2

The Abstract Setting

We are interested in discrete structures like graphs, hypergraphs, and simplicial complexes, and we want to study them via spectral properties of Laplace type operators defined on them. These operators perform some kind of local averaging. The eigenvalues and eigenfunctions of such operators can be obtained as critical values and points of Rayleigh quotients. Symmetry groups of the underlying structures have representations on eigenspaces.

Let us start abstractly.

2.1 A Category Theoretical Perspective

A category consists of *objects* $A, B, C, \ldots$ and *morphisms* $f\colon A \to B, g\colon B \to C, \ldots$ between them. (Because of their graphical representation, morphisms are also called *arrows*.) Morphisms can be composed if the codomain of one is the domain of the others, for example, in the case at hand, we can form $g \circ f\colon A \to C$. The composition is associative, and every object carries (at least) the identity morphism, which is neutral for compositions. A *functor* from one category to another maps objects to objects and morphisms between objects to morphisms between their images.

For an introduction to category theory, see, for instance, [147] or any other relevant textbook.

2.1.1 The Static Picture

We start with basic category $\mathbf{I}$ with just two objects $\boldsymbol{0}, \boldsymbol{1}$ and, besides their identity morphisms, only a single morphism

$$\boldsymbol{1} \longrightarrow \boldsymbol{0}$$

and a *functor*

$$H\colon \mathbf{I} \to \mathbf{Sets}$$

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into the category of sets. Thus, we have two sets R and U and a map

$$h\colon R \to U. \tag{2.1.1}$$

We shall usually assume that the sets R and U are finite.

For our purposes, we shall take

$$U = \mathcal{P}(V),$$

the set of subsets of a set V. We note that $\mathcal{P}(V)$ defines a simplicial complex (see Definition 2.3.1), its Čech complex. The vertices of this complex are the elements of $\mathcal{P}(V)$, that is, the subsets of V. Such subsets $V_0, V_1, \ldots, V_k$ define a k-simplex whenever

$$V_0 \cap \ldots \cap V_k \neq \emptyset.$$

Turning to the other set involved, R, we shall interpret an $r \in R$ as a *relation*. Thus, to each relation $r \in R$, we associate a set of elements of V. Without loss of generality, we can assume that $V = \bigcup_{r\in R} h(r)$.

This is a basic *hypergraph*. Thus, a hypergraph is a cover of a vertex set V. If every $h(r) \subset V$ has precisely two elements, we have an ordinary *graph*.

This is not yet enough structure for us, however. So, we turn to Section 2.1.2.

2.1.2 The Dynamical Picture

We consider the basic category **J** with two objects and two nontrivial arrows

$$\boldsymbol{1} \rightrightarrows \boldsymbol{0}$$

and a *functor*

$$H\colon \mathbf{J} \to \mathbf{Sets}.$$

Thus, we have two sets R and U and two maps

$$h_i, h_t\colon R \to U. \tag{2.1.2}$$

This is a *directed* graph, where R is the edge set and U is the vertex set, and the two arrows h_i, h_t map an edge to its initial and terminal vertex, respectively.

For our purposes, again

$$U = \mathcal{P}(V)$$

is the set of subsets of a set V.

This gives us more structure, and we can interpret a relation $r \in R$ as a *hyperedge*. That is, to r, we associate two sets of elements of V, the input

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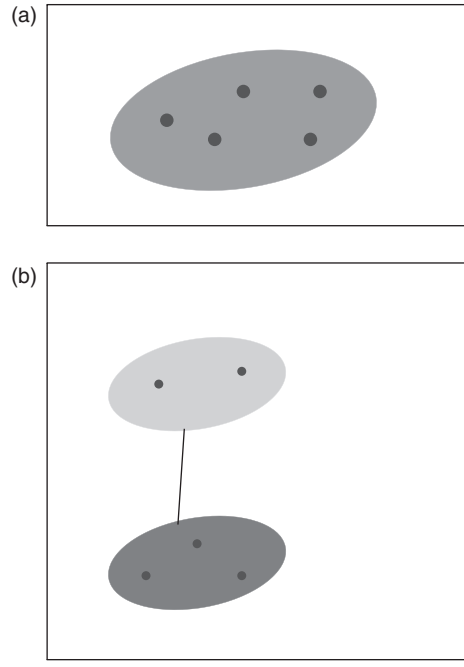


Figure 2.1 An ordinary (a) and an oriented (b) hypergraph.

and the output of the hyperedge. We then have a *directed hypergraph*. Thus, a directed hypergraph consists of two families of subsets of a vertex set V.

We get an induced map

$$h_i(r) \mapsto h_t(r)$$

that maps an input of a hyperedge to its output. This, however, in general does **not** induce a simplicial map of the Čech complexes $\Sigma_i \to \Sigma_t$. Therefore, we should construct other invariants than those coming from these Čech complexes. We shall usually ignore the direction, that is, also consider the reverse direction and speak of *orientations*. In [153], we called this a *chemical hypergraph*.

In short, in an *oriented hypergraph*, one hyperedge can carry two orientations, corresponding to two directions.

In a *directed* hypergraph, a hyperedge goes in only one direction.

The difference between an ordinary and an oriented hypergraph is illustrated Figure 2.1. The hyperedge between the top and the bottom vertex set does not have a direction, so this is not a directed hypergraph.

Figure 2.2 shows a hypergraph with overlapping hyperedges.

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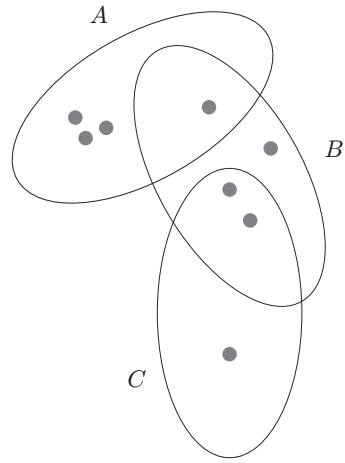


Figure 2.2 A hypergraph with overlapping hyperedges.

The hyperedges A and B have one vertex in common; the hyperedges B and C have two. There is no vertex that is in both A and C, and hence, in particular, none simultaneously contained in all three.

Another example, the hyperflower, introduced in Definition 4.9.9 and depicted in Figure 4.2, will play an important role in Chapter 4.9 and Section 11.5.

2.1.3 Probing a (Dynamical) Structure

We associate the Hilbert space $L^2(U)$ of functions to target U in (2.1.1) or (2.1.2). Giving a weight w_u to each $u \in U$, we define a scalar product for functions on U

$$(f,g)_U := \sum_u w_u f(u) g(u)$$

and let R operate on $L^2(U)$.

We can then consider *differences:* For a function $f : U \to \mathbb{R}$ and $h_i, h_t : R \to U$, we put

$$\delta f(r) := \sum_{u \in h_i(r)} f(u) - \sum_{v \in h_t(r)} f(v).$$

When we exchange $h_i(r)$ and $h_t(r)$, we get a $-$ sign. Therefore, we should consider oriented relations r. We denote the change of orientation, that is, exchanging $h_i(r)$ and $h_t(r)$, by a $-$ sign

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$$r \mapsto -r.$$

δ satisfies an important property, *closedness:* If

$$h_t(r_1) = h_i(r_2) \quad \text{and } h_t(r_2) = h_i(r_3) \quad \text{and } h_t(r_3) = h_i(r_1)$$

then

$$\delta f(r_1) + \delta f(r_2) + \delta f(r_3) = 0.$$

Thus, δ annihilates cycles.

As the next step, we also define a scalar product on functions on R

$$(\gamma, \omega)_R := \sum w_r \gamma(r)\omega(r)$$

to obtain the adjoint δ^*

$$(f, \delta^*\gamma)_U = (\delta f, \gamma)_R.$$

Since δ is antisymmetric, that is, $\delta f(-r) = -\delta f(r)$, δ^* vanishes on symmetric γ. Thus, if $\gamma(-r) = \gamma(r)$, $\delta^*(\gamma) = 0$. Hence, we may assume antisymmetry,

$$\gamma(-r) = -\gamma(r).$$

(If we wanted to include symmetric γ, we should choose a non-antisymmetric δ.)

We interpret

$$\delta f(r) := \sum_{u \in h_i(r)} f(u) - \sum_{v \in h_t(r)} f(v)$$

as a *boundary operator*. δ and its adjoint δ^*

$$(f, \delta^*\gamma)_U = (\delta f, \gamma)_R$$

allow us to define the *Laplacians*

$$\Delta_U := \delta^*\delta \colon L^2(U) \to L^2(U)$$
$$\Delta_R := \delta\delta^* \colon L^2(R) \to L^2(R)$$

Since

$$(\Delta_U f, g)_U = (\delta f, \delta g)_R = (f, \Delta_U g)_U$$
$$(\Delta_R \gamma, \omega)_R = (\delta^*\gamma, \delta^*\omega)_U = (\gamma, \Delta_R \omega)_R$$

Δ_U and Δ_R are both symmetric and non-negative and therefore have real, non-negative eigenvalues. Also, their nonzero eigenvalues coincide.

Let us summarize the steps that yield these operators [153].

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1. We give weights to the hyperedges. In the simplest case, each hyperedge has weight 1, and therefore each vertex v acquires weight hypdeg $v := \sum_{h:\, v\in h}(|h|-1)$, $|h|$ being the number of vertices contained in the hyperedge h;
2. define a scalar product for functions defined on hyperedges and a scalar product for functions defined on vertices, based on the weights;
3. define the boundary operator for functions defined on vertex sets, mapping them to functions defined on hyperedges;
4. find the adjoint operator based on the scalar products and
5. define the Laplace operators as the two different compositions of the boundary operator and its adjoint.

2.2 Relations and Norms

We now develop an analytical perspective, as opposed to the categorical one mentioned earlier. Now, the formal setting will be more concrete than before, but the analytical one will be more general. As before, we consider a set V. In the sequel, it will be usually finite, but for the moment, we only need it to be a compact space. (Most of what we are going to say also applies to infinite or noncompact spaces, but we do not want to address the analytical issues involved with that.) On this set, we consider observables, that is, functions $f: V \to \mathbb{R}$.

And as before, we furthermore assume that the elements of V can stand in relations. The observables should help us explore those relations. For instance, when V is finite (in which case, we write $|V|$ for the number of its elements), the simplest such relations are binary neighborhood relations, $v \sim v'$, and whenever such a relation holds, we may consider the difference $\delta f(v,v') := f(v') - f(v)$. The relations could be weighted, with weights $w_{vv'} \in \mathbb{R}$, offering us the notational convenience to put $w_{vv'} = 0$ when v and v' are not related. This convention will often be used in the sequel, usually without being explicitly declared. We call V *connected* if for any two $v, v' \in V$, we can find $v_0 = v, v_1, \ldots, v_m = v'$ with $w_{v_{i-1}v_i} \neq 0$ for $i = 1, \ldots, m$. We sometimes also write $w(v,v')$ instead of $w_{vv'}$.

