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Non-commutative L^p-spaces

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1. *Introduction.* The spaces L^1 and L^2 of unbounded operators associated with a *regular gauge space* (von Neumann algebra equipped with a faithful normal semi-finite trace) are defined by Segal(5) definitions 3·3, 3·7. The spaces L^p $(1 < p < \infty, p \neq 2)$ are defined by Dixmier(2) as the abstract completions of their bounded parts. Dixmier makes use of the Riesz convexity theorem to prove the Hölder inequality, and the uniform convexity, and hence reflexivity, of L^p $(2 < p < \infty)$.

Here, we define the L^p-spaces concretely as spaces of unbounded operators, give a more direct and elementary proof of the Hölder inequality, and prove that the dual of L^p is L^q $(1 < p < \infty, 1/p + 1/q = 1)$ by means of a Radon–Nikodym theorem.

Let $\mathscr{R}$ be a semifinite von Neumann algebra and let m be a faithful semifinite normal trace on $\mathscr{R}$. Given Segal's definition, (5) 3·3, of the space $L^1 = L^1(\mathscr{R}, m)$ of m-integrable unbounded operators, and of the extension of m to L^1, we make the following definition.

1·1. *Definition.* Let $1 \leqslant p < \infty$ and denote by $\mathscr{M} = \mathscr{M}(\mathscr{R})$ the algebra of measurable operators affiliated with $\mathscr{R}$ ((5), definition 2·1). Define

$$L^p = L^p(\mathscr{R}, m) = \{A \in \mathscr{M} : |A|^p \in L^1\},$$

$$\|A\|_p = (m(|A|^p))^{1/p} \quad (A \in L^p).$$

This is consistent with (5), definitions 3·3, 3·4, 3·7 and 3·8, in the cases $p = 1, 2$, by (5) corollary 11·2, corollary 12·11, and corollary 12·14.

It is not immediate that L^p is a vector space and that $A \to \|A\|_p$ is a norm; this will follow from the Hölder inequality (Theorem 3·4), which will be deduced from an inequality (Theorem 3·3) involving *rearrangements* of operators. The latter inequality is proved by Garling, ((3) proposition 4), in the case where $\mathscr{R} = \mathscr{B}(\mathscr{H})$ and L^1 is the usual trace class of compact operators, and is stated for bounded operators by Grothendieck ((4) Corollaire 2(6)) as a special case of a more general inequality ((4) théorème 1).

2. *Preliminaries.* Throughout, $\mathscr{R}$ is a von Neumann algebra on a Hilbert space $\mathscr{H}$, m is a faithful semifinite normal trace on $\mathscr{R}$, $\mathscr{F} = \mathscr{F}(\mathscr{R}, m)$ is the ideal of *elementary operators* ((5) definition 3·1), $\mathscr{L} = \mathscr{L}(\mathscr{R})$ is the algebra of *locally measurable operators* ((7) definition 2·2), and $\mathscr{S}^+$ denotes $\{A \in \mathscr{S} : A \geqslant 0\}$ for any subset $\mathscr{S}$ of $\mathscr{L}$. A *simple* operator is an operator of the form $\sum_{i=1}^{k} \lambda_i E_i$ with $\lambda_i \in \mathscr{C}$, E_i projections in $\mathscr{F}$, and $E_i E_j = 0$ if $i \neq j$. We introduce the algebra $\mathscr{K} = \mathscr{K}(\mathscr{R}, m)$ of *totally measurable operators*.

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2·1. *Definition.* An operator $A \in \mathscr{L}$ is *totally measurable* if there is a projection $E \in \mathscr{F}$ such that $A(I-E) \in \mathscr{R}$. The subalgebra $\mathscr{C}$ of $\mathscr{K}$ is defined by: $A \in \mathscr{C}$, if and only if for every $\epsilon > 0$, there is a projection $E \in \mathscr{F}$ such that $\|A(I-E)\|_\infty < \epsilon$, where $\| \ \|_\infty$ denotes the usual operator norm on $\mathscr{R}$.

An application of (7) lemma 1·1 shows that, if $A \in \mathscr{L}$ and $|A| = \int_0^\infty \lambda \, dE(\lambda)$, is the spectral resolution of $|A| = (A^*A)^{\frac{1}{2}}$, then the following conditions are equivalent:

(i) $A \in \mathscr{K}$,

(ii) $m(I-E(\lambda_0)) < \infty$ for some $\lambda_0 > 0$,

(iii) $m(I-E(\lambda)) < \infty$ for some $\lambda_0 > 0$ and all $\lambda \geqslant \lambda_0$,

(iv) $m(I-E(\lambda)) \to 0$ as $\lambda \to \infty$.

Also, $A \in \mathscr{C}$ if and only if $m(I-E(\lambda)) < \infty$ for all $\lambda > 0$.

2·2. *Definition.* The *rearrangement* $\tilde{A}$ of an operator $A \in \mathscr{K}$ is the function $\tilde{A} :]0, \infty[\to [0, \infty[$ defined by

$$\tilde{A}(\alpha) = \inf\{\lambda \in [0, \infty[: m(I-E(\lambda)) \leqslant \alpha\},$$

where

$$|A| = \int_0^\infty \lambda \, dE(\lambda).$$

$\tilde{A}$ is decreasing and continuous on the right. We have

$$\mathscr{C} = \{A \in \mathscr{K} : \tilde{A}(\alpha) \to 0 \quad \text{as} \quad \alpha \to \infty\}$$

and, by (7) lemma 1·1,

2·3. PROPOSITION. $\tilde{A}(\alpha) = \inf\{\|A(I-E)\|_\infty : E \in F, m(E) \leqslant \alpha\}$.

