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GAPS BETWEEN HEXAGONAL AND SQUARE LATTICES

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Abstract

Let $\triangle$ denote the integers represented by the quadratic form $x^2 + 3y^2$ and $\square_2$ denote the numbers represented as a sum of two squares. For a nonzero integer a, let $S(\triangle, \square_2, a)$ be the set of integers n such that $n \in \triangle$ and $n + a \in \square_2$. We conduct a census of $S(\triangle, \square_2, a)$ in short intervals by showing that there exists a constant $H_a > 0$ with

$$\#S(\triangle, \square_2, a) \cap [x, x + H_a \cdot x^{5/6} \cdot \log^{19} x] \geq x^{5/6-\varepsilon}$$

for large x. To derive this result and its generalisation, we use a theorem of Tolev ['On the remainder term in the circle problem in an arithmetic progression', *Proc. Steklov Inst. Math.* **276** (2012), 261–274].

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1. Introduction

For $n \in \mathbb{Z}$, let $r_2(n)$ denote the number of representations of n as a sum of two squares. In 1932, Estermann [5] showed that whenever a is a nonzero integer, one obtains the asymptotic formula

$$\sum_{n \leq x} r_2(n) \cdot r_2(n + a) = C_a \cdot x + O(x^{\tau+\varepsilon}) \tag{1.1}$$

for $\tau = 11/12$, $C_a > 0$, which depends only on a, and any $\varepsilon > 0$. Later, with Weil's work on Kloosterman sums, Hooley provided an improvement to (1.1) with $\tau = 5/6$ [6, page 282]. Using properties of modular groups, Chamizo showed that $\tau = 2/3$ [3, 4]. For two sets of integers W_1 and W_2, denote

$$S(W_1, W_2, a) = \{n : n \in W_1,\ n + a \in W_2\}.$$

We use the notation $\square_2$ to denote the numbers that are a sum of two squares. When a is a nonzero integer, we may survey the number of elements of the set $S(\square_2, \square_2, a)$

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in intervals. In this direction, Hooley [7] showed that there exist constants C, D depending on $a \neq 0$ and with $0 < C < D$, so that for large $x \in \mathbb{R}$,

$$\frac{C \cdot x}{\log x} \leq \#S(\square_2, \square_2, a) \cap [1, x] \leq \frac{D \cdot x}{\log x}.$$

Since

$$r_2(n) = 4 \cdot \sum_{d|n} \chi_4(d), \tag{1.2}$$

where χ_4 is the primitive character modulo 4 [5, (33)], for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ so that $r_2(n) \leq C_\varepsilon \cdot n^\varepsilon$. Thus, a quick application of the asymptotic formula (1.1) with $\tau = 2/3$ leads us to a lemma that counts elements of $S(\square_2, \square_2, a)$ in short intervals.

LEMMA 1.1. *If β and $\varepsilon > 0$, then $\#(S(\square_2, \square_2, a) \cap [x, x + x^{2/3+\beta}]) \geq x^{2/3+\beta-\varepsilon}$ for sufficiently large x.*

The following lemma serves to bound the largest gaps.

LEMMA 1.2. *There exists a constant $C_a > 0$ such that there is an element of $S(\square_2, \square_2, a)$ in the interval $[x, x + C_a \cdot x^{1/2}]$ whenever $x \geq 1$.*

PROOF. If $a = 2^t a'$ for some nonnegative integer t and odd a', then $g \in S(\square_2, \square_2, a')$ implies $2^t \cdot g \in S(\square_2, \square_2, 2^t \cdot a')$, since the product of two sums of two squares is itself a sum of two squares. Thus, it is enough to prove the lemma for odd values of a. Now observe that when s is an integer,

$$\left(s^2 + \left(\frac{a-1}{2}\right)^2\right) + a = s^2 + \left(\frac{a+1}{2}\right)^2. \qquad \square$$

Let $\triangle$ denote the set of integers represented by the quadratic form $x^2 + 3y^2$. We similarly examine the density of the set $S(\triangle, \square_2, a)$.