Weights induce us to consider $\delta f(v,v') := w_{vv'}(f(v') - f(v))$. When the relations are symmetric, that is, $w_{vv'} = w_{v'v}$, and non-negative, i.e. $w_{vv'} \geq 0$ for all v, v', things become simpler. (When the weights are non-symmetric, one might have to consider observables $f: V \to \mathbb{C}$.) When V is a differentiable manifold (possibly with boundary), as the δ-operator, we may consider the differential $df(v)$ and apply it to tangent vectors.

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We could also have relations not between individual elements but between subsets of V. When $V_1, V_2 \subset V$ are related, we may consider differences of the form $\sum_{v\in V_2} f(v) - \sum_{v'\in V_1} f(v')$, and again, this can be refined by weights if desired. We can also work with higher order differentials as on simplicial complexes or on differentiable manifolds. In particular, in that case, we might have weights $w(v_1,\dots,v_k)$ between k vertices, for all values of k that are relevant in the context in question.

To become quantitative, we also need norms $\|.\|$, both on the set of observables on V and on their differentials on the set E of relations. We may then consider quotients

$$\frac{\|\delta f\|_E}{\|f\|_V}, \tag{2.2.1}$$

assuming, of course, that $\|f\|_V \neq 0$. Here, we have used the fact that both the observables and their differentials constitute vector spaces. Since norms are homogeneous, that is, $\|\lambda g\| = |\lambda|\|g\|$ when $\lambda \in \mathbb{R}$, we may normalize $\|f\|_V = 1$, that is, consider

$$\|\delta f\|_E \text{ on the unit sphere } \{\|f\|_V = 1\}. \tag{2.2.2}$$

It turns out to be advantageous to generalize this further, by not taking the norms as such, but some functions of them. The simplest possibility consists in raising them to power p and consider

$$\frac{\|\delta f\|_E^p}{\|f\|_V^p}, \tag{2.2.3}$$

for some choice of $1 \leq p \leq \infty$. The choice $p = 2$ is the most convenient one, and we have used in (2.2.3) to define the Laplace operators. It occurs naturally when the norms come from scalar products. In that case, the resulting eigenvalue problems become *linear*, which is of course a great mathematical advantage. Nevertheless, as we shall see, other choices of p, and in particular $p = 1$, that is, the original formulation (2.2.1), while more difficult to analyze mathematically, can provide important insight.

The basic idea then consists in looking for the critical values and the critical points of this functional (2.2.3) on the unit sphere, that is, with the normalization $\{\|f\|_V = 1\}$. These critical values will constitute the spectra that we are interested in.

We can of course also take more general monotonically increasing functions F, G and consider

$$\frac{F(\|\delta f\|_E)}{G(\|f\|_V)}. \tag{2.2.4}$$

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Again, suitable choices of F and G will let us obtain insight about the underlying structures.

2.3 Complexes and Orientations

We explained in Section 2.2 that we want to consider differences

$$\delta f(v,v') = f(v') - f(v) \tag{2.3.1}$$

of observations at points v, v' that are related or more general versions thereof. Now, such a difference (2.3.1) has a sign that changes if we interchange v and v'. While this does not matter if we take norms, as in (2.2.1), (2.2.3), and (2.2.4), for an analysis of the underlying structure of relations, this seems relevant and important.

Thus, the relations are naturally *directed*, that is, of the form $v' \to v$, and we may speak of v' as the *tail* and v as the *head* of the relation underlying (2.3.1).

In the *symmetric* case, we could then simply consider both differences $df(v,v') = f(v') - f(v)$ and $df(v',v) = f(v) - f(v')$, which would then give us a factor of 2 in (2.2.1) but would otherwise not change much.

Nevertheless, even in this case, it turns out that we should not neglect signs and consider oriented relations of the form $v' \to v$. This becomes clear when we also consider higher order relations. In the simplest case, such higher order relations can be expressed geometrically in terms of a *simplicial complex*.

Definition 2.3.1. Let V be a (usually finite) set of vertices. A *simplicial complex* Σ on V is a subset of its power set, $\Sigma \subset \mathcal{P}(V)$, that is, closed under taking subsets, that is, for a *simplex* $C \in \Sigma$, any of its subsets $C' \subset C$ is also a simplex in Σ. The empty set is also considered as a simplex.

A simplex C with $q+1$ vertices is called a *q-simplex* (where we consider q as its dimension), its subsimplices are called its *faces*, and its $(q-1)$-dimensional faces are called its *facets*. The dimension of a simplicial complex is the largest dimension among its simplices. A one-dimensional simplicial complex is a *graph*.

Let Σ_q be the collection of the q-simplices of Σ. We let $C_q = C_q(\Sigma)$ be the Abelian group with coefficients in $\mathbb{R}$ generated by the elements of Σ_q. We also write $C^q = C^q(\Sigma)$ for the linear functions from C_q to $\mathbb{R}$.

We usually assume that a simplicial complex is *connected*. This means that for any two of its nonempty simplices s, s', there exists a chain of simplices $s_0 = s, s_1, \ldots, s_m = s'$ such that any two adjacent simplices in this chain have at least one vertex in common.

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We usually and naturally assume that all elements of the vertex set V participate in the simplicial complex Σ. Thus, Σ_0 is the vertex set V. We also note that we assume that all the vertices constituting a simplex are different from each other.

We shall also need *orientations* of simplices.

Definition 2.3.2. An *orientation* of a q-simplex is an ordering of its vertices up to even permutation. An odd permutation of the vertices changes an oriented q-simplex σ_q into the oppositely oriented simplex $-\sigma_q$.

We require that $f \in C^q$ satisfies

$$f(-\sigma_q) = -f(\sigma_q), \tag{2.3.2}$$

for every oriented q-simplex.

For instance, for $q = 1$, we have

$$f(v_1, v_0) = -f(v_0, v_1).$$

Orientations introduce signs, and we can cancel terms with opposite signs. This will lead to the simplifications that (co)homology provides.

Definition 2.3.3. Let $f \in C^{q-1}$. We then define its *coboundary* $\delta f \colon C^q \to \mathbb{R}$ as

$$\delta f(v_0, v_1, \ldots, v_q) = \sum_{i=0}^{q} (-1)^i f(v_0, \ldots, \hat{v_i}, \ldots, v_q), \tag{2.3.3}$$

where a $\hat{}$ over a vertex means that it is omitted.

When we want to emphasize the dimension, we write

$$\delta^q \colon C^q \to C^{q+1}. \tag{2.3.4}$$

Equation (2.3.3) applies to any q-simplex with vertices $v_0, \ldots, v_q$. On the right-hand side, we sum over its facets. By the defining property of a simplicial complex, they are all contained in Σ_{q-1}, and so, f is defined for all of them. We then extend (2.3.3) by linearity to C^q.

The basic property of this coboundary operator, which provides the foundation for cohomology theory and which is easily checked, is

Proposition 2.3.1.

$$\delta^{q+1} \circ \delta^q = 0 \quad \textit{for } q = 0, 1, \ldots \tag{2.3.5}$$

or written more shortly

$$\delta \circ \delta = 0. \tag{2.3.6}$$

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To see the mechanism, let us consider the case $q = 0$ and a simplex with vertices v_0, v_1, v_2. Then,

$$\begin{aligned}\delta^1 \circ \delta^0 f(v_0, v_1, v_2) &= \delta^0 f(v_1, v_2) - \delta^0 f(v_0, v_2) + \delta^0 f(v_0, v_1) \\ &= f(v_2) - f(v_1) - (f(v_2) - f(v_0)) + f(v_1) - f(v_0) \\ &= 0.\end{aligned} \tag{2.3.7}$$

Assuming (2.3.2), we can rewrite (2.3.7) as

$$\begin{aligned}\delta f(v_1, v_2) + \delta f(v_2, v_0) + \delta f(v_0, v_1) = f(v_2) - f(v_1) + f(v_0) \\ - f(v_2) + f(v_1) - f(v_0) = 0,\end{aligned}$$

which makes the algebraic structure clearer.

Definition 2.3.4. The q-th *cohomology group* of the simplicial complex Σ is

$$H^q(\Sigma) := \ker \delta^q / \operatorname{im} \delta^{q-1}. \tag{2.3.8}$$

The dimension $b_q(\Sigma)$ of $H^q(\Sigma)$ is called the q-th *Betti number* of Σ.

Remark 2.3.1. We can also let $C_q(\Sigma, \mathbb{F})$ be the linear space with coefficients in an Abelian group $\mathbb{F}$, generated by the elements of Σ_q, and let $C^q(\Sigma, \mathbb{F})$ be the linear functions from $C_q(\Sigma, \mathbb{F})$ to $\mathbb{F}$, and then we can define the cohomology group $H^q(\Sigma, \mathbb{F})$ in the same way. It is usual to take $\mathbb{F}$ to be a commutative ring (e.g., the integer ring $\mathbb{Z}$) or even a field (e.g., the field $\mathbb{C}$ of the complex numbers or the finite field $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$). As an interesting example, we refer to [254] for Cheeger constants defined on a simplicial complex which use the cohomology over the finite field $\mathbb{Z}_2$. In this book, we mainly work with real coefficients.

2.4 (Oriented) Hypergraphs and their Higher Order Analogs

2.4.1 Generalizations

The concept of a simplicial complex obviously generalizes that of a graph. Further generalizations are possible and meaningful and will be treated in this book. In fact, we have already introduced (oriented) hypergraphs in Section 2.1. Before discussing that structure in more detail, to motivate what follows, let us consider two examples where such structures naturally arise in applications.

1. Social relations and interactions: Social network theory considers binary relations or interactions between persons, and they are thus naturally modeled by a graph. However, often, more than two persons constitute a unit

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of interaction. For instance, when we consider relations between scientists, we can consider research groups or coauthorships of scientific papers as the basic relations. In particular, when we consider groups of coauthors, they often overlap, as a scientist can write different papers with different coauthors. Moreover, when we look at research groups, we also naturally encounter higher order units, such as research institutes or university departments.

2. Networks of chemical reactions. Here, usually, a set of inputs, ingredients, or educts is transformed by a reaction into a set of outputs or products, which can then, in turn, become inputs for further reactions. The sets of inputs and outputs connected by a reaction need not necessarily be disjoint, as there could be catalyzers that participate in the reaction (in fact, may enable it) but are not changed by it.