Proof. $I-E(\lambda) \prec E$ implies $m(I-E(\lambda)) \leqslant m(E)$. The next result summarizes some basic facts about rearrangements.

2·4. PROPOSITION.

(i) $A^{*\sim} = |A|^\sim = \tilde{A}$ $(A \in \mathscr{K})$,

(ii) $(AX)^\sim \leqslant \|X\|_\infty \tilde{A}$ $(A \in \mathscr{K}, X \in \mathscr{R})$,

(iii) $(A+B)^\sim(\alpha+\beta) \leqslant \tilde{A}(\alpha) + \tilde{B}(\beta)$ $(A, B \in \mathscr{K}, \alpha, \beta > 0)$,

(iv) $(AB)^\sim(\alpha+\beta) \leqslant \tilde{A}(\alpha).\tilde{B}(\beta)$ $(A, B \in \mathscr{K}, \alpha, \beta > 0)$,

(v) $\tilde{A} \leqslant \tilde{B}$ *if* $A, B \in \mathscr{K}$, $0 \leqslant A \leqslant B$,

(vi) $(A^p)^\sim = (\tilde{A})^p$ *if* $A \in \mathscr{K}$, $A \geqslant 0$, $1 \leqslant p < \infty$,

(vii) $A \in \mathscr{R}$ *if and only if* $A \in \mathscr{K}$ *and* $\tilde{A}(0+) = \lim_{\alpha \downarrow 0} \tilde{A}(\alpha) = \sup_{\alpha > 0} \tilde{A}(\alpha) < \infty$,
in which case $\|A\|_\infty = \tilde{A}(0+)$,

(viii) $A \in L^p$ $(1 \leqslant p < \infty)$ *if and only if* $A \in \mathscr{K}$ *and* $\int_0^\infty (\tilde{A})^p < \infty$,
in which case $\|A\|_p = \left(\int_0^\infty (\tilde{A})^p \right)^{1/p}$.

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Proof. (i) Let $A = W|A|$ be the polar decomposition of A and

$$|A| = \int_0^\infty \lambda dE(\lambda)$$

the spectral resolution of $|A|$. Then

$$|A^*| = (AA^*)^{\frac{1}{2}} = W|A|W^* = \int_0^\infty \lambda dF(\lambda),$$

where $F(\lambda) = WE(\lambda)\,W^* + (I - WW^*)$, so that $m(I-F(\lambda)) = m(I-E(\lambda))$.

(ii) By Proposition 2·3.

(iii) There exist projections E, F s.t.

$$\|A(I-E)\|_\infty = \alpha, \quad m(E) \leqslant \alpha,$$
$$\|B(I-F)\|_\infty = \beta, \quad m(F) \leqslant \beta.$$

Let $G = E \vee F, I - G = (I-E) \wedge (I-F)$.

Then $\|(A+B)(I-G)\|_\infty \leqslant \alpha+\beta, \quad m(G) \leqslant \alpha+\beta$.

(iv) With E, F as in (iii), let

$$G = F \vee [B^*E(\mathscr{H})], \quad I - G = (I-F) \wedge [B^{-1}(I-E)(\mathscr{H})].$$

Then $\|AB(I-G)\|_\infty \leqslant \alpha\beta, m(G) \leqslant \alpha+\beta$.

(v) By Proposition 2·3.

(vi) If $A = \int_0^\infty \lambda dE(\lambda), A^p = \int_0^\infty \lambda^p dE(\lambda) = \int_0^\infty \mu dF(\mu)$, where $F(\mu) = E(\mu^{1/p})$.

(vii) $A \in \mathscr{R}$ if and only if $E(\lambda) = I$ for all $\lambda \geqslant \lambda_0 = \|A\|_\infty$.

(viii) By (vi), it is sufficient to prove the case $p = 1$. This follows from (5) corollaries 12·6 and 12·14 and the fact that

$$\int_0^\infty \tilde{A}(\alpha)\,d\alpha = \int_0^\infty m(I-E(\lambda))\,d\lambda = \int_0^\infty \lambda\,dm(E(\lambda)) \quad \text{if} \quad |A| = \int_0^\infty \lambda dE(\lambda).$$

We obtain the following properties of the L^p-"norm".

2·5. Proposition. *Let* $1 \leqslant p < \infty$.

(i) *If* $A \in L^p$ *and* $\|A\|_p = 0$, *then* $A = 0$,

(ii) $\|A^*\|_p = \||A|\|_p = \|A\|_p$,

(iii) $\|AX\|_p \leqslant \|X\|_\infty \|A\|_p \quad (A \in L^p, X \in \mathscr{R})$,

(iv) $\|A\|_p \leqslant \|B\|_p$ *if* $A, B \in L^p, 0 \leqslant A \leqslant B$.

Proof.

(i) By (5) corollary 12·3.

(ii) By Proposition 2·4 (i) and (vi).

(iii) By Proposition 2·4 (ii) and (vi).

(iv) By Proposition 2·4 (iii) and (vi).

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2·6. *Definition.* The expression

$$\begin{aligned}\sigma(A-B) &= \inf_{E\in\mathscr{F}} \max\{\|(A-B)(I-E)\|_\infty, m(E)\}\\ &= \inf_{\lambda>0} \max\{\lambda, m(I-E(\lambda))\}\\ &= \inf_{\alpha>0} \max\{(A-B)^\sim(\alpha), \alpha\} \quad (A,B\in\mathscr{K}),\end{aligned}$$

where $|A-B| = \int_0^\infty \lambda\, dE(\lambda)$, defines a metric on $\mathscr{K}$. Convergence in this metric coincides with *convergence in measure* as defined in (6) section 3, where it is proved that the algebraic operations are continuous. It can be shown, as in (7) theorem 3·4, that $\mathscr{K}$ is complete in this metric.

