean width at least four. Hence we can assume that the I_is are pairwise disjoint intervals, and moreover that $I_1 < I_2 < I_3$. Let the union of the intervals assigned to the vertices of the connected $G - A_2$ be I; $I \cap I_1 \neq \emptyset$ and $I \cap I_3 \neq \emptyset$. Hence I covers I_2, proving that the width is at least four. □

Lemma 5.4. *Any representation of the graphs F_3, T_1, T_2, D_4, S, D_3 has width at least four.*

Proof. Figure 9 marks the sets A_1, A_2, A_3, showing that the previous lemma is applicable. □

Lemma 5.5. *Any representation of E has width at least four.*

Proof. Let ϱ be a representation of E. A and the B_is $(i = 1, 2, 3, 4, 5)$ are cliques in E (see Figure 9); hence the intervals assigned to the vertices in each set are pairwise intersecting. Let $I = \cap_{v \in A} \varrho(v)$ and $I_i = \cap_{v \in B_i} \varrho(v)$ $(i = 1, 2, 3, 4, 5)$. These intervals must be pairwise disjoint, otherwise the width is at least four.

Assume that $I_1 < I$. The existence of two independent edges between A and B_1 implies that each point of $[r(I_1), l(I)]$ is covered by at least two intervals assigned to vertices of $A \cup B_1$. Hence, if the width is less than four, there is no I_i between I and I_1. A similar argument is true for I_2. Hence $I_1 < I < I_2$. Furthermore, there are no other I_is in the interval $[l(I_1), r(I_2)]$, *i.e.*, I_3, I_4 and I_5 fall into two categories: left from I_1 and right from I_2. One of these categories, say the former, contains two of these intervals. We can assume that two intervals from I_3, I_4 and I_5 are on the left side of I_1. Let $\widehat{B_3}$ be the vertex set we obtain by adding to it the only neighbour of B_3. Let $J_3 = \cup_{v \in \widehat{A_3}} \varrho(v)$. J_3 covers I_3 and I. The same can be said for indices 4 and 5. This proves that I_1 will be covered four times. □

Lemma 5.6. *Any representation of U has width at least four.*

Proof. Let ϱ be a representation of U. A_1, A_2, the B_is $(i = 1, 2, 3, 4)$ and C are cliques in U; hence the intervals assigned to the vertices of each set are pairwise intersecting. Let

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$I_i = \cap_{v \in A_i} \varrho(v)$ $(i = 1, 2)$, $J_i = \cap_{v \in B_i} \varrho(v)$ $(i = 1, 2, 3, 4)$, and $I = \cap_{v \in C} \varrho(v)$. We can assume that these intervals are pairwise disjoint.

First we consider the order of J_1, I_1 and I_2. As in the previous proof, we can assume that $J_1 < I_1 < I_2$ and there are no other J_is in the interval $[l(J_1), r(I_2)]$. As before, one can show that $J_2 \not\geq I_2$ (hence $I_2 < J_1$). Furthermore, J_3 and J_4 cannot be on the left side of J_1. So

$$J_2 < J_1 < I_1 < I_2 < (J_3 \text{ and } J_4), \tag{5.1}$$

in some order.

Now we take a look at I. $U - C$ spans a connected subgraph of U; hence all the I_is and J_is are on the same side of I. This leaves two possible positions for I in (5.1): first or last. Each possibility implies width four, proving our claim. □

6. Concluding remarks

In this article we have presented another way of analysing graphs with path-width at most two: namely, these are graphs which can be represented with intervals, such that any point is covered by at most three intervals. The same idea can be applied to tree-width.

The operations described in Section 3 are indeed path-width-preserving, except possibly O_6. Hence these reductions can also be used for the classification of graphs with greater path-width.

There are other possibilities for introducing minor-monotone graph parameters. We might assign to each vertex of a graph an arc of a circle, such that adjacent vertices correspond to intersecting arcs. The width of an arc-representation and the arc-width of a graph can be defined naturally. Since the union of two arcs is an arc (we consider the whole circle as an arc), one can easily prove minor-monotonicity of the introduced graph parameter.

One should consider tree-width and the newly introduced arc-width parameter in the same way as we did path-width in this paper. Our remarks on arc-width will be published in a separate report.

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