These two examples suggest somewhat different generalizations of graphs or simplicial complexes, as we had already discussed in Section 4.9.

1. Instead of edges connecting two vertices, constituting a graph, we can look at *hyperedges* connecting arbitrary numbers of vertices. We can then look at higher order relations connecting hyperedges, and if we want, we can iterate this.
2. Instead of vertices connected by edges, we can look at *hypervertices*, that is, collections of vertices, connected by edges. In contrast to the first generalization above, where a single collection of vertices can be connected by a hyperedge, here we would naturally have two collections of vertices that are connected by an edge. This will naturally lead us to the concept of an oriented hypergraph (already introduced from a different perspective in Section 4.9). Since we could also consider graphs with self-loops, that is, where an edge is allowed to connect a vertex with itself, we should also admit the possibility that these two collections of vertices are not necessarily disjoint.

   We may then also consider relations involving more than two hypervertices.

Construction 1 simply generalizes edges (two-element sets) to hyperedges (multielement sets), while construction 2 just extends vertices (single-element sets) to hypervertices (multielement sets). Essentially, however, the difference is not so much in the content but in the conceptualization. Construction 2 naturally gives oriented hypergraphs. It has the disadvantage that in many cases, the resulting graph, with hypervertices as its vertices, is disconnected. On the other hand, one can perhaps more naturally encode restrictions on connections between hypervertices. Here is an example: Take as hypervertices the members

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of the powerset of some finite set S, that is, all the nonempty subsets of S. Then, connect any two of them by an edge whenever their intersection is empty.

We shall formally define those two types of generalizations in the next two sections. In brief, the point of view for the first type will be to consider hyperedges instead of edges, where a hyperedge may connect more than two vertices. For the second type, we consider hypervertices instead of vertices. A hypervertex is a collection of vertices. An edge then connects two hypervertices.

2.4.2 Hypergraphs and Hypersimplical Complexes

Here, we explore the first type of generalization suggested in the previous section.

Definition 2.4.1. A *hypergraph* is a pair $\Gamma = (V, E)$, where V is a finite set of nodes or vertices, and E is a multiset of elements $e \subseteq V$ called *hyperedges*. Given $v \in V$, we define its *degree*, denoted $\deg v$, as the number of hyperedges containing v.

Since E is a multiset, different hyperedges can contain exactly the same nodes; however, we label the hyperedges so that we can treat E as a set, and we write $E = \{e_1, \ldots, e_M\}$.

The notion of hypergraph can be generalized by iterating this principle.

Definition 2.4.2. An *ℓ-th order hypergraph* is a sequence $\Upsilon = (\Upsilon_1, \ldots, \Upsilon_\ell)$ such that, for each $q = 1, \ldots, \ell - 1$, $(\Upsilon_q, \Upsilon_{q+1})$ is a hypergraph.

Remark 2.4.1. Higher order hypergraphs can have interesting empirical interpretations. For instance, taking up an example from the previous section, we can consider a third-order hypergraph that represent researchers, as follows:

- The elements of Υ_1 are researchers;
- The elements of Υ_2 are research groups; and
- The elements of Υ_3 are research institutes.

We can also consider the hyperedges of some hypergraph as the basic elements of a higher structure:

Definition 2.4.3. Given a hypergraph $\Gamma = (V, E)$, a *hypersimplicial complex* on Γ is a simplicial complex on E.

We now consider hypergraphs with the additional structure that each vertex–hyperedge incidence is given a coefficient.

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Definition 2.4.4. Let $\mathcal{C}$ be a set such that $0 \in \mathcal{C}$. A *hypergraph with coefficients in* $\mathcal{C}$ is a triple $\Gamma = (V, E, \psi)$ where $\Gamma = (V, E)$ is a hypergraph and $\varphi\colon V \times E \to \mathcal{C}$ such that

$$\varphi(v, e) \neq 0 \iff v \in e$$

is the *incidence function*. If $V = \{v_1, \dots, v_n\}$ and $E = \{e_1, \dots, e_m\}$, the *incidence matrix* of Γ is the $n \times m$ matrix $\mathcal{I} = (\mathcal{I}_{ij})$ whose entries are

$$\mathcal{I}_{ij} \coloneqq \varphi(v_i, e_j).$$

We may adopt the implicit convention that $\varphi(v, e) = 0$ means that $v \notin e$.

Definition 2.4.5. A *hypergraph with real coefficients* is a hypergraph with coefficients in $\mathbb{R}$. In this case, the *degree* of a vertex $v \in V$ is defined as

$$\deg v \coloneqq \sum_{e \in E} \varphi(v, e)^2.$$

A *weighted hypergraph* is a hypergraph with real coefficients such that $w(e) \coloneqq \sqrt{\varphi(v, e)} > 0$ does not depend on v, for each $e \in E$ and $v \in e$.

In this case, in particular,

$$\deg v = \sum_{e \in E\colon v \in e} w(e).$$

Definition 2.4.6. A *complex unit hypergraph* is a hypergraph with coefficients in the complex unit circle.

We adopt the following definitions and notations for any hypergraph Γ with vertex set V and hyperedge set E, with or without coefficients.
For the maximal (minimal) degree of the vertices in Γ, we use the symbol $\overline{d} = \overline{d}(\Gamma)$ ($\underline{d} = \underline{d}(\Gamma)$). Given $e \in E$, we denote its *cardinality*, that is, the number of vertices contained in e, by $|e|$. For the maximal (minimal) cardinality of the hyperedges in Γ, we use the symbol $\overline{c} = \overline{c}(\Gamma)$ ($\underline{c} = \underline{c}(\Gamma)$). We say that Γ is *d-regular* if $\deg v = d$ for each $v \in V$; it is *c-uniform* if $|e| = c$ for each $e \in E$. We say that Γ is *connected* if, for every pair of vertices $v, v' \in V$, there exists a *path* that connects v and v', that is, there exist $\hat{v}_1, \dots, \hat{v}_k \in V$ and $\hat{e}_1, \dots, \hat{e}_{k-1} \in E$ such that $\hat{v}_1 = v$, $\hat{v}_k = v'$, and $\{\hat{v}_i, \hat{v}_{i+1}\} \subseteq \hat{e}_i$ for each $i = 1, \dots, k-1$.

2.4.3 Oriented Hypergraphs and Oriented Hypersimplical Complexes

We now formalize the second type of generalization suggested in Section 2.4.1.

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Definition 2.4.7. An *oriented hypergraph* is a hypergraph $\Gamma = (V, E, \varphi)$ with coefficients in $\{-1, 0, +1\}$. If $\varphi(v, e) = 1$ (respectively, $\varphi(v, e) = -1$), v is said to be an *output* (respectively, *input*) for e. If $v \neq v'$ and

$$\varphi(v, e) = \varphi(v', e) \neq 0 \quad (\text{respectively}, \varphi(v, e) = -\varphi(v, e) \neq 0),$$

the vertices v and v' are *co-oriented* (respectively, *antioriented*) in e.

By Definition 2.4.5, the degree of a vertex in an oriented hypergraph is

$$\deg v = \sum_{e \in E} \varphi(v, e)^2 = |\{e \in E \colon v \in e\}|.$$

On an oriented hypergraph, any process (such as a random walk or a diffusion process) is allowed to go in either direction, that is, from the inputs to the outputs or from the outputs to the inputs. Thus, calling sets of vertices in- or outputs are just labels. When we want to distinguish between them, we consider

Definition 2.4.8. A *directed hypergraph* is an oriented hypergraph with the additional assumption that hyperedges are *directed*, from the inputs toward the outputs.

On a directed hypergraph, processes can only follow the directions. Therefore, a random walker or a diffusion process on a directed hypergraph can only go from the inputs toward the outputs.

Remark 2.4.2. Our definition of directed hypergraph is slightly more general than that in [104].

We now come to the definition of hypersimplicial complex.

Definition 2.4.9. Let V again be a finite set. An *oriented hypersimplicial complex* is a simplicial complex on some $\mathcal{A} \subset \mathcal{P}(V)$, the power set of V. The elements of $\mathcal{A}$ are called *hypervertices*, to distinguish them from the vertices, the elements of V. All notions then are defined analogously to Definition 2.3.1, with the prefix *hyper* and the qualification *oriented* (to distinguish it from the structures introduced in Section 2.4.2).

In particular, a one-dimensional oriented hypersimplicial complex is an *oriented hypergraph*.

The additional structural refinement that an oriented hypersimplicial complex offers when compared to an ordinary simplicial complex is that different zero-dimensional hypersimplices, that is, hypervertices, can share some elements of V, that is, vertices. We refer to the example in Section 2.4.1 where hypervertices are sets of educts or products of chemical reactions.

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Thus, when we have a function $g\colon V \to \mathbb{R}$, we can define a cumulative function via

$$f(V_0) = \sum_{v \in V_0} g(v) \quad \text{for } V_0 \in \mathcal{A}. \tag{2.4.1}$$

Again, for functions f defined on $\mathcal{A}$ and extended linearly as before to linear combinations, we can form the differences

$$\delta f(V_0, V_1) = \sum_{v' \in V_1} f(v') - \sum_{v \in V_0} f(v), \tag{2.4.2}$$

for $V_0, V_1 \in \mathcal{A}$. Equation (2.3.3) can then be extended analogously.

2.5 Scalar Products and Laplace Operators

2.5.1 Simplicial Complexes

We consider a simplicial complex Σ. We recall that, concerning the orientations, according to (2.3.2), we have for any $\phi \in C^q$, where C^q are the linear functions on chains of q-dimensional (hyper)simplices,

$$\phi(-\sigma_q) = -\phi(\sigma_q), \tag{2.5.1}$$

that is, changing the orientation yields a minus sign.

We then choose positive definite inner products $(\cdot,\cdot)_q$ on the C^q.

Definition 2.5.1. The adjoint $(\delta^q)^*\colon C^{q+1} \to C^q$ of the coboundary operator δ^q is defined by

$$(\delta^q f_1, f_2)_{q+1} = (f_1, (\delta^q)^* f_2)_q,$$

for $f_1 \in C^q$ and $f_2 \in C^{q+1}$.