2·7. PROPOSITION. *The following are equivalent*:

(i) *The sequence* (A_n) *in* $\mathscr{K}$ *converges in measure to* $A\in\mathscr{K}$,

(ii) $m(I-E_n(\lambda))\to 0$ *for each* $\lambda>0$, *where* $|A-A_n| = \int_0^\infty \lambda\, dE_n(\lambda)$,

(iii) $(A-A_n)^\sim(\alpha)\to 0$ *for each* $\alpha>0$.

2·8. *Definition.* A sequence (A_n) in $\mathscr{K}$ converges m-nearly everywhere (m.n.e.) to $A\in\mathscr{K}$ if, for each $\epsilon>0$, there exist projections $E_n\in\mathscr{R}$, such that

$$E_n\uparrow I, \quad \|(A-A_n)E_n\|_\infty<\epsilon \quad \text{and} \quad m(I-E_n)\to 0.$$

The above definition is in conflict with (5) definition 3·3 but it is, in fact, the concept actually used in (5) theorem 11, lemma 12·1 and corollary 13·1.

Convergence m.n.e. implies both convergence in measure and convergence locally nearly everywhere, each of which implies convergence locally in measure ((7) definitions 3·1, 5·1). The proof of (6) theorem 3·1 shows that if $A_n\to A$ in measure, then (A_n) has a subsequence that converges to A m.n.e.

We use later the following version of Fatou's lemma. The proof given in (6) theorem 4·10 is incorrect; the proof given here is derived from that of (5) theorem 13.

2·9. THEOREM. *Let* (A_α) *be a net in* L^1 *such that* A_α *converges locally in measure to* $A\in\mathscr{L}$, *and* $\sup_\alpha \|A_\alpha\|_1<\infty$. *Then* $A\in L^1$ *and* $\|A\|_1\leqslant \varliminf_\alpha \|A_\alpha\|_1$.

Proof. First suppose that $\mathscr{R}$ has countably decomposable centre $\mathscr{Z}$, so that the topology of convergence locally in measure is metrizable, and let (A_n) be a sequence in L^1, such that $\sup_n \|A_n\|_1<\infty$ and A_n converges locally nearly everywhere to $A\in\mathscr{L}$. As in the proof of (7) theorem 3·4 or 5·2, we can find projections $Q_k\in\mathscr{R}$, such that

$$Q_k\uparrow I, \quad A_nQ_k\in\mathscr{R} \quad \text{if} \quad n\geqslant k,$$

$$AQ_k\in\mathscr{R} \quad \text{and} \quad \lim_{n\to\infty}\|(A_n-A)Q_k\|_\infty=0 \quad \text{for each} \quad k.$$

For any projection $P\in\mathscr{F}$ with $P\leqslant Q_k$, we have

$$|\|AP\|_1-\|A_nP\|_1|\leqslant \|AP-A_nP\|_1\leqslant \|(A-A_n)Q_k\|_\infty . m(P),$$

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$$m(|A|P) \leqslant \|AP\|_1 = \lim_n \|A_n P\|_1 \leqslant \overline{\lim_n} \|A_n\|_1.$$

By (5) lemma 13·1 (extended to locally measurable operators),

$$m(Q_k|A|Q_k) = \sup\{m(|A|P): P \in \mathscr{F}, \quad P \leqslant Q_k\},$$

and

$$m(|A|) = \sup_k m(Q_k|A|Q_k).$$

It follows that

$$A \in L^1 \quad \text{and} \quad \|A\|_1 = m(|A|) \leqslant \overline{\lim_n} \|A_n\|_1.$$

This still holds if $A_n \to A$ locally in measure, since then every subsequence of (A_n) has a subsequence that converges locally nearly everywhere to A ((7) section 5). Thus, if $\mathscr{Z}$ is countably decomposable, the ball $\{A \in L^1: \|A\|_1 \leqslant \lambda\}$ is (sequentially) closed in $\mathscr{L}$ in the (metric) topology of convergence locally in measure.

In the general case, suppose that (A_α), A are as in the statement of the theorem and let (Z_β) be a set of projections in $\mathscr{Z}$ with each Z_β countably decomposable in $\mathscr{Z}$ and $Z_\beta \uparrow I$. Then for each β, $\lim_\alpha A_\alpha Z_\beta = AZ_\beta$, locally in measure, and, by the first part,

$$m(|A|Z_\beta) = \|AZ_\beta\|_1 \leqslant \overline{\lim_\alpha} \|A_\alpha Z_\beta\|_1 \leqslant \overline{\lim_\alpha} \|A_\alpha\|_1.$$

Since $m(|A|) = \sup_\beta m(|A|Z_\beta)$, the result follows.

3. *Inequalities.* The next result is a first step towards Theorems 3·3, 3·4.

3·1. Proposition. *Let* $A = \sum_{i=1}^{k} \lambda_i E_i$, $B = \sum_{j=1}^{l} \mu_j F_j$ *be positive simple operators, where* $0 < \lambda_k < \lambda_{k-1} < \ldots < \lambda_1$, $0 < \mu_l < \mu_{l-1} < \ldots < \mu_1$, *and* E_i, F_j *are projections in* $\mathscr{F}$, *with* $E_i E_{i'} = 0$ $(i \neq i')$, $F_j F_{j'} = 0$ $(j \neq j')$. *Then* $m(AB) \leqslant \int A^\sim B^\sim$.

