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BOOK REVIEW

Multidimensional Continued Fractions

By Fritz Schweiger
Oxford Science Publications

The field of multidimensional continued fraction algorithms (MCFAs) is at the magnificent crossroads of number theory and ergodic theory. The number theoretical aspects covered in this book are convergence, periodicity and Diophantine approximations. Moreover, it covers their dynamical aspects such as ergodic and metric properties. The field is a rich source of examples in ergodic theory and many dynamical systems have underlying MCFAs. There has been much progress made on the subject in the last few decades and it deserves texts such as the present one. So far the books dedicated to MCFAs are a survey by Brentjes [**B**], which concerns their arithmetical and computational aspects, and an earlier book by Schweiger [**S**], which regards their ergodic and metric properties. There is an intersection between Schweiger's new book and his earlier one.

The regular continued fraction expresses an irrational number s as

$$s = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots}},$$

where $a_0, a_1, \ldots \in \mathbb{Z}$ and $a_n \geq 1$, for $n \geq 1$. The continued fraction expansion is a classical mathematical notion, it deserves attention in its own right, appears in a variety of mathematical branches and is an important mathematical tool. The MCFAs appeared when mathematicians tried to extend some properties of the continued fraction expansion. The first MCFA was defined by Jacobi. A classical property of the continued fractions is Lagrange's Theorem which characterizes the quadratic surds: an irrational number is a quadratic surd if, and only if, its continued fraction expansion is ultimately periodic. Jacobi made an attempt to characterize cubic irrationals as well, proposing a higher-dimensional extension known as the Jacobi–Perron algorithm.

Another classical property of the continued fraction expansion is Diophantine approximations. The continued fraction expansion gives the best rational approximations of a real number. Beginning with a geometrical interpretation of continued fractions, Poincaré proposed an extension of the classical algorithm aiming to obtain Diophantine approximations for n-uples of real numbers.

Jacobi and Poincaré were not successful in their projects, nevertheless they developed marvellous ideas which have many applications. Their proposals were so natural that, independently, similar generalizations of the continued fractions appeared. To understand the reason behind the multitude of examples, we recall that the continued fraction is the

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extension of the Euclidean algorithm (EA) to real numbers. We write the EA as the homogeneous algorithm

$$f : (x_1, x_2) \in \mathbb{R}_+^2 \mapsto \begin{cases} (x_1 - x_2, x_2), & \text{if } x_1 \geq x_2, \\ (x_1, x_2 - x_1), & \text{otherwise.} \end{cases}$$

Let $p_1,\ p_2 \in \mathbb{Z}_+$ and d be their greatest common divisor. There exists $k \geq 0$ such that $f^k(p_1, p_2) = (0, d)$ or $(d, 0)$. Inducing f in a suitable set yields the map

$$g : (x_1, x_2) \in \mathbb{R}_+^2 \mapsto \begin{cases} (x_1 - [x_1/x_2]x_2, x_2), & \text{if } x_1 \geq x_2, \\ (x_1, x_2 - [x_2/x_1]x_1), & \text{otherwise,} \end{cases}$$

where $[s]$ means the integer part of s. For dynamical purposes it is adequate sometimes to renormalize f and g. We can project on the simplex $\{(x_1, x_2) \in \mathbb{R}_+^2 : x_1 + x_2 = 1\}$ to keep the two-dimensional structure. To relate the EA with the continued fraction expansion another renormalization is convenient, we set for $0 < s < 1$,

$$s \equiv (1, s) \mapsto g(1, s) = (1/s - [1/s], 1) \equiv 1/s - [1/s]$$

and the Gauss map appears. The Jacobi–Perron algorithm is a generalization of the above map:

$$(u_1, \ldots, u_n) \in \ ]0, 1[^n \mapsto (u_2/u_1 - [u_2/u_1], \ldots, u_n/u_1 - [u_n/u_1], 1/u_1 - [1/u_1]).$$

Letting $1 \leq i < j \leq n$, we have a naive extension of f to $\mathbb{R}^n$,

$$f_{ij} : (x_1, \ldots, x_n) \in \mathbb{R}_+^2 \mapsto \begin{cases} (\ldots, x_i - x_j, \ldots, x_j, \ldots), & \text{if } x_i \geq x_j, \\ (\ldots, x_i, \ldots, x_j - x_i, \ldots), & \text{otherwise.} \end{cases}$$

Examples of homogeneous MCFAs are finite compositions of the above maps, chains of these maps or suitable well-defined Poincaré first return maps of the resulting maps. For computational purposes sometimes it is useful to permute the coordinates and in this case new combinatorial phenomena occur; for example, if instead of f we consider the map $(x_1, x_2) \mapsto f(x_2, x_1)$.

Using $E_1 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $E_2 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, the elementary matrices of $SL(2, \mathbb{Z})$, the definition of f becomes

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}_+^2 \mapsto \begin{cases} E_1^{-1}x, & \text{if } x_1 \geq x_2, \\ E_2^{-1}x, & \text{otherwise.} \end{cases}$$

In the n-dimensional case the maps f_{ij} have a similar description. The arithmetical properties of $SL(2, \mathbb{Z})$ make its linear action easier to study than that of $SL(n, \mathbb{Z})$, for $n \geq 3$. This is the core of the missing good Diophantine approximation properties of the MCFAs.

The EA is related to the horocycle flow; this is the work begun by Artin [**A**] and Hedlund [**He**]. Relating flows and MCFAs, an interesting example was discovered by Veech [**V**]. His work considers an inducing algorithm for interval exchange transformations called Rauzy induction. The algorithm describes how to stack rectangles over the intervals to build a flow, or a measured foliation, on an orientable surface of negative

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Euler characteristic, and *vice versa* (for non-orientable surfaces one uses flipped intervals). The algorithm is a MCFA and gives a fruitful approach to study the dynamics of interval exchange transformations and measured foliations.

This book is devoted to MCFAs in their renormalized version. Roughly speaking the text is divided into two parts. In the first part basic theoretical notions are presented. The notion of fibred systems is used and there is an entire chapter dedicated to their ergodic properties. The basic tools of ergodic theory are developed, such as Rényi's condition which ensures the existence of an ergodic finite invariant measure. Next, numerous examples are discussed for several chapters. The algorithms have many names: Brun, Guting, Parry–Daniels, Selmer, subtractive, fully subtractive, etc. It would have been very nice if the author could have shown some links between these algorithms. Unfortunately it appears that each case is treated independently. One may therefore regret that there is no apparent unifying principle in the book. It is slightly annoying that, although the subject is rich in elementary motivations (flows, relatively prime numbers, geometrical description, etc.) only a few of them are shown in the text.

Some chapters are very difficult to read, as a lot of the work is left to the reader. A typical example occurs in Chapter 11 which deals with interval exchange transformations: the techniques developed previously to derive the density of the invariant measure are not applied to the case. Even restricting himself to the exchanges of three intervals, the author proposes an approach based on Veech's paper **[V]** to which the reader is asked to refer.

Considering the goal that it wants to achieve, which is to give an account of a great variety of topics in ergodic theory and number theory of MCFAs, the book will be very useful as a reference. The subject matter is well selected; it is also a valuable source of open questions and a good guide for a very scattered literature, due to the wide scope of results that it covers.

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Arnaldo Nogueira
Instituto de Matemática
Universidade Federal do Rio de Janeiro
Caixa Postal 68530
21945-970 Rio de Janiero
RYJ
Brazil

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