We then have the arrows

$$C^{q-1} \underset{\delta^{q-1*}}{\overset{\delta^{q-1}}{\rightleftarrows}} C^q \underset{\delta^{q*}}{\overset{\delta^q}{\rightleftarrows}} C^{q+1}. \tag{2.5.2}$$

Lemma 2.5.1.

$$(\operatorname{im} \delta^q)^\perp = \ker(\delta^q)^*, \tag{2.5.3}$$

$$(\operatorname{im} (\delta^q)^*)^\perp = \ker \delta^q. \tag{2.5.4}$$

Proof If $\delta^q f = 0$, then

$$((\delta^q)^* g, f) = (g, \delta^q f) = 0 \text{ for all } g \in C^{q+1},$$

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and conversely, if this vanishes for all $g \in C^{q+1}$, then $\delta^q f = 0$, because the scalar product is positive definite, showing (2.5.4). The proof of (2.5.3) is of course analogous. □

Lemma 2.5.2.

$$\operatorname{im} \delta^{q-1} \cap \operatorname{im} (\delta^q)^* = \{0\}. \tag{2.5.5}$$

Proof If $f = \delta^{q-1} h = (\delta^q)^* g$, then

$$(f, f) = (\delta^{q-1} h, (\delta^q)^* g) = (\delta^q \delta^{q-1} h, g) = 0$$

by (2.3.5), hence $f = 0$. □

Corollary 2.5.1.

$$C^q = \operatorname{im} \delta^{q-1} \oplus \operatorname{im} (\delta^q)^* \oplus (\ker \delta^q \cap \ker(\delta^{q-1})^*, \tag{2.5.6}$$

□

where $\oplus$ indicates orthogonality with respect to the scalar product $(\cdot, \cdot)$. Equation (2.5.6) is a discrete analogue of the Hodge decomposition in Riemannian geometry (see [150]).

Proof By the proof of Lemma 2.5.2, $\operatorname{im} \delta^{q-1}$ and $\operatorname{im} (\delta^q)^*$ are orthogonal, and by Lemma 2.5.1, any element that is orthogonal to both of them has to be contained in both $\ker \delta^q$ and $\ker(\delta^{q-1})^*$. □

We now define the following three operators on C^q:

Definition 2.5.2. (i) The *q-dimensional up Laplace operator* or simply *q-up Laplace operator* of the (hyper)simplicial complex Σ is

$$L^q{}_{up} := (\delta^q)^* \delta^q.$$

(ii) The *q-dimensional down Laplace operator* or *q-down Laplace operator* is

$$L^q{}_{down} := \delta^{q-1} (\delta^{q-1})^*.$$

(iii) The *q-dimensional Laplace operator* or *q-Laplace operator* is

$$L^q := L^q{}_{up} + L^q{}_{down} = (\delta^q)^* \delta^q + \delta^{q-1} (\delta^{q-1})^*.$$

Note that we usually omit the argument Σ, that is, write L^q instead of $L^q(\Sigma)$.

For $q = 0$, that is, when we look at the operators on the vertices of a simplicial complex, we have $L^0{}_{down} = 0$, and hence

$$L^0 = L^0{}_{up}. \tag{2.5.7}$$

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Similarly, for $q = \dim \Sigma$, the up-Laplacian vanishes, and

$$L^{\dim \Sigma} = L^{\dim \Sigma}{}_{down}. \tag{2.5.8}$$

By construction, we have

Lemma 2.5.3. *The operators L^q_{up}, L^q_{down}, and L^q are self-adjoint.*

□

Also,

Lemma 2.5.4. *The operators $L = L^q_{up}, L^q_{down}, L^q$ are non-negative, that is, they satisfy*

$$(Lf, f) \geq 0 \text{ for all } f \in C^q. \tag{2.5.9}$$

Proof We have

$$(L^q_{up} f, f) = ((\delta^q)^* \delta^q f, f) = (\delta^q f, \delta^q f) \geq 0 \tag{2.5.10}$$

$$(L^q_{down} f, f) = (\delta^{q-1}(\delta^{q-1})^* f, f) = ((\delta^{q-1})^* f, (\delta^{q-1})^* f) \geq 0 \tag{2.5.11}$$

$$(L^q f, f) = (\delta^q f, \delta^q f) + ((\delta^{q-1})^* f, (\delta^{q-1})^* f) \geq 0. \tag{2.5.12}$$

□

In particular, from (2.5.10) to (2.5.12),

Corollary 2.5.2.

$$L^q_{up} f = 0 \text{ if and only if } \delta^q f = 0 \tag{2.5.13}$$

$$L^q_{down} f = 0 \text{ if and only if } (\delta^{q-1})^* f = 0 \tag{2.5.14}$$

$$L^q f = 0 \text{ if and only if } \delta^q f = 0 \text{ and } (\delta^{q-1})^* f = 0. \tag{2.5.15}$$

Since the operators L^q_{up}, L^q_{down} and L^q are self-adjoint, non-negative operators on finite-dimensional Hilbert spaces, we have

Theorem 2.5.1. *The eigenvalues of the operators $L^q_{up}(\Sigma)$, $L^q_{down}(\Sigma)$, and $L^q(\Sigma)$ are real and non-negative.*

Corollary 2.5.2 characterizes the eigenvalue 0. The other eigenvalues then are positive.

Definition 2.5.3. The collection of the eigenvalues (counted with multiplicity) of a self-adjoint operator on a finite dimensional Hilbert space is called its *spectrum*.

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We now look at scalar products on the spaces of cochains, as needed for the definition of the Laplace operators. Here, we only consider positive inner products, and when we shall speak about a scalar product in the sequel, we shall always assume that it is positive definite.

Each oriented simplex σ generates a cochain, consisting of the real multiples of the function f_σ with $f_\sigma(\sigma) = 1$, $f_\sigma(\tau) = 0$ for all oriented simplices $\tau \neq \pm\sigma$ (and of course $f_\sigma(-\sigma) = -1$). We assume that the cochains generated by different simplices are orthogonal to each other. This restricts the possible scalar products. A scalar product with this property can be obtained from a *weight function* w that associates to every simplex σ a positive real number.

Definition 2.5.4. A *weighted simplicial complex* (Σ, w) is a simplicial complex with a non-negative weight $w(\sigma)$ for every simplex.

A positive inner product on the space C^q can be written in terms of a positive weight function w as

$$(f,g)_q = \sum_{\sigma\in\Sigma_q} w(\sigma) f(\sigma) g(\sigma), \tag{2.5.16}$$

where Σ_q is the space of q-simplices. We shall, however, also systematically consider cases where some of the weights may vanish. In that case, the corresponding product (2.5.16) will of course no longer be positive.

In the sequel, we shall write $\operatorname{sgn}(\sigma,\sigma') = \pm 1$ for two orientations of a simplex when those orientations coincide/differ. We also write $\partial\sigma$ for the cellular boundary of a simplex, that is, for the collection of its facets.

By simple linear algebra, the q-up Laplace operator is then given by

$$\begin{aligned}(L^q_{up} f)([\sigma]) = &\sum_{\substack{\rho\in\Sigma_{q+1}:\\ \sigma\in\partial\rho}} \frac{w(\rho)}{w(\sigma)} f([\sigma]) \\ &+ \sum_{\substack{\sigma'\in\Sigma_q:\ \sigma\neq\sigma',\\ \sigma,\sigma'\in\partial\rho}} \frac{w(\rho)}{w(\sigma)} \operatorname{sgn}([\sigma],\partial[\rho]) \operatorname{sgn}([\sigma'],\partial[\rho]) f([\sigma']),\end{aligned} \tag{2.5.17}$$

and the q-down Laplace operator is

$$\begin{aligned}(L^q_{down} f)([\sigma]) = &\sum_{\tau\in\partial\sigma} \frac{w(\sigma)}{w(\tau)} f([\sigma]) \\ &+ \sum_{\sigma':\ \sigma\cap\sigma'=\tau} \frac{w(\sigma')}{w(\tau)} \operatorname{sgn}([\tau],\partial[\sigma]) \operatorname{sgn}([\tau],\partial[\sigma']) f([\sigma']).\end{aligned} \tag{2.5.18}$$

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For our purposes, however, we need some relation between the weights in different dimensions.

Definition 2.5.5. *The degree* of a q-simplex σ of Σ is

$$\deg \sigma := \sum_{\rho \in \Sigma_{q+1} :\, \sigma \in \partial \rho} w(\rho). \tag{2.5.19}$$

Definition 2.5.6. If the weight function w on Σ satisfies

$$w(\sigma) = \deg \sigma, \tag{2.5.20}$$

for every $\sigma \in \Sigma_q$, we call the Laplace operator defined on the cochain complex of Σ the *weighted normalized combinatorial Laplace operator*. If, in addition, the weights of the facets of Σ are equal to 1, then the Laplace operator is called the *normalized combinatorial Laplace operator*.

When (2.5.20) holds, (2.5.17) simplifies, and the normalized combinatorial up-Laplacian is given by

$$\begin{aligned}(L^q_{up} f)([\sigma]) &= f([\sigma]) \\ &+ \frac{1}{\deg \sigma} \sum_{\substack{\sigma' \in \Sigma_q :\, \sigma' \neq \sigma, \\ \sigma, \sigma' \in \partial \rho}} w(\rho)\, \mathrm{sgn}([\sigma], \partial[\rho])\, \mathrm{sgn}([\sigma'], \partial[\rho])\, f([\sigma']).\end{aligned} \tag{2.5.21}$$

2.5.2 Relative Complexes

Let Σ be a simplicial complex and Σ', a subcomplex. Then, the q-th cochain group of the pair (Σ, Σ') is

$$C^q(\Sigma, \Sigma') := \{f \in C^q(\Sigma) | f([F]) = 0, \text{ for every } F \in \Sigma_q(\Sigma')\},$$

and the coboundary operator $\delta^q : C^q(\Sigma) \to C^{q+1}(\Sigma)$ induces a homomorphism of relative groups $\delta^q_{rel} : C^q(\Sigma, \Sigma') \to C^{q+1}(\Sigma, \Sigma')$ by

$$\delta^q_{rel}(f)([\bar{F}]) = f(\partial_q[\bar{F}]), \quad \text{for } f \in C^q(\Sigma, \Sigma'). \tag{2.5.22}$$

Definition 2.5.7. The q-th *cohomology group* of the pair (Σ, Σ_0) is

$$H^q(\Sigma, \Sigma_0) := \ker \delta^q_{rel} / \mathrm{im}\, \delta^{q-1}_{rel}. \tag{2.5.23}$$

(For relative cohomology, see for instance [128] or [203].)

When we have inner products on the $C^q(\Sigma, \Sigma')$, we can define the corresponding Laplace operators.