Proof. We have

$$\tilde{A} = \sum_{i=1}^{k} \lambda_i \chi[\alpha_{i-1}, \alpha_i[= \sum_{i=1}^{k} (\lambda_i - \lambda_{i+1}) \chi[0, \alpha_i[,$$

$$\tilde{B} = \sum_{j=1}^{l} \mu_j \chi[\beta_{j-1}, \beta_j[= \sum_{j=1}^{l} (\mu_j - \mu_{j+1}) \chi[0, \beta_j[,$$

where

$$\alpha_i = m\left(\sum_{r=1}^{i} E_r\right) \quad (i = 1, \ldots, k),$$

$$\beta_j = m\left(\sum_{s=1}^{j} F_s\right) \quad (j = 1, \ldots, l),$$

$$\alpha_0 = \beta_0 = \lambda_{k+1} = \mu_{l+1} = 0.$$

Thus

$$\int \tilde{A}\tilde{B} = \sum_{i=1}^{k} \sum_{j=1}^{l} (\lambda_i - \lambda_{i+1})(\mu_j - \mu_{j+1}) \min(\alpha_i, \beta_j).$$

But

$$m(AB) = \sum_{i=1}^{k} \sum_{j=1}^{l} \lambda_i \mu_j m(E_i F_j) = \sum_{i=1}^{k} \sum_{j=1}^{l} (\lambda_i - \lambda_{i+1})(\mu_j - \mu_{j+1}) M_{ij},$$

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where $$M_{ij} = \sum_{r=1}^{i}\sum_{s=1}^{j} m(E_r F_s) = m\left(\sum_{r=1}^{i} E_r . \sum_{s=1}^{j} F_s\right) \leqslant \min(\alpha_i, \beta_j),$$

since

$$m(PQ) \leqslant \min(m(P), m(Q))$$

for projections $P, Q \in \mathscr{F}$.

3·2. PROPOSITION. *Let* $A, B \in \mathscr{F}^+$. *Then* (i) $m(AB) \leqslant \int \tilde{A}\tilde{B}$, (ii) $\|AB\|_1 \leqslant \int \tilde{A}\tilde{B}$.

Proof. (i) Choose sequences $A_n = \sum_{i=1}^{k_n} \lambda_{i,n} E_{i,n}$, $B_n = \sum_{j=1}^{l_n} \mu_{j,n} F_{j,n}$ of positive simple operators such that

$$A_n \leqslant A, \quad B_n \leqslant B \quad \text{and} \quad \|A - A_n\|_\infty \to 0, \quad \|B - B_n\|_\infty \to 0.$$

Then

$$\tilde{A}_n \leqslant \tilde{A}, \quad \tilde{B}_n \leqslant \tilde{B}, \quad m(A_n B_n) \leqslant \int \tilde{A}_n \tilde{B}_n \leqslant \int \tilde{A}\tilde{B},$$

and $m(AB) = \lim m(A_n B_n)$, since the restriction of m to the finite von Neumann algebra $E\mathscr{R}E$ is norm-continuous, where $E = [A(\mathscr{H})] \vee [B(\mathscr{H})] \in \mathscr{F}$.

(ii) Let $AB = V|AB|$ be the polar decomposition of AB, so that

$$\|AB\|_1 = m(|AB|) = m(V^*AB) = m(ABV^*).$$

By (1) 1·6·10 Lemma 2,

$$\begin{aligned}(m(ABV^*))^2 &\leqslant m(A|BV^*|) . m(A|(BV^*)^*|) \\ &\leqslant \int \tilde{A}|BV^*|^\sim . \int \tilde{A}|(BV^*)^*|^\sim \\ &\leqslant (\int \tilde{A}\tilde{B})^2.\end{aligned}$$

3·3. THEOREM. *Let* $A, B \in \mathscr{K}$ *and suppose that* $\int \tilde{A}\tilde{B} < \infty$. *Then* $AB \in L^1$ *and* $\|AB\|_1 \leqslant \int \tilde{A}\tilde{B}$.

Proof. Since $AB \in L^1$ iff $|A||B^*| \in L^1$, and $\|AB\|_1 = \||A||B^*|\|_1$, and also $\tilde{A} = |\tilde{A}|$, $\tilde{B} = |B^*|^\sim$, we may assume that $A, B \geqslant 0$. First suppose that both, $A, B \in \mathscr{C}$.

$$A = \int_0^\infty \lambda\, dE(\lambda), \quad B = \int_0^\infty \lambda\, dF(\lambda)$$

with $$m(I - E(\lambda)) < \infty, \quad m(I - F(\lambda)) < \infty \quad \text{for all} \quad \lambda > 0,$$

and let $$A_n = A(E(n) - E(1/n)), \quad B_n = B(F(n) - F(1/n)) \in \mathscr{F}.$$

Then $\|A_n B_n\|_1 \leqslant \int \tilde{A}_n \tilde{B}_n \leqslant \int \tilde{A}\tilde{B}$, and $\|AB\|_1 \leqslant \varliminf \|A_n B_n\|_1$, by Theorem 2·9, since $A_n \to A$ and $B_n \to B$ metrically nearly everywhere, and so $A_n B_n \to AB$ in measure.