LEMMA 1.3. *If $\chi_4(a) = 1$ or $\chi_3(a) = -1$, then $\#(S(\triangle, \square_2, a) \cap [1, x]) \gg x/\log^{3/2} x$.*

PROOF. Let $\mathbb{P}$ denote the set of primes. If $\chi_4(a) = 1$, then by [9, Theorem 1],

$$\#\{n : 1 \leq n \leq x, \text{ there exists } x, y \in \mathbb{Z} \text{ with } n = 4(x^2 + 3y^2),\ n + a \in \mathbb{P}\} \gg x/\log^{3/2} x.$$

If $\chi_3(a) = -1$, then by [9, Theorem 1],

$$\#\{n : 1 \leq n \leq x, \text{ there exists } x, y \in \mathbb{Z} \text{ with } n = 9(x^2 + y^2),\ n - a \in \mathbb{P}\} \gg x/\log^{3/2} x.$$

Since primes $\equiv 1 \bmod 3$ are in $\triangle$ and primes $\equiv 1 \bmod 4$ are in $\square_2$, the proof is complete. $\square$

As in Lemma 1.1, we can count elements of the set $S(\triangle, \square_2, a)$ in short intervals.

COROLLARY 1.4. *Let $\varepsilon > 0$. There exists a constant $H_a > 0$ such that for sufficiently large x, $\#(S(\triangle, \square_2, a) \cap [x, x + H_a \cdot x^{5/6} \cdot \log^{19} x]) \geq x^{5/6-\varepsilon}$.*

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Integers that are represented by the quadratic form x^2+xy+y^2 are the same as those for x^2+3y^2 [11]. For a positive integer n, denote by $R_2(n)$ the number of solutions to the equation $n=x^2+xy+y^2$ and by χ_3 the nontrivial character modulo 3. Analogously to (1.2), we have the classical formula (see [8, (7.2)])

$$R_2(n)=6\cdot\sum_{d|n}\chi_3(d),$$

which is a key ingredient in the proof of Corollary 1.4. When ψ is a character modulo k ($k>1$), define

$$F_\psi(n)=\sum_{d|n}\psi(d).$$

For natural numbers a,k, denote by $P(a,k)$ the unique integer $t\mid a$ so that if a prime p divides t, then $p\mid k$ and $\gcd(a/t,k)=1$. The next theorem is a 'local version' of a theorem by Müller [12, Theorem 1].

THEOREM 1.5. *Let ψ and ρ be primitive characters modulo $k>1$. If $a\geq 1$, then*

$$\sum_{n\leq x}F_\psi(n)\cdot F_\rho(n+a)=M_{\psi,\rho}(a)\cdot x+O(x^{5/6+\varepsilon}),$$

where

$$M_{\psi,\rho}(a)=C_{\psi,\rho}(a)+\left\{k^{-1}\sum_{b|P(a,k)}b^{-1}\sum_{j=1}^{k}\psi(j)\cdot\rho\left(\frac{a}{b}+j\right)\right\}\cdot C_{\bar\psi,\bar\rho}(a)$$

and

$$C_{\psi,\rho}(a)=\frac{L(1,\rho)\cdot L(1,\psi)}{L(2,\rho\cdot\psi)}\sum_{d|a}\psi(d)\cdot\rho(d)\cdot d^{-1}.$$

By setting $\psi=\rho=\chi_3$, Theorem 1.5 allows the study of the set $S(\triangle,\triangle,a)$ in short intervals. However, this theorem cannot be employed to derive even a weaker version of Corollary 1.4. The obstacle is that the characters χ_4 and χ_3 (or χ_6) cannot be used simultaneously in Theorem 1.5: they require a common modulus k (where $12\mid k$), and lifting them to such a modulus destroys the primitivity in the theorem's hypothesis.

For a positive integer q, set

$$\eta_a(q):=\#\{1\leq\alpha,\beta\leq q:\ \alpha^2+\beta^2\equiv a\ (\mathrm{mod}\ q)\}.$$

THEOREM 1.6. *Let ψ be a real nontrivial character modulo $b\geq 4$, with b even. If $a\neq 0$ is an integer, then*

$$\sum_{n\leq x,\,(n,b)=1}F_\psi(n)\cdot F_{\chi_4}(n+a)=\beta(\psi,a)\cdot\eta^*(\psi,a)\cdot x+O(x^{5/6}\cdot\log^{19}x),\tag{1.3}$$

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where

$$\beta(\psi,a)=\sum_{d=1}^{\infty}\frac{\psi(d)\cdot\eta_a(d)}{d^2}>0 \quad \textit{and} \quad \eta^*(\psi,a)=\sum_{j=1,\,\psi(j-a)=1}^{b}\frac{\pi\cdot\eta_j(b)}{2b^2}.$$