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Definition 2.5.8. The *Laplace operators* of a pair (Σ,Σ') are

$$L^q_{up} := L^q_{up}(\Sigma,\Sigma') := (\delta^q_{rel})^*\delta^q_{rel}, \quad L^q_{down} := L^q_{down}(\Sigma,\Sigma') := \delta^{q-1}_{rel}(\delta^{q-1}_{rel})^*. \tag{2.5.24}$$

The *Laplace operator* is

$$L^q := L^q(\Sigma,\Sigma') := L^q_{up} + L^q_{down}. \tag{2.5.25}$$

Again, and obviously, these operators depend on the inner products. As before, we shall consider such products that are defined through weights on the simplices. And we shall allow for some of the weights to vanish.

We shall not only consider positive-valued weight functions but also permit values equal to 0. Thus, we consider weight functions

$$w\colon \bigcup_i \Sigma_q \to \mathbb{R}^+ \cup \{0\}. \tag{2.5.26}$$

When some of the weights are 0, we say that (Σ,w) is a *degenerate weighted simplicial complex*. The preceding constructions carry over to degenerate simplicial complexes, with the difference that $(\delta^q)^*$ is a formal adjoint, defined as the standard adjoint on its nondegenerate counterpart and extended by zero on simplices of weight zero, that is,

$$(\delta^q)^*\bar{f}([F]) = \begin{cases} \sum_{\substack{\bar{F}\in\Sigma_{q+1}\colon \\ F\in\partial\bar{F}}} \frac{w(\bar{F})}{w(F)}\operatorname{sgn}([F],\partial[\bar{F}])f([F]) & \text{for } w(F)\neq 0, \\ 0 & \text{for } w(F)=0. \end{cases} \tag{2.5.27}$$

On degenerate simplicial complexes, the appropriate version of the discrete Hodge Theorem says that the kernel of the q-Laplace operator is isomorphic to the direct sum of the q-th homology group of a complex and the vector space generated by q-simplices of weight zero.

A graph with loops constitutes a special case of a weighted graph. For the graph Laplace operator, a loop effects only the degree of the corresponding vertex. The same is true for simplicial complexes. We can define *simplicial complexes with loops* in analogy to graphs with loops:

Definition 2.5.9. Let $[\Sigma]$ be an oriented simplicial complex. If we allow oriented faces in which a vertex appears more than once, to be elements of $[\Sigma]$, then we say that $[v_0,\dots,v_i,v_{i+1}\dots,v_i,v_{j+1},v_{q+1}]$ is a *loop* on the simplex

$$[v_0,\dots,v_i,v_{i+1}\dots,\hat{v_i},v_{j+1},v_{q+1}].$$

The boundary of a loop is zero. Thus, the Laplace operator defined on a simplicial complex $[\Sigma]$ with loops coincides with the one defined on Σ. For the degree of a face, we have to add the weight of the loop to its total degree.

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Figure 2.3 In (b), we see the proper difference of graph (a) and its subgraph Σ'.

Therefore, simplicial complexes with loops are already contained in the definition of weighted complexes, and we need not treat them separately. We include simplicial complexes with loops here because they will arise naturally when we consider simplicial maps, which do not preserve dimensionality.

Definition 2.5.10. A weighted simplicial complex $(\Sigma', w_{\Sigma'})$ is a *subcomplex* of (Σ, w_Σ) if Σ' is a subcomplex of Σ and $w_{\Sigma'}(F) \leq w_\Sigma(F)$ for every simplex F of Σ'.

Definition 2.5.11. A *proper difference* of the weighted simplicial complexes (Σ, w_Σ) and $(\Sigma', w_{\Sigma'})$ is a degenerate weighted simplicial complex (Θ, w_Θ), such that $\Theta \equiv \Sigma$, $w_\Theta := w_\Sigma - w_{\Sigma'}$, and $\Theta' = \{F \in \Sigma \mid w_\Theta(F) > 0\}$ is a simplicial complex.

As an example, we can take a subgraph of an unweighted graph. We delete all the edges of the subgraph. The remaining edges have weight 1, those that are deleted get weight 0. The degrees, that is, the weights of the vertices, are decreased by how many edges they lose.

In Figure 2.3, Σ is the triangle in graph (a). Every vertex has degree, hence weight 2. Σ' is the bottom edge with its two end points, which have weight 1 in Σ', and the proper difference Θ is graph (b), where the two lower vertices now have weight $2 - 1 = 1$ (and the bottom edge has weight 0 and is hence not drawn).

2.5.3 Laplace Operators as Averaging Operators

We consider the case of pairwise relations, that is, in the terminology of Section 2.5.1, the case $q = 0$, vertices connected by edges. In particular, $L^0_{down} = 0$, and so for $q = 0$, (2.5.21) becomes

$$L^0 f(v) = f(v) - \frac{1}{\deg v} \sum_{\rho=[v,v']} w(\rho) f(v'), \tag{2.5.28}$$

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with $\deg v = \sum_{\rho=[v,v']} w(\rho)$ as the normalizing factor. The operator L^0 is the Laplacian of the graph formed by the vertices and edges of a simplicial complex. In (2.5.28), we compare the value of f at v with the average of the values of f at all other points connected to v by some edge. This is the important *averaging property* of the zero-dimensional Laplacian.

This averaging property can also be taken as the essence of a general definition of Laplace operators. Let Ω be a domain in $\mathbb{R}^n$ or some more abstract space, and $w\colon \Omega \times \Omega \to \mathbb{R}$, a non-negative, symmetric function. The quantity $w(x,y)$ can be interpreted as some kind of edge weight between the points x, y for any pair $(x,y) \in \Omega \times \Omega$. When x, y are vertices in a graph, the integrals become sums, and we are back in the case of (2.5.28). We define the average $\bar{w}\colon \Omega \to \mathbb{R}$ of w by

$$\bar{w}(x) = \int_\Omega w(x,y)dy$$

and assume that $\bar{w}$ is positive almost everywhere. On a graph, while w is an edge function, $\bar{w}$ would be a vertex function, $\bar{w}(x)$ being the degree of the vertex x with edge weights $w(x,y)$. We first use $\bar{w}(x)$ and $w(x,y)$ to define the L^2-norms for functions $u\colon \Omega \to \mathbb{R}$ and vector fields p, that is, $p\colon \Omega \times \Omega \to \mathbb{R}$,

$$(u_1,u_2)_{L^2} := \int_\Omega u_1(x)u_2(x)\bar{w}(x)dx$$
$$(p_1,p_2)_{L^2} := \int_{\Omega\times\Omega} p_1(x,y)p_2(x,y)w(x,y)dxdy$$

and the corresponding norms $|u|$ and $|p|$.

The discrete derivative of a function $u\colon \Omega \to \mathbb{R}$ is defined by

$$Du(x,y) = u(y) - u(x). \tag{2.5.29}$$

Even though Du does not depend on w, it is in some sense analogous to a gradient, as we shall see (2.5.32). Its pointwise norm then is

$$|Du|(x) = \left(\frac{1}{\bar{w}(x)} \int_\Omega (u(y) - u(x))^2 w(x,y)dy\right)^{\frac{1}{2}}. \tag{2.5.30}$$

The *divergence* of a vector field $p\colon \Omega \times \Omega \to \mathbb{R}$ is defined by

$$\operatorname{div} p(x) := \frac{1}{\bar{w}(x)} \int_\Omega (p(x,y) - p(y,x))w(x,y)dy. \tag{2.5.31}$$

Note that, in contrast to Du for a function u, the divergence of a vector field depends on the weight w. For $u\colon \Omega \to \mathbb{R}$ and $p\colon \Omega \times \Omega \to \mathbb{R}$, we then have the integration by parts formula

$$(Du,p)_{L^2} = -(u, \operatorname{div} p)_{L^2}. \tag{2.5.32}$$

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With the vector field Du and the divergence operator div, we can define a Laplacian for functions

$$\Delta u(x) := -\mathrm{div}\,(Du) = u(x) - \frac{1}{\bar{w}(x)} \int_\Omega u(y) w(x,y) dy\,, \tag{2.5.33}$$

which in the case of a graph coincides with the Laplacian (2.5.28). Again, the principle is to compare the value of a function at a point with the average of the values on some neighborhood. For such a principle on Riemannian manifolds and more general metric spaces, see [143, 144].

2.5.4 Oriented Hypergraphs

We consider an oriented hypergraph Γ with vertex set V and hyperedge set E. For a vertex v, as in Section 2.4.3, we let

$$\deg v := |\text{hyperedges containing } v| \tag{2.5.34}$$

and, again, we assume that $\deg v > 0$ for all $v \in V$. As before, this *degree* will serve as a normalizing factor. Of course, one could choose a different such factor, but our choice here is in agreement with the conventions of Section 2.5.1. We then obtain the corresponding *scalar products*.

Definition 2.5.12. For $f, g : V \to \mathbb{R}$, let

$$(f,g)_V := \sum_{v \in V} \deg v \cdot f(v) \cdot g(v). \tag{2.5.35}$$

For $\omega, \gamma : E \to \mathbb{R}$, let

$$(\omega,\gamma)_E := \sum_{e \in E} \omega(e) \cdot \gamma(e). \tag{2.5.36}$$

The *boundary operator* then maps functions on vertices to functions on hyperedges that change their sign upon a change of orientation, as always.

Definition 2.5.13. For $f : V \to \mathbb{R}$ and $e \in E$, let

$$\delta f(e) := \sum_{v^j \text{ output of } e} f(v^j) - \sum_{v_i \text{ input of } e} f(v_i). \tag{2.5.37}$$

Thus, the convention is that inputs will have a $-$ sign, while outputs will have a $+$ sign.

We then have

Lemma 2.5.5. *The adjoint of the operator δ with respect to the scalar products (2.5.35), (2.5.36) is*

$$\delta^*(\gamma)(v) = \frac{\sum_{h_{out} :\, v \text{ output}} \gamma(h_{out}) - \sum_{h_{in} :\, v \text{ input}} \gamma(h_{in})}{\deg v}. \tag{2.5.38}$$

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We then obtain our *Laplace operators*

Lemma 2.5.6. *The Laplace operator for functions $f : V \to \mathbb{R}$ on the vertex set V of an oriented hypergraph is given by*

$$L^0 f(v) := \delta^*\delta = \frac{\sum_{h_{in}:\, v \text{ input}} \left(\sum_{v' \text{ input of } h_{in}} f(v') - \sum_{z' \text{ output of } h_{in}} f(z') \right)}{\deg v} - \frac{\sum_{h_{out}:\, v \text{ output}} \left(\sum_{\hat{v} \text{ input of } h_{out}} f(\hat{v}) - \sum_{\hat{z} \text{ output of } h_{out}} f(\hat{z}) \right)}{\deg v}. \tag{2.5.39}$$

The Laplacian for functions $\gamma : E \to \mathbb{R}$ on the hyperedge set E, with $\gamma(h^+) = -\gamma(h^-)$ under a change of orientation, is given by

$$L^1\gamma(h) := \delta\delta^* = \sum_{v_i \text{ input of } h} \frac{\sum_{h_{in}:\, v_i \text{ input}} \gamma(h_{in}) - \sum_{h_{out}:\, v_i \text{ output}} \gamma(h_{out})}{\deg v_i} - \sum_{v^j \text{ output of } h} \frac{\sum_{h'_{in}:\, v^j \text{ input}} \gamma(h'_{in}) - \sum_{h'_{out}:\, v^j \text{ output}} \gamma(h'_{out})}{\deg v^j}. \tag{2.5.40}$$

2.6 Rayleigh Quotients and their Generalizations

2.6.1 Rayleigh Quotients

The eigenvalues of our Laplace operators admit a variational characterization described in the following general theorem.