Suppose next that $A \in \mathscr{K} \setminus \mathscr{C}$ (the case where $B \in \mathscr{K} \setminus \mathscr{C}$ is similar). Then

$$\lambda_0 = \lim_{\alpha \to \infty} \tilde{A}(\alpha) = \inf\{\lambda : m(1 - E(\lambda)) < \infty\} > 0,$$

and since $\int \tilde{A}\tilde{B} < \infty$, $\int \tilde{B} < \infty$, whence $B \in L^1$, Let

$$A_1 = AE(\lambda_0) + \lambda_0(I - E(\lambda_0)),$$

$$A_2 = (A - \lambda_0 I)(I - E(\lambda_0)) = \int_{\lambda_0}^\infty (\lambda - \lambda_0)\, dE(\lambda_0) = \int_0^\infty \lambda\, dE(\lambda + \lambda_0),$$

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so that

$$A = A_1 + A_2, \quad A_1 \in \mathscr{R}, \quad \|A_1\|_\infty = \lambda_0,$$

and

$$\tilde{A}_2 = \tilde{A} - \lambda_0, \quad A_2 \in \mathscr{C} \quad (\text{thus } \mathscr{K} = \mathscr{R} + \mathscr{C}).$$

Then $\int \tilde{A}_2 \tilde{B} < \infty$, $A_2 B \in L^1$, hence $AB = A_1 B + A_2 B \in L^1$, and

$$\begin{aligned} \|AB\|_1 &\leqslant \|A_1\|_\infty \|B\|_1 + \|A_2 B\|_1 \\ &\leqslant \lambda_0 \int \tilde{B} + \int \tilde{A}_2 \tilde{B} \\ &= \int \tilde{A}\tilde{B}. \end{aligned}$$

3·4. THEOREM. *If $A \in L^p$, $B \in L^q$, $p, q > 1$, $1/p + 1/q = 1$, then*

$$AB \in L^1 \quad \textit{and} \quad \|AB\|_1 \leqslant \|A\|_p \|B\|_q.$$

Proof. $\|AB\|_1 \leqslant \int \tilde{A}\tilde{B} \leqslant (\int \tilde{A}^p)^{1/p} (\int \tilde{B}^q)^{1/q} = \|A\|_p \|B\|_q < \infty$.

3·5. LEMMA. *Let $p > 1$, $1/p + 1/q = 1$, $A \in \mathscr{L}$. If $AB \in L^1$ for every $B \in \mathscr{F}$ and $\sup\{|m(AB)| : B \in \mathscr{F}, \|B\|_q \leqslant 1\} = N < \infty$, then $A \in L^p$ and $\|A\|_p \leqslant N$.*

Proof. Let $A = W|A|$, $|A| = \int_0^\infty \lambda \, dE(\lambda)$. We first show that $m(I - E(\lambda)) < \infty$ for all $\lambda > 0$, so that $A \in \mathscr{C}$. Suppose that $\lambda > 0$, $m(I - E(\lambda)) = \infty$, and choose a projection $E \in \mathscr{F}$, such that $E \leqslant I - E(\lambda)$ and $m(E) > \lambda^{-p} N^p$. Then $EAE \geqslant \lambda E$ and

$$N \geqslant \frac{m(AE)}{\|E\|_q} = \frac{m(EAE)}{(m(E))^{1/q}} \geqslant \lambda (m(E))^{1-1/q} > N,$$

a contradiction. Now let

$$B_n = |A|^{p-1}(E(n) - E(1/n))\, W^* \in \mathscr{F}.$$

Then

$$AB_n = |B_n|^q = W|A|^p (E(n) - E(1/n))\, W^*,$$

$$N \geqslant \frac{m(AB_n)}{\|B_n\|_q} = (m(|A|^p (E(n) - E(1/n))))^{1/p},$$

and, since

$$|A|^p (E(n) - E(1/n)) \uparrow |A|^p \quad \text{m.n.e.},$$

$$|A|^p \in L^1 \quad \text{and} \quad m(|A|^p) \leqslant N^p.$$

3·6. COROLLARY. *If $A \in L^p$, where $p > 1$, and $1/p + 1/q = 1$, then*

$$\begin{aligned} \|A\|_p &= \sup\{|m(AB)| : B \in L^q, \quad \|B\|_q \leqslant 1\} \\ &= \sup\{\|AB\|_1 : B \in L^q, \quad \|B\|_q \leqslant 1\}. \end{aligned}$$

Proof. If $A \in L^p$ and $\|B\|_q \leqslant 1$, then $|m(AB)| \leqslant \|AB\|_1 \leqslant \|A\|_p$. The converse follows from 3·5.

3·7. THEOREM. (i) *If $A, B \in L^p$ then $A + B \in L^p$ and $\|A + B\|_p \leqslant \|A\|_p + \|B\|_p$. It follows that L^p is a vector space and that $A \to \|A\|_p$ is a norm on L^p.*

(ii) *$\mathscr{F}$ is dense in L^p.*

(iii) *L^p is complete.*

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Proof. (i) If $A, B \in L^p$, then $A+B \in L^p$, by 2·4 (iii), (viii). It now follows from (3·6) that $\|A+B\|_p \leqslant \|A\|_p + \|B\|_p$.

(ii) If $A \in L^p$, let $A = W|A|$, $|A| = \int \lambda dE(\lambda)$ and let $A_n = W|A|(E(n)-E(1/n))$. Then $A_n \in \mathscr{F}$ and $\|A-A_n\|_p = m(|A|^p(1-E(n)+E(1/n))) \to 0$.