Theorem 1.6 implies Corollary 1.4. With χ_6 being the nontrivial real character modulo 6, by Lemma 1.3 and (1.2),

$$\begin{aligned}\sum_{n\le x,\,(n,6)=1}R_2(n)\cdot r_2(n+a)&=24\cdot\sum_{\substack{n\le x,\,(n,6)=1\\ n\ge\max\{1,1-a\}}}F_{\chi_3}(n)\cdot F_{\chi_4}(n+a)\\&=24\cdot\sum_{\substack{n\le x,\,(n,6)=1\\ n\ge\max\{1,1-a\}}}F_{\chi_6}(n)\cdot F_{\chi_4}(n+a).\end{aligned}$$

Thus,

$$\sum_{n\le x,\,(n,6)=1}R_2(n)r_2(n+a)=24\cdot\beta(\chi_6,a)\cdot\eta^*(\psi,a)\cdot x+O(x^{5/6}\cdot\log^{19}x).$$

The quantity $\eta^*(\chi_6,a)$ takes values from the set $\{\eta^*(\chi_6,0),\eta^*(\chi_6,1),\dots,\eta^*(\chi_6,5)\}$ and each such value is positive. Noting that $\beta(\psi,a)>0$, as proved in Theorem 1.6, we conclude, by differencing the above asymptotic formula at suitable endpoints, that the remainder of the proof of Corollary 1.4 follows as in the deduction of Lemma 1.1.

When K is a quadratic number field with discriminant D, there exists a primitive character χ_K modulo D so that the number of ideals of norm n is given by the expression $F_{\chi_K}(n)$ [12]. Denote

$$\begin{aligned}\diamond_K&:=\{n:\ n\in\mathbb{N},\ \text{there exists an ideal } I\subseteq K \text{ with } [O_K:I]=n\}\\&=\{n:n\in\mathbb{N},\ F_{\chi_K}(n)>0\}.\end{aligned}$$

Then, with Theorem 1.6, a result similar to Corollary 1.4 can be obtained for the set $S(\diamond_K,\square_2,a)$. Deep results of Iwaniec [9, Theorem 1] examine $S(Q,\mathbb{P},a)$, when Q is a set containing values assumed by a quadratic form and $\mathbb{P}$ is the set of primes. Further, Moree and Riele [11] compare the densities of $\triangle$ and $\square_2$. Let $B_j(x)$ denote the number of integers v that satisfy $v\le x$ and v is represented by the quadratic form x^2+jy^2. Then, $B_1(x)\ge B_3(x)$ [11, Theorem 1]. Similar to Lemma 1.2, which attempts to study the largest gaps between elements of $S(\square_2,\square_2,a)$, we obtain a weaker gap result for sets of the form $S(\triangle,\square_2,a)$.

THEOREM 1.7. *Let $x\ge 1$. There exists a constant $D_a>0$ such that there is an element of $S(\triangle,\square_2,a)$ in the interval $[x,x+D_a\cdot x^{\Upsilon(a)}]$, where*

$$\Upsilon(a)=\begin{cases}1/2 & \text{for } a\in\{n^2-3m^2:\ n,m\in\mathbb{Z}\},\\ 5/8 & \text{for } a\in\mathbb{Z}\setminus\{n^2-3m^2:\ n,m\in\mathbb{Z}\}.\end{cases}$$

A generalisation of Theorem 1.7 can be obtained, but we do not pursue that here.

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2. Notation

We adopt the usual notation: (u, v) denotes the greatest common divisor of the integers u and v; for a prime p and an integer w, $v_p(w)$ is the largest integer t with $p^t \mid w$ (defined to be ∞ when $w = 0$); $\phi(\cdot)$ is the Euler totient function and $\zeta(\cdot)$ is the Riemann zeta function.

Let U, V and W be complex quantities dependent on parameters $\mathbf{P}$ from some set, along with (possibly) $d, q \in \mathbb{N}$ and $x \in \mathbb{R}$. Unless otherwise specified, we write $U \ll V$, $V \gg U$ or $U = O(V)$ if there exists an effective constant $C > 0$, independent of d, q and x, such that $|U| \le C \cdot V$ for all choices of d, q and x (with other variables fixed), whenever $V \neq 0$. Further, we write $U = V + O(W)$ if $U - V \ll W$.

3. Proof of Theorem 1.6

We interpret a result of Tolev [13, (10)]. Put

$$S_{q,a}(x) := \sum_{\substack{n \le x \\ n \equiv a \pmod q}} F_{\chi_4}(n) = \pi \frac{\eta_a(q)}{4q^2} x + R_{q,a}(x).$$

Then,

$$R_{q,a}(x) \ll (q^{1/2} + x^{1/3})(a, q)^{1/2} \tau^4(q) \log^4 x. \tag{3.1}$$

With the notation $F_\psi(\cdot) = A(\cdot)$ and $F_{\chi_4}(\cdot) = B(\cdot)$, the main expression we wish to evaluate asymptotically is

$$J(x) := \sum_{n \le x,\ (n,b)=1} A(n) \cdot B(n + a).$$

We need two lemmas.