Theorem 2.6.1 (Courant-Fischer-Weyl min-max principle)**.** *Let H be an N-dimensional vector space with a positive definite scalar product $(\cdot,\cdot)$. Let $\mathcal{H}_k$ be the family of all k-dimensional subspaces of H. Let $A : H \to H$ be a self-adjoint linear operator. Then the eigenvalues $\lambda_1 \leq \ldots \leq \lambda_N$ of A can be obtained by*

$$\lambda_k = \min_{H_k \in \mathcal{H}_k} \max_{g(\neq 0) \in H_k} \frac{(Ag,g)}{(g,g)} = \max_{H_{N-k+1} \in \mathcal{H}_{N-k+1}} \min_{g(\neq 0) \in H_{N-k+1}} \frac{(Ag,g)}{(g,g)}. \tag{2.6.1}$$

The vectors g_k realizing such a min-max or max-min then are corresponding eigenvectors, and the min-max spaces H_k are spanned by the eigenvectors for the eigenvalues $\lambda_1, \ldots, \lambda_k$, and analogously, the max-min spaces H_{N-k+1} are spanned by the eigenvectors for the eigenvalues $\lambda_k, \ldots, \lambda_N$.

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Thus, we also have

$$\lambda_k = \min_{g\in H,(g,g_j)=0 \text{ for } j=1,\dots,k-1} \frac{(Ag,g)}{(g,g)} = \max_{g\in H,(g,g_\ell)=0 \text{ for } \ell=k+1,\dots,N} \frac{(Ag,g)}{(g,g)}. \tag{2.6.2}$$

In particular,

$$\lambda_1 = \min_{g\in H} \frac{(Ag,g)}{(g,g)}, \qquad \lambda_N = \max_{g\in H} \frac{(Ag,g)}{(g,g)}. \tag{2.6.3}$$

Definition 2.6.1.

$$R_A(g) := \frac{(Ag,g)}{(g,g)} \tag{2.6.4}$$

is the *Rayleigh quotient* (or the *Rayleigh–Ritz quotient*) of g.

We shall usually implicitly assume that $g \neq 0$ in (2.6.4), without always explicitly mentioning it; else, we use the convention that $R_A(g) = 0$ if $g = 0$.

With this convention, we have the following obvious but useful factorization result.

Lemma 2.6.1. *Let $A, B\colon H \to H$ be self-adjoint linear operators, with $A = a^\star a$, $B = b^\star b$. Then*

$$R_A(g) = \frac{(ag,ag)}{(g,g)}, \tag{2.6.5}$$

$$R_{AB}(g) = \frac{(abg,abg)}{(bg,bg)}\frac{(bg,bg)}{(g,g)} = R_{AB}(bg)R_B(g). \tag{2.6.6}$$

□

In fact, without loss of generality, we may assume $(g,g) = 1$ in (2.6.1). Then, we get rid of the denominators, and the variational problems reduce to problems on the unit sphere in H (or more precisely, since g and $-g$ always yield the same value, on the corresponding projective space). That perspective will also be adopted in Chapter 5.

Proof We want to find an orthonormal basis of H consisting of eigenfunctions of A,

$$u_k,\ k = 1,\dots,N.$$

This is achieved as follows. We iteratively define, with $H_1 := H$,

$$H_k := \{g \in H\colon (g,u_i) = 0 \text{ for } i \leq k-1\}, \tag{2.6.7}$$

we put

$$\lambda_k := \inf_{g\in H_k\setminus\{0\}} \frac{(Ag,g)}{(g,g)}, \tag{2.6.8}$$

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and we claim that those are the eigenvalues, which thus can be obtained as these infima. Since $H_k \subset H_{k-1}$, we have

$$\lambda_k \geq \lambda_{k-1}. \tag{2.6.9}$$

We may find a function u_k that realizes the infimum in (2.6.8), that is,

$$\lambda_k = \frac{(Au_k, u_k)}{(u_k, u_k)}. \tag{2.6.10}$$

Since then for every $\phi \in H_k, t \in \mathbb{R}$,

$$\frac{(A(u_k + t\phi), u_k + t\phi)}{(u_k + t\phi, u_k + t\phi)} \geq \lambda_k, \tag{2.6.11}$$

the derivative of that expression with respect to t vanishes at $t = 0$, and we obtain, using (2.6.10) and the self-adjointness of A,

$$0 = (Au_k, \phi) - \lambda_k(u_k, \phi) \tag{2.6.12}$$

for all $\phi \in H_k$; in fact, this even holds for all $\phi \in H$, and not only for those in the subspace H_k, since for $i \leq k-1$

$$(u_k, u_i) = 0 \tag{2.6.13}$$

and

$$(Au_k, u_i) = (u_k, Au_i) = \lambda_i(u_k, u_i) = 0, \tag{2.6.14}$$

since $u_k \in H_k$. Thus, if we also recall (2.6.12),

$$(Au_k, \phi) = \lambda_k(u_k, \phi) \tag{2.6.15}$$

for all $\phi \in H$ whence

$$Au_k = \lambda_k u_k. \tag{2.6.16}$$

Since, as noted (2.6.13), we may require

$$(u_k, u_k) = 1 \tag{2.6.17}$$

for $k = 1, \ldots, N$, and since u_k is mutually orthogonal by construction, we have constructed an orthonormal basis of H consisting of eigenfunctions of A.

Thus, we may expand any $g \in H$ as

$$g = \sum_k (g, u_k) u_k. \tag{2.6.18}$$

We then also have

$$(g, g) = \sum_k (g, u_k)^2, \tag{2.6.19}$$

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since u_k satisfies

$$(u_j, u_k) = \delta_{jk}, \tag{2.6.20}$$

which is the condition for being an orthonormal basis. Finally, using (2.6.19) and (4.1.46), we obtain

$$(Ag, g) = \sum_k \lambda_k (g, u_k)^2. \tag{2.6.21}$$

We have (2.6.8),

$$\lambda_k = \min \left\{ \frac{(Ag,g)}{(g,g)} : \ g \neq 0, (g, u_j) = 0 \text{ for } j = 1, \dots, k-1 \right\}. \tag{2.6.22}$$

Dually,

$$\lambda_k = \max \left\{ \frac{(Ag,g)}{(g,g)} : \ g \neq 0 \text{ linear combination of } u_j \text{ with } j \leq k \right\}. \tag{2.6.23}$$

That maximum is realized when g is a multiple of the k-th eigenfunction, and so is the minimum in (2.6.22).

When L is a $k+1$-dimensional subspace, there is some g in L satisfying

$$(g, u_j) = 0 \text{ for } j = 0, \dots, k-1. \tag{2.6.24}$$

Equations (2.6.19) and (2.6.21) yield

$$\frac{(Ag,g)}{(g,g)} = \frac{\sum_{j \geq k} \lambda_j (g, u_j)^2}{\sum_{j \geq k} (g, u_j)^2} \geq \lambda_k. \tag{2.6.25}$$

This gives us

$$\max_{g \in L \setminus \{0\}} \frac{(Ag,g)}{(g,g)} \geq \lambda_k. \tag{2.6.26}$$

This yields the second equality in (2.6.2), and the first follows by dualizing the argument. □

2.6.2 Critical Points and p-Laplacians

We take another look at the quotient (2.6.5), that is,

$$R(g) = \frac{(ag, ag)}{(g,g)} \tag{2.6.27}$$

for the self-adjoint linear operator $A = a^\star a \colon H \to H$. In fact, $a \colon H \to H_1$ may map H into another Hilbert space H_1, and the product (ag, ag) will then be taken in H_1.

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By homogeneity, instead of looking for critical points of $R_A(g)$ via a minimaxing scheme as in Section 2.6.1, by homogeneity, we look for critical points of (ag, ag) under the condition $(g, g) = 1$. Equivalently, we can introduce a Lagrange multiplier λ and search for critical points of

$$(ag, ag) - \lambda((g, g) - 1). \tag{2.6.28}$$

And a critical g_λ then has to satisfy $(g_\lambda, g_\lambda) = 1$ and

$$\frac{d}{dt}_{|t=0} ((a(g_\lambda + t\varphi), a(g_\lambda + t\varphi)) - \lambda(g_\lambda + t\varphi, g_\lambda + t\varphi)) = 0 \tag{2.6.29}$$

for all $\varphi \in H$, and therefore

$$(a\varphi, ag_\lambda) - \lambda(\varphi, g_\lambda) = 0\,, \tag{2.6.30}$$

and hence, the eigenvalue equation

$$Ag_\lambda = \lambda g_\lambda, \tag{2.6.31}$$

that is, the Lagrange multiplier becomes an eigenvalue. Conversely, when g_λ solves (2.6.31), it is a critical point of (2.6.27) with critical value λ.

Equation (2.6.31) is, of course, a linear equation, but we now want to generalize this in a nonlinear fashion. When we write $\|g\| = (g, g)^{1/2}$, the Rayleigh quotient (2.6.27) becomes

$$R(g) = \frac{\|ag\|^2}{\|g\|^2},$$

and we can then naturally consider

$$R_p(g) = \frac{\|ag\|^p}{\|g\|^p} \tag{2.6.32}$$

for some exponent p, which we first restrict to $1 < p < \infty$, even though subsequently the cases $p = 1$ and $p = \infty$ will be of particular interest.