(iii) Let (A_n) be a cauchy sequence in L^p. By (6) theorem 3·2, (A_n) is cauchy in measure and, as in the proof of (5) theorem 13, has a m.n.e.–convergent subsequence. To save notation, we assume (A_n) is itself m.n.e.–convergent to $A \in \mathscr{C}$. Let $B \in L^q$. Then $\|A_nB - A_mB\|_1 \leqslant \|A_n - A_m\|_p \|B\|_q$ so that (A_nB) is cauchy in L^1, and since $A_nB \to AB$ m.n.e., $AB \in L^1$ and $A_nB \to AB$ in L^1. If $\|B\|_q \leqslant 1$, then

$$|m(AB)| = \lim |m(A_nB)| \leqslant \underline{\lim} \|A_n\|_p \|B\|_q,$$

and it follows by 3·5 that $A \in L^p$ and $\|A\|_p \leqslant \underline{\lim} \|A_n\|_p$. Finally, if $B \in L^q$ and $\|B\|_q \leqslant 1$, then

$$|m(A_n - A)B| = \lim_{m\to\infty} |m(A_n - A_m)B| \leqslant \lim_{m\to\infty} \|A_nB - A_mB\|_1 \leqslant \lim_{m\to\infty} \|A_n - A_m\|_p \|B\|_q,$$

whence

$$\|A_n - A\|_p = \sup\{|m(A_n - A)B| : B \in L^q, \|B\|_q \leqslant 1\} \leqslant \lim_{m\to\infty} \|A_n - A_m\|_p,$$

so that $A_n \to A$ in L^p.

3·8. *Remark.* The full strength of theorem 2·9 is not needed for the proof of theorem 3·7 (iii) or that of (5) theorem 13; all that is necessary is the result that if (A_n) is a cauchy sequence in L^1 and $A_n \to 0$ m.n.e., then $\|A_n\|_1 \to 0$. A proof can be given on the lines of (5) theorem 11, making use of (5) lemma 13·1.

3·9. THEOREM. *Let $p > 1$, $1/p + 1/q = 1$. If $A \in \mathscr{L}$ and $AB \in L^1$ for all $B \in L^q$, then $A \in L^p$.*

Proof. We show that the mapping $B \mapsto AB$ $(L^q \to L^1)$ has closed graph, and is therefore continuous. There exists then a number N, such that $\|AB\|_1 \leqslant N\|B\|_q$ $(B \in L^q)$, and the result follows from 3·5. Suppose that $B_n \to B$ in L^q and $AB_n \to C$ in L^1. Then $B_n \to B$ locally in measure, hence $AB_n \to AB$ locally in measure, but $AB_n \to C$ locally in measure, whence $C = AB$.

4. *Duality.* It follows from the results of the preceding section that the mapping $A \mapsto f_A$, where $f_A(B) = m(AB)$ $(B \in L^p)$, is a linear isometry of L^q $(q > 1,\ 1/p + 1/q = 1)$ onto a closed subspace of the norm-dual of L^p. We show now that if f is any continuous linear functional on L^p, then $f = f_A$ for some $A \in L^q$. The construction of the operator A requires the following result, which is a form of the Radon–Nikodym theorem.

4·1. THEOREM. *Let f be an ultraweakly continuous linear functional on $\mathscr{R}$. Then there exists $A \in L^1$ such that*

$$f(B) = m(AB) \quad (B \in \mathscr{R}).$$

Proof. It is shown in (1) 1·6·10 that the mapping $A \mapsto f_A$, where $f_A(B) = m(AB)$ $(B \in \mathscr{R})$ is a linear isometry of $\mathscr{F}$, equipped with the L^1-norm, onto a dense subspace

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of the ultraweak dual $\mathscr{R}_*$ of $\mathscr{R}$. Let $f \in \mathscr{R}_*$. Then there exists a sequence (A_n) in $\mathscr{F}$, such that $f = \lim f_{A_n}$. Let $A = \lim A_n \in L^1$. Then for any $B \in \mathscr{R}$, $AB = \lim A_n B \in L^1$, and $f(B) = \lim f_{A_n}(B) = \lim m(A_n B) = m(AB)$.

4·2. THEOREM. *Suppose that we are given, for each projection $E \in \mathscr{F}$, an operator $A_E \in L^1$ such that*

(i) $A_F = A_E F$ *whenever* $E, F \in \mathscr{F}$, $F \leqslant E$,

(ii) $E.A_E$ *is self-adjoint for each* $E \in \mathscr{F}$.

Then there exists $A = A^* \in \mathscr{L}$, such that $A_E = AE$ for each $E \in \mathscr{F}$.

Proof. Define an operator A_0 by

$$\mathscr{D}A_0 = \bigcup_{E \in \mathscr{F}} (\mathscr{D}_{A_E} \cap E(\mathscr{H})),$$

$$A_0 x = A_E x \quad (x \in \mathscr{D}_{A_E} \cap E(\mathscr{H}), E \in \mathscr{F}).$$

The operator A_0 is well defined and linear, for if $x \in (\mathscr{D}_{A_E} \cap E(\mathscr{H})) \cap (\mathscr{D}_{A_F} \cap F(\mathscr{H}))$, then, since

$$A_E(E \wedge F) = A_F(E \wedge F) = A_{E \wedge F},\ x \in \mathscr{D}_{A_{E \wedge F}} \cap (E \wedge F)(\mathscr{H}) \quad \text{and} \quad A_E x = A_{E \wedge F} x = A_F x;$$

and if $$x \in \mathscr{D}_{A_E} \cap E(\mathscr{H}) \quad \text{and} \quad y \in \mathscr{D}_{A_F} \cap F(\mathscr{H}),$$

then, since

$$A_E = A_{E \vee F} E \quad \text{and} \quad A_F = A_{E \vee F} F,$$

$$x, y, x+y \in \mathscr{D}_{A_{E \vee F}} \cap (E \vee F)(\mathscr{H}) \quad \text{and} \quad A_{E \vee F}(x+y) = A_{E \vee F} x + A_{E \vee F} y.$$

Similarly,

$$\begin{aligned} \langle A_0 x, y \rangle &= \langle A_{E \vee F} x, (E \vee F) y \rangle = \langle (E \vee F) A_{E \vee F} x, y \rangle \\ &= \langle x, (E \vee F) A_{E \vee F} y \rangle = \langle (E \vee F) x, A_{E \vee F} y \rangle \\ &= \langle x, A_0 y \rangle, \end{aligned}$$

so that A_0 is symmetric.