LEMMA 3.1. *If $\psi(n) = -1$, then $A(n) = 0$.*

PROOF. Observe that A is the convolution of ψ and 1, and thus, is multiplicative. On primes p, with $\psi(p) = -1$, we have that $A_b(p^j) = 0$ if j is an odd number. Hence, the lemma follows. □

Extend the character ψ to the real numbers by setting $\psi(w) = 0$ whenever $w \notin \mathbb{Z}$. When $\psi(n) = 1$ and $d \mid n$, we observe that $\psi(n/d) = \psi(d)$, and this symmetry leads to the following lemma.

LEMMA 3.2. *If $\psi(n) = 1$, then*

$$A(n) = \Big(2 \sum_{d < \sqrt{n}} \psi(d)\Big) + \psi(\sqrt{n}).$$

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With Lemmas 3.1 and 3.2, we write

$$\begin{aligned} J(x) &= \sum_{n\le x,\ \psi(n)=1} A(n)\cdot B(n+a) = \sum_{n\le x,\ \psi(n)=1} \Big(\sum_{d|n} \psi(d)\Big)\cdot B(n+a) \\ &= \sum_{n\le x,\ \psi(n)=1} \Big(2\cdot\Big(\sum_{d|n,\ d<\sqrt{n}} \psi(d)\Big) + \psi(\sqrt{n})\Big)\cdot B(n+a) \\ &= \sum_{n\le x,\ \psi(n)=1} \Big(2\cdot \sum_{d|n,\ d<\sqrt{n}} \psi(d)\Big) B(n+a) + O(x^{1/2+\varepsilon}). \end{aligned}$$

This implies

$$J(x) = \sum_{1\le d<\sqrt{x}} 2\psi(d)\Big(\sum_{\substack{d^2<n\le x\\ d|n,\ \psi(n)=1}} B(n+a)\Big) + O(x^{1/2+\varepsilon}).$$

For $(b,d)=1$, let $V(d)$ be the set of residue classes modulo bd so that if r is a congruence class modulo bd, then $r\equiv a \pmod d$ and $\psi(r-a)=1$. Note that $V(d)$ has $\phi(b)/2$ residue classes. List the residue classes as $r(1,d), r(2,d), \dots, r(\phi(b)/2,d)$. Then,

$$J(x) = \sum_{1\le d<\sqrt{x}} 2\psi(d) \sum_{j=1}^{\phi(b)/2} \Big(\sum_{\substack{m\equiv r(j,d) \pmod{bd}\\ d^2+a<m\le x+a}} B(m)\Big) + O(x^{1/2+\varepsilon}).$$

Let

$$Q(j;x) = \sum_{\substack{m\equiv r(j,d) \pmod{bd}\\ d^2+a<m\le x+a}} B(m).$$

By the bound (3.1),

$$Q(j;x) = \pi\frac{\eta_{r(j,d)}(bd)}{4b^2d^2}(x-d^2) + O(((bd)^{1/2}+x^{1/3})\cdot (r(j,d),bd)^{1/2}\cdot \tau^4(bd)\cdot \log^4 x).$$

Since $a\neq 0$, we have $(r(j,d),bd)\le |ab|$ and so

$$Q(j;x) = \pi\frac{\eta_{r(j,d)}(bd)}{4b^2d^2}(x-d^2) + O(x^{1/3}\cdot \tau^4(d)\cdot \log^4 x),$$

where the implied constant in the O-notation is independent of d in the range $1\le d<\sqrt{x}$. Using $\sum_{1\le d<\sqrt{x}} \tau^4(d) \ll \sqrt{x}\cdot \log^{15} x$ [14] (also see [10]), we find

$$\begin{aligned} J(x) &= \Big(\sum_{1\le d<\sqrt{x}} 2\psi(d) \sum_{j=1}^{\phi(b)/2} \pi\frac{\eta_{r(j,d)}(bd)}{4b^2d^2}(x-d^2)\Big) + O(x^{5/6}\cdot \log^{19} x) \\ &= \frac{\pi}{2b^2}\cdot\Big(\sum_{1\le d<\sqrt{x},\ (d,b)=1} \psi(d) \sum_{j=1}^{\phi(b)/2} \frac{\eta_{r(j,d)}(bd)}{d^2}(x-d^2)\Big) + O(x^{5/6}\cdot \log^{19} x). \end{aligned}$$