Analogously to (2.6.28), we introduce a Lagrange multiplier λ to require $\|g\|^p = 1$ and search for critical points of

$$\|ag\|^p - \lambda(\|g\|^p - 1). \tag{2.6.33}$$

Analogously to (2.6.30), critical points now have to satisfy

$$(a\varphi, (ag_\lambda, ag_\lambda)^{\frac{p}{2}-1} ag_\lambda) - \lambda(\varphi, (g_\lambda, g_\lambda)^{\frac{p}{2}-1} g_\lambda) = 0\,,$$

and hence the nonlinear eigenvalue equation

$$a^\star((ag_\lambda, ag_\lambda)^{\frac{p}{2}-1} ag_\lambda) = \lambda(g_\lambda, g_\lambda)^{\frac{p}{2}-1} g_\lambda\,, \tag{2.6.34}$$

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or when written in terms of norms,

$$A_p g_\lambda := a^\star(\|ag_\lambda\|^{p-2} a g_\lambda) = \lambda \|g_\lambda\|^{p-2} g_\lambda. \tag{2.6.35}$$

In the spirit of the minimaxing scheme of Section 2.6.1, there is also a variational characterization of these nonlinear eigenvalues. We discuss here only the case where the Hilbert spaces H, H_1 are finite dimensional, of dimension n, so that we can identify them with $\mathbb{R}^n$ with its Euclidean structure (although in Section 3.3, the infinite dimensional case will occur). The appropriate concept is (see for instance [152, 277])

Definition 2.6.2. The *Krasnoselskii* $\mathbb{Z}_2$ *genus* of a centrally symmetric set S in $\mathbb{R}^n$ is

$$\text{genus}(S) := \begin{cases} \min\{k \in \mathbb{Z}^+ : \exists \text{ odd continuous } h : S \setminus \{\mathbf{0}\} \to \mathbb{S}^{k-1}\} & \text{if } S \setminus \{\mathbf{0}\} \neq \emptyset, \\ 0 & \text{if } S \setminus \{\mathbf{0}\} = \emptyset. \end{cases}$$

For $k \geq 1$, we let $\text{Gen}_k := \{S \subset \mathbb{R}^n : S \text{ centrally symmetric with genus } (S) \geq k\}$.

Here, $\mathbb{Z}_2$ is the symmetry group for the reflection $j(x) = -x$ of $\mathbb{R}^n$, and *centrally symmetric* simply means invariance under j. The typical example of such a centrally symmetric set is the sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$.

Lemma 2.6.2. *The genus of the sphere* $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ *is* n.

Proof The identity map $\mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is odd and continuous. Therefore, the genus is at most n. But it follows from the Borsuk-Ulam Theorem [37] that there exists no such odd continuous map from $\mathbb{S}^{n-1}$ to $\mathbb{S}^{k-1}$ for $k < n$. □

Lemma 2.6.3. *1. The genus of* $\mathbb{R}^n \backslash \mathbb{R}^m$ *is* $n - m$ $(n \geq m)$.
2. If $S_1 \subset S_2$, *then* $\text{genus}(S_1) \leq \text{genus}(S_2)$
3. If $\text{genus}(S) \geq k$ *for* $S \subset \mathbb{R}^n$, *then* $S \cap \mathbb{R}^m \neq \emptyset$ *for* $m \geq n - k$

Proof For $x = (x^1, \ldots, x^n) \in \mathbb{R}^n \backslash \mathbb{R}^m$, we have $(x^{m+1})^2 + \ldots (x^n)^2 \neq 0$, where we let $\mathbb{R}^m = \{(y^1, \ldots, y^n) \in \mathbb{R}^n : y^{m+1} = \cdots = y^n = 0\}$, and so the map

$$(x^1, \ldots, x^n) \mapsto \left(\frac{x^{m+1}}{(x^{m+1})^2 + \ldots (x^n)^2}, \ldots, \frac{x^n}{(x^{m+1})^2 + \ldots (x^n)^2} \right) \in \mathbb{R}^n \backslash \mathbb{R}^m$$

is continuous and odd on $\mathbb{R}^n \backslash \mathbb{R}^m$. Hence, the genus is $\leq n - m$. The opposite inequality follows from the argument of Lemma 2.6.2 (but will not be needed in the sequel). This proves the first statement. The second one is a direct consequence of the definition because a continuous map restricted to a subset of its

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domain remains continuous. The third statement is a consequence of the two others. □

Thus, the idea is to perform minimaxing over families that (in the sense of Definition 2.6.2) have the topology of spheres. According to the general theory of Lusternik and Schnirelman, presented for instance in [152, 277], for $k = 1, \ldots, n$, the constants

$$\lambda_k(A_p) := \inf_{S \in \mathrm{Gen}_k} \sup_{f \in S \setminus \{\mathbf{0}\}} R_p(f) \tag{2.6.36}$$

then are eigenvalues of A_p. They satisfy

$$\lambda_1 \leq \ldots \leq \lambda_n$$

and, if $\lambda = \lambda_{k+1} = \ldots = \lambda_{k+l}$ for $0 \leq k < k + l \leq n$, then

$$\mathrm{genus}(\{\text{eigenfunctions corresponding to } \lambda\}) \geq l.$$

In the preceding, we have required $p > 1$, because if we took $p = 1$, (2.6.34) would become

$$a^\star \frac{a g_\lambda}{\|a g_\lambda\|} = \lambda \frac{g_\lambda}{\|g_\lambda\|}, \tag{2.6.37}$$

and these quotients become undefined at points where $g_\lambda = 0$. (In our applications, the Hilbert space H and H_1 are function spaces, and the functions may then have zeroes without being identically 0.)

In order to handle this issue, we define the operator A_p for $p = 1$ as a set-valued map. The Rayleigh quotients, as their name indicates, are quotients of functionals

$$\frac{F(f)}{G(f)},$$

and when both F and G are differentiable, a critical point g satisfies the eigenvalue equation

$$\nabla F(g) - \frac{F(g)}{G(g)} \nabla G(g) = 0$$

for the eigenvalue $\lambda = \frac{F(g)}{G(g)}$ as we have explored earlier. In the nondifferentiable case, as in [130], we use the generalized gradient of Clarke [63], defining

Definition 2.6.3. The *generalized gradient* of a functional $R\colon H \to \mathbb{R}$ at $g \in H$ is

$$\partial R(g) := \{g^* \in H\colon \lim_{f \to g, t \to 0} \sup \frac{R(f + t\phi) - R(f)}{t} \geq (g^*, \phi) \text{ for all } \phi \in H\}. \tag{2.6.38}$$

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$g \in H$ is a *critical point* of R if

$$0 \in \partial R(g). \tag{2.6.39}$$

[130] then shows

Lemma 2.6.4. *Let F, G be convex, Lipschitz continuous, even and positively p-homogeneous for $p \geq 1$ (i.e., $F(\mu f) = \mu^p F(f)$ for all $\mu \geq 0$, and the same for G). Then, a necessary condition that f is a critical point of $\frac{F(f)}{G(f)}$ is that*

$$0 \in \partial F(f) - \lambda \partial G(f), \tag{2.6.40}$$

where $\lambda = \frac{F(f)}{G(f)}$. When G is continuously differentiable at f, this is also sufficient.

In fact, in the present situation, instead of the generalized gradient ∂, we can also and will use the *subgradient*, see for instance [88, 239].

Definition 2.6.4. $R\colon \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is called *subdifferentiable* at g if there exists some $g^* \in H$ such that

$$R(h) - R(g) \geq (h - g, g^*)\,, \tag{2.6.41}$$

and the set of all g^* with this property is called the *subgradient* $\nabla R(g)$.

(When we ignore the lim sup in (2.6.38) and put $f = g, t\phi = h - g$, then (2.6.38) reduces to (2.6.41).)

Thus, the subgradient of a subdifferentiable R at g consists of the slopes of all affine linear functions that are not larger than R and agree with R at g. Our eigenvalue equation then becomes

$$0 \in \nabla F(g_\lambda) - \lambda \nabla G(g_\lambda). \tag{2.6.42}$$

Thus, we take (2.6.40) or (2.6.42) as our eigenvalue equation for the operator A_1 when $\frac{F(f)}{G(f)}$ is the Rayleigh quotient $R_1(f)$. This condition will be spelled out in more detail in Sections 3.3 and 7.2. We should, however, point out that the setting there will be slightly different. We shall consider spaces of functions $f\colon \Sigma \to \mathbb{R}$, and instead of working with the p-th powers of the L^2-norm, we shall consider the L^p-norms. But the abstract setting and the problems to be overcome will remain the same.

2.7 Symmetries

Let V be some set with relations $\omega(v_1, \ldots, v_k)$ for certain values of k as in Section 2.2. We abbreviate that structure as (V, ω).

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Definition 2.7.1. An *automorphism* of (V,ω) is a bijection $\sigma\colon V \to V$ with the property that

$$\omega(\sigma(v_1),\dots,\sigma(v_k)) = \omega(v_1,\dots,v_k) \tag{2.7.1}$$

for all relations. For such an automorphism σ and $f\colon V \to \mathbb{R}$, we put

$$\sigma^* f(v) = f(\sigma(v)).$$

Remark 2.7.1. This concept also applies to hypersimplicial complexes, with obvious modifications.

Obviously, the automorphism of (V,ω) forms a group, its *automorphism group*. We then have the obvious, but important

Lemma 2.7.1. *If σ is an automorphism of (V,ω), then for all Laplacians*

$$L(\sigma^*\eta)(v) = \sigma^*(L\eta)(v) \tag{2.7.2}$$

for all $v \in V$ and all η to which it can be applied, for instance to $f\colon V \to \mathbb{R}$ when the relations hold for $k = 2$, that is, for pairs of vertices.

Proof This holds because L is a linear operator. □

Thus, Laplacians commute with automorphisms σ. In particular, the spectrum is not affected by automorphisms, and eigenspaces stay invariant. We conclude

Corollary 2.7.1. *Laplacian eigenspaces constitute representations of the automorphism group of (V,ω).* □

We can use Lemma 2.7.1 to decompose the spectrum of L. For simplicity, we consider the case $k = 2$, and hence, the space of functions $f\colon V \to \mathbb{R}$ and an automorphism τ of (V,ω) with

$$\tau^2 = \mathrm{id}. \tag{2.7.3}$$

Then, τ has two possible eigenvalues, ± 1, on the space of functions $f\colon V \to \mathbb{R}$, and L leaves those two eigenspaces $L_\pm$ invariant. Also, $V = V_0 + V_1$ where $\tau(v_0) = v_0$ precisely if $v_0 \in V_0$. That is, V_0 is the set of those vertices that are fixed by τ. Moreover, we can write $V_1 = V' \cup V''$ where V', V'' are disjoint and $\tau(V') = V''$. Since also $\tau(V'') = V'$ because of (2.7.3), V' and V'' play symmetric roles.