Further, if U is any unitary operator in the commutant $\mathscr{R}'$ of $\mathscr{R}$, and $x \in \mathscr{D}_{A_E} \cap E(\mathscr{H})$, then

$$Ux = UEx = EUx \in \mathscr{D}_{A_E} \cap E(\mathscr{H})$$

and $A_0 Ux = A_E Ux = UA_E x = UA_0 x$, proving that $A_0 \eta \mathscr{R}$. Evidently $A_E = A_0 E$ for each projection $E \in \mathscr{F}$. Since A_0 is symmetric, it has a symmetric closure $A \eta \mathscr{R}$, and $A_E = AE$ for each projection $E \in \mathscr{F}$. Since $\mathscr{D}_{A_E} \cap E(\mathscr{H})$ is dense in $E(\mathscr{H})$ for each E, and $\bigcup_{E \in \mathscr{F}} E(\mathscr{H})$ is dense in $\mathscr{H}$, $\mathscr{D}_{A_0}$ and therefore $\mathscr{D}_A$ is dense in $\mathscr{H}$, and so A has a polar decomposition $A = W|A|$, $W \in \mathscr{R}$. Let $|A| = \int_0^\infty \lambda \, dE(\lambda)$, then $|A| \eta \mathscr{R}$, $E(\lambda) \in \mathscr{R}$ for all λ.

Let $\{P_\alpha\}$ be a set of minimal non-zero projections in $\mathscr{R}$ that is maximal, subject to the condition that the central carriers $Q_\alpha = C(P_\alpha)$ are mutually orthogonal. Then $Q_\alpha \mathscr{R}$ is a type I factor for each α and $(I - \Sigma Q_\alpha)\mathscr{R}$ contains no minimal non-zero projections. We prove that AQ_α is bounded for each α and that $A(I - \Sigma Q_\alpha) \in L^1 + \mathscr{R}$, so that $A \in \mathscr{L}$ by (7) theorem 2·1·2 (iii).

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If E is a minimal non-zero projection with $E \leqslant Q_\alpha$, then $A_E = A_E E$ is bounded, since the only possible spectral projections of $|A_E| = E|A_E|E$ are 0, E, $I-E$, I. If E is a projection with $E \leqslant Q_\alpha$, $m(E) < \infty$, then there exist minimal non-zero projections $E_1, \ldots, E_n$, such that

$$E = \sum_{i=1}^{n} E_i,$$

so that

$$A_E = A_E E = \sum_{i=1}^{n} A_E E_i = \sum_{i=1}^{n} A_{E_i}$$

is bounded.

For each, α, there is a Hilbert space $\mathscr{H}_\alpha$ and a *-isomorphism ϕ_α of $\mathscr{R}Q_\alpha$ onto $\mathscr{B}(\mathscr{H}_\alpha)$. Define an operator A_α on $\mathscr{H}_\alpha$ by $A_\alpha(x) = \phi_\alpha(A_E)(x)$, where E is a projection such that $E \leqslant Q_\alpha$, $m(E) < \infty$, $\phi_\alpha(E)x = x$. By a repetition, with simplification, of the argument of the first part of this proof, A_α is well defined on all of $\mathscr{H}_\alpha$, linear, symmetric, and hence bounded, and satisfies $A_\alpha \phi_\alpha(E) = \phi_\alpha(A_E)$, $\phi_\alpha^{-1}(A_\alpha)E = A_E$ for all projections E such that $E \leqslant Q_\alpha$, $m(E) < \infty$. (The set of all such projections is mapped by ϕ_α onto the set of all projections of finite rank defined on $\mathscr{H}_\alpha$.) Since $AQ_\alpha E = A_E$ for all such projections, the closed operator AQ_α coincides with the bounded operator $\phi_\alpha^{-1}(A_\alpha)$ on the dense subset $\bigcup\{E(Q_\alpha\mathscr{H}) : E \leqslant Q_\alpha, m(E) < \infty\}$ of $Q_\alpha(\mathscr{H})$, and so AQ_α is bounded.

For the rest of the proof to save notation, we may assume that $\mathscr{R}$ contains no minimal non-zero projections. Suppose that $m(I-E(\lambda)) = \infty$ for all $\lambda > 0$. Let $\lambda_0 = 1$, and choose, inductively, λ_n such that $\lambda_n \geqslant 2^n$ and $m(E(\lambda_n) - E(\lambda_{n-1})) \geqslant 2^{-n}$. Since $\mathscr{R}$ contains no minimal non-zero projections, we can choose projections $F_n \in \mathscr{R}$ such that $F_n \leqslant E(\lambda_n) - E(\lambda_{n-1})$ and $m(F_n) = 2^{-n}$, and let $F = \sum_{n=1}^{\infty} F_n$. Then $m(F) = 1$ and $F|A|F = FW^*AF = FW^*A_F \in L^1$, and