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Since $\eta_s(\cdot)$ is multiplicative for each $s \in \mathbb{Z}$,

$$J(x) = \frac{\pi}{2b^2} \cdot \Big(\sum_{1 \le d < \sqrt{x}} \psi(d) \sum_{j=1,\, \psi(j-a)=1}^{b} \frac{\eta_j(b)\eta_a(d)}{d^2}(x - d^2) \Big) + O(x^{5/6} \cdot \log^{19} x).$$

Thus,

$$J(x) = \frac{\pi}{2b^2} \cdot \Big(\sum_{j=1,\, \psi(j-a)=1}^{b} \eta_j(b) \sum_{1 \le d < \sqrt{x}} \frac{\psi(d)\eta_a(d)}{d^2}(x - d^2) \Big) + O(x^{5/6} \cdot \log^{19} x). \tag{3.2}$$

3.1. Behaviour of $\eta_a(d)$. Put $\lambda_a(d) := \eta_a(d)/d$. The next two lemmas follow from [2, (2.22), (2.23) and (2.24)].

LEMMA 3.3. *If $p \equiv 1 \pmod 4$ is a prime such that $\nu_p(a) > 0$, then*

$$\lambda_a(p^j) = \begin{cases} 1 + j(1 - 1/p) & \text{for } 1 \le j \le \nu_p(a), \\ (1 + \nu_p(a))(1 - 1/p) & \text{for } j \ge \nu_p(a) + 1. \end{cases}$$

LEMMA 3.4. *If $p \equiv 3 \pmod 4$ is a prime such that $\nu_p(a) > 0$, then*

$$\lambda_a(p^j) = \begin{cases} 1/p & \text{for } 1 \le j \le \nu_p(a),\ j \text{ odd}, \\ 1 & \text{for } 1 \le j \le \nu_p(a),\ j \text{ even}, \\ 1 + 1/p & \text{for } j \ge \nu_p(a) + 1,\ \nu_p(a) \text{ even}, \\ 0 & \text{for } j \ge \nu_p(a) + 1,\ \nu_p(a) \text{ odd}. \end{cases}$$

The next lemma follows from [2, (2.20) and (2.21)].

LEMMA 3.5. *Let p be an odd prime such that $p \nmid a$. If j is a positive integer, then*

$$\lambda_a(p^j) = 1 - \chi_4(p)/p.$$

Following the proof of [2, Lemma 2.8], we also see that when $j \ge 1$,

$$0 \le \lambda_a(2^j) \le 4. \tag{3.3}$$

COROLLARY 3.6. *Let $n \in \mathbb{N}$. There exists a constant $K_a > 0$ so that $0 \le \lambda_a(n) \le K_a$.*

PROOF. The result follows by Lemmas 3.3, 3.4, the estimate (3.3) and noting that λ_a is a multiplicative function. □

Let λ_a^{-1} be the convolution of λ_a with the Möbius function μ, that is, $\lambda_a^{-1} = \lambda_a * \mu$. Note that

$$\lambda_a(n) = \sum_{d \mid n} \lambda_a^{-1}(d).$$

LEMMA 3.7. *We have*

$$\lambda_a^{-1}(n) = \frac{f_{\lambda_a}(n)}{n},$$

where $|f_{\lambda_a}(z)| \le C_a$ whenever z is an odd number and $C_a > 0$ is some positive constant.

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PROOF. Note that $\lambda_a^{-1} = \lambda_a * \mu$ is a multiplicative function. If p is an odd prime divisor of a, then by Lemmas 3.3 and 3.4, $\lambda_a^{-1}(p^j) = \lambda_a(p^j) - \lambda_a(p^{j-1}) = 0$ whenever $j \geq \nu_p(a) + 2$. Put $D_a := \max_{d|a^2} |\lambda_a^{-1}(d)|$. Observe that if $p \mid a$, then $\nu_p(z) \leq \nu_p(a) + 1$ or else $\lambda_a^{-1}(z) = 0$. If $\lambda_a^{-1}(z) \neq 0$, then $\lambda_a^{-1}(z) = \lambda_a^{-1}(r)\lambda_a^{-1}(q)$, where $z = rq$ with $r \mid a^2$ and $(a, q) = 1$, so that

$$|\lambda_a^{-1}(z)| \leq D_a |\lambda_a