Without loss of generality, we assume that V' (and hence also V'') is connected, as otherwise we can rearrange the decomposition of V_1 and/or write τ as the composition of several such automorphisms.

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Lemma 2.7.2. *There is a $|V'|$-dimensional space of functions $f : V \to \mathbb{R}$ consisting of eigenfunctions of L that vanish on V_0 and that are antisymmetric on V' and V'', that is, $f(v'') = -f(v')$ if $v'' = \tau(v') \in V''$ for $v' \in V'$, and a remaining $(|V'|+|V_0|)$-dimensional space of eigenfunctions that are symmetric on V' and V'', that is, $f(v'') = f(v')$.*

Proof The first class of functions are those that are eigenfunctions of τ for the eigenvalue -1, where the second class has eigenvalue 1. By Lemma 2.7.1, these are unions of eigenspaces of L. Since $|V''| = |V'|$ and $V = V_0 \cup V' \cup V''$, this generates the space of all functions on V. □

Definition 2.7.2. An *induced substructure* or a *motif* $(\hat{V}, \omega)$ consists of some nonempty vertex set $\hat{V} \subset V$ and the same relations between its elements as in V.

Let $\hat{V}$ be such a motif. We then have the induced Laplacian

$$L_{V,\hat{V}} f(v) = f(v) - \frac{1}{\deg_V v} \sum_{v'} \omega_{vv'} f(v'), \tag{2.7.4}$$

where $\deg_V v$ denotes the degree of v in V.

Definition 2.7.3. The motif V' with vertex set V' is a *duplicated motif* if V' and V'' are disconnected.

Lemma 2.7.3. *Let V' be a duplicated motif in V, and let $v_0 \in V_0$ be a neighbor of some $v' \in V'$, that is, $\omega_{v',v_o} \neq 0$. Then, v_0 is also a neighbor of $v'' = \tau(v') \in V''$.*

Proof Since $v_0 \in V_0$ is fixed by the automorphism τ, and since $\omega_{v',v_o} = \omega_{\tau(v'),v_o}$, the claim follows. □

Lemma 2.7.4. *Let V' be a duplicated motif in V. Then, we find a basis of eigenfunctions of the Laplacian L of V of functions f satisfying either*

1.

$$L_{V,V'} f(v) = \begin{cases} f(v) & \text{for } v \in V' \text{ (and also for } v \in V'') \\ -f(\tau(v)) & \text{for } v \in V'' \\ 0 & \text{for } v \in V_0 \end{cases} \tag{2.7.5}$$

2. or

$$f(\tau(v)) = f(v) \text{ for } v \in V'. \tag{2.7.6}$$

The latter eigenfunctions are those of V^τ obtained as the quotient of V by τ, with the induced weights.

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Proof If v_0 in V_0 and $f(v_0) = 0$ and if f is antisymmetric, then also $Lf(v_0) = 0$ because the contributions from its neighbors v' and $v'' = \tau(v')$ (see Lemma 2.7.3) cancel each other in $Lf(v_0)$. The result then follows from Lemma 2.7.2 because under our assumptions, a neighbor w of $v \in V'$ is contained either in V' or in V_0, in which case, for an antisymmetric f, $f(z) = 0$, and therefore, we can restrict the computation in (2.7.5) to the induced Laplacian, that is, consider only the vertices in V'. □

For the case of hypergraphs with real coefficients, the role of symmetries was studied in [154], and we have used that reference for our exposition.

2.8 Homomorphisms

As in Section 2.7, let U, V be sets with relations $\omega_U(u_1, \dots, u_k)$ and $\omega_V(v_1, \dots, v_k)$, respectively, for certain values of k as in Section 2.2.

Definition 2.8.1. A *homomorphism* of $h\colon (U, \omega_U) \to (V, \omega_V)$ is a map $h\colon U \to V$ with the property that

$$\omega_V(h(u_1), \dots, h(u_k)) = \omega_U(u_1, \dots, u_k) \tag{2.8.1}$$

for all relations. For such a homomorphism h and $f\colon V \to \mathbb{R}$, we put

$$h^* f(u) = f(h(u)).$$

When $\hat{\Gamma}$ and Γ are (unweighted) graphs,[1] a homomorphism $h\colon \hat{\Gamma} \to \Gamma$ thus maps the vertices of $\hat{\Gamma}$ to vertices of Γ and preserves edges, that is, $h(v_1) \sim h(v_2)$ if and only if $v_1 \sim v_2$. A special case is

Definition 2.8.2. Let $\hat{\Gamma}$ and Γ be graphs. A *covering* is a homomorphism $\pi\colon \hat{\Gamma} \to \Gamma$ that maps the edges emanating from each vertex v of $\hat{\Gamma}$ bijectively onto the edges emanating from its image $\pi(v)$ in Γ.

We also say that the *stars* of vertices are mapped bijectively. Importantly, we do not require that π induces a bijection between the vertex sets. When Γ is connected, then each vertex of Γ must have the same number of preimages under h, and if that number is k, we speak of a k-covering. See [6] and the references therein for more detailed discussions of coverings. We shall analyze 2-coverings in Section 4.6.

[1] The definitions should be clear but will be explicitly formulated in Section 4.1.

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2.9 General Setting

We shall work in the context of a general discrete structure. Let $\mathcal{M} := (M_1, \dots, M_\ell)$ be a sequence of multisets. For each $q = 1, \dots, \ell$, consider a Hilbert space $C(M_q)$ equipped with a scalar product $(\cdot,\cdot)_q$, and consider two operators

$$C(M_q) \underset{\beta^q}{\overset{\delta^q}{\rightleftarrows}} C(M_{q+1}).$$

We call $(\mathcal{M}, (\delta^q)_{q\geq 1}, (\beta^q)_{q\geq 1})$ a *discrete structure* because it contains some groups of objects and the interaction relations between them.

Most of the discrete objects considered in this book can be recovered by this general structure, including (directed, oriented, weighted) hypergraphs, simplicial complexes, and Lefschetz complexes. Moreover, we shall use this to formalize new objects we are interested in, such as hypersimplicial complexes.

We consider **Laplace operators** of the following type.

(i) The *q-up Laplace operator* of $\mathcal{M}$ is

$$L^q_{up} := \beta^q \delta^q,$$

(ii) The *q-down Laplace operator* of $\mathcal{M}$ is

$$L^q_{down} := \delta^{q-1}\beta^{q-1}, \text{and}$$

(iii) The *q-Laplace operator* of $\mathcal{M}$ is

$$L^q := L^q{}_{up} + L^q{}_{down}.$$

The **linear case** is the most important one. Suppose that both β^q and δ^q are linear. Then, the nonzero eigenvalues of L^q_{down} and L^{q-1}_{up} coincide. Also, since $(L^q_{up})^* = (\delta^q)^*(\beta^q)^*$ has the same eigenvalues with L^q_{up}, in principle, we only need to study L^q_{up}.

Example 2.9.1. Recall, from Section 2.4.3, that a directed hypergraph is an oriented graph $\Gamma = (V, E, \varphi)$ with the additional assumption that hyperedges are directed, from the inputs toward the outputs. In this case, each hyperedge e can be seen as a directed edge from its input set $\text{in}(e)$ toward its output set $\text{out}(e)$. Hence, E can be seen as the edge set of a directed graph whose vertex set is some $\mathcal{A} \subset \mathcal{P}(V)$. Consider

$$C(V) \xrightarrow{\delta^0} C(\mathcal{A}) \xrightarrow{\delta^1} C(E)\ ,$$

where $\delta^0 f(A) := \sum_{v\in A} f(v)$ and $\delta^1 g(e) := g(\text{out}(e)) - g(\text{in}(e))$. By letting $\delta := \delta^1\delta^0$, we have $\delta f(e) = \sum_{v'\in \text{out}(e)} f(v') - \sum_{v\in \text{in}(e)} f(v)$. Let $\beta\colon C(E) \to$

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$C(V)$ be defined by $\beta q(v) := \sum_{e:\, v\in \mathrm{out}(e)} q(e)$. Then, the Laplacian can be defined as $L = \beta\delta$.

In many instances, we shall also consider **non-linear cases**. We can also take β^q and δ^q as nonlinear operators, as shown by the next two examples.

Example 2.9.2. Let $\Gamma = (V,E)$ be a k-uniform hypergraph with nodes $v_1,\dots,v_N$. Let $\delta f(e) := \prod_{v\in e} f(v)$, and let $\beta g(v) := \sum_{e\in E:\, v\in e} g(e)$. Then, the nonlinear operator $\beta\delta$ is related to the tensors associated with Γ that we shall treat in Chapter 11. In fact, $\beta\delta$ is the k-th order N-dimensional tensor T such that $T_{i_1,\dots,i_k} = 1$ if $\{v_{i_1},\dots,v_{i_k}\} \in E$, and $T_{i_1,\dots,i_k} = 0$ otherwise. Also, the corresponding *H-eigenvalue problem* can be formulated as $\beta\delta f = \lambda f^k$.

Example 2.9.3. Let $\Gamma = (V,E,\varphi)$ be a directed hypergraph. Let

$$\delta f(e) := \max_{v\in \mathrm{out}(e)} f(v) - \min_{v'\in \mathrm{in}(e)} f(v'),$$

and consider a set-valued map β such that $\beta\delta f$ coincides with the subgradient of $\|\delta f\|_p^p := \sum_{e\in E} |\delta f(e)|^p$. Then, $\beta\delta$ coincides with the Lovász p-Laplacian for hypergraphs that we shall treat in Chapters 6 and 8.

In many instances, we shall also consider **linear and adjoint cases**. In this book, the most important case is where δ^q is linear and $\beta^q = (\delta^q)^*$. Hence, as suggested by (2.2.4), we shall consider the critical points of functionals in the form of

$$f \mapsto \frac{F(\delta f)}{G(f)},$$

where $F\colon C(M_{q+1}) \to \mathbb{R}$ and $G\colon C(M_q) \to \mathbb{R}$ are locally Lipschitz functions. Thus, we shall study the general eigenvalue problem

$$0 \in \nabla_f F(\delta f) - \lambda \nabla_f G(f),$$

where ∇_f indicates the Clarke derivative at f (see Definition 2.6.3). Since

$$\nabla_f F(\delta f) = \delta^* \nabla F(\delta f) = \delta^* \circ \nabla F \circ \delta(f),$$

it is convenient to set $\beta = \delta^*$ (or $\beta = \delta^* \circ \nabla F$). That is, the additional assumption $\beta = \delta^*$ helps us to study the eigenvalue problems we are interested in by variational methods.

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