$$\infty > m(F|A|F) = \sum_{n=1}^{\infty} m(F_n|A|F_n).$$

But

$$|A|(E(\lambda_n) - E(\lambda_{n-1})) \geqslant \lambda_{n-1}(E(\lambda_n) - E(\lambda_{n-1})) \geqslant 2^{n-1}(E(\lambda_n) - E(\lambda_{n-1})),$$

$$F_n|A|F_n \geqslant 2^{n-1}F_n, \quad m(F_n|A|F_n) \geqslant 2^{n-1}m(F_n) = \tfrac{1}{2},$$

yielding a contradiction. It follows that $m(I-E(\lambda)) < \infty$ for some $\lambda > 0$, and

$$|A| = |A|E(\lambda) + |A|(1-E(\lambda)) \in \mathscr{R} + L^1.$$

Since A is symmetric and locally measurable, $A = A^*$.

4·3. THEOREM. *Let f be a linear functional on $\mathscr{F}$, such that*

(i) $f(B^*) = \overline{f(B)}$ $(B \in \mathscr{F})$,

(ii) *for each projection $E \in \mathscr{F}$, the restriction of f to the ultraweakly closed right ideal $E\mathscr{R}$ is ultraweakly continuous.*

Then there exists an operator $A = A^* \in \mathscr{L}$ such that $f(B) = m(AB)$ $(B \in \mathscr{F})$.

Proof. For each projection $E \in \mathscr{F}$, the mapping $B \to EB$ $(\mathscr{R} \to E\mathscr{R})$ is ultraweakly continuous, and so $B \to f(EB)$ defines an ultraweakly continuous linear functional

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on $\mathscr{R}$. By Theorem 4·1 there exists an operator $A_E \in L^1$ such that $f(EB) = m(A_E B)$ $(B \in \mathscr{R})$. If $F \leqslant E$,

$$m(A_F B) = f(FB) = f(EFB) = m(A_E FB) \quad (B \in \mathscr{R}) \quad \text{whence} \quad A_F = A_E F.$$

Also

$$m(EA_E B) = m(A_E BE) = f(EBE) = \overline{f(EB^*E)} = \overline{m(A_F B^* E)}$$

$$= m(EBA_E^*) = m(A_E^* EB) \quad (B \in \mathscr{R}),$$

and so

$$EA_E = A_E^* E = (EA_E)^*.$$

By Theorem 4·2, there exists $A = A^* \in \mathscr{L}$ such that $A_E = AE$ $(E \in \mathscr{F})$. If $B \in \mathscr{F}$, then $E = [B(\mathscr{H})] \in \mathscr{F}$ and $B = EB$,

$$f(B) = f(EB) = m(A_E B) = m(AEB) = m(AB).$$

4·4. THEOREM. *Let f be a continuous linear functional on L^p $(p > 1)$. Then there exists $A \in L^q$ $(1/p + 1/q = 1)$, such that $f(B) = m(AB)$ $(B \in L^p)$.*

Proof. By considering the real and imaginary parts $g = \frac{1}{2}(f + f^*)$, $h = (1/2i)(f - f^*)$ of $f = g + ih$, where $f^*(B) = \overline{f(B^*)}$ $(B \in L^p)$, it is clear that we may assume $f(B) = f(B^*)$ $(B \in L^p)$.

Let E be a projection in $\mathscr{F}$, and let (A_α) be a net in $\mathscr{R}$ such that

$$\|A_\alpha\|_\infty \leqslant 1, \quad A_\alpha = A_\alpha E \quad \text{and} \quad A_\alpha \to 0 \quad \text{strongly.}$$

Then $|A_\alpha| = E|A_\alpha|\,E \to 0$ strongly and $|A_\alpha|^p = E|A_\alpha|^p E \to 0$ strongly,

since $|A_\alpha|^p \leqslant |A_\alpha|$. Hence, since m is strongly continuous on

$$\{A \in \mathscr{R}; \|A\|_\infty \leqslant 1,\ A = EAE\}, \quad m(|A_\alpha|^p) \to 0, \|A_\alpha\|_p \to 0, \quad \text{and} \quad f(A_\alpha) \to 0.$$

Thus f is strongly continuous at 0 on $\{A \in \mathscr{R}: \|A\|_\infty \leqslant 1,\ A = AE\}$, hence ultrastrongly and therefore ultraweakly continuous on $\mathscr{R}E = \{A \in \mathscr{R};\ A = AE\}$.

Since the mapping $B \to B^*$ is ultraweakly continuous on $\mathscr{R}$ and $(E\mathscr{R})^* = \mathscr{R}E$, f is ultraweakly continuous on $E\mathscr{R}$. By Theorem 4·3 there exists $A \in L$ such that $f(B) = m(AB)$ $(B \in \mathscr{F})$. Let $B \in L^p$, and let (B_n) be a sequence in $\mathscr{F}$, such that $B_n \to B$ in L^p. Then $B_n \to B$ in measure and so $AB_n \to AB$ locally in measure. We have

$$\|AB_n - AB_m\|_1 = \sup\{|m(A(B_n - B_m)X)| : X \in \mathscr{R}, \|X\|_\infty \leqslant 1\}$$

$$= \sup\{|f((B_n - B_m)X)| : X \in \mathscr{R}, \|X\|_\infty \leqslant 1\}$$

$$\leqslant \|f\| \cdot \|B_n - B_m\|_p,$$

hence $AB_n \to AB$ in L^1, and $f(B) = \lim f(B_n) = \lim m(AB_n) = m(AB)$.

By Lemma 3·5, $A \in L^q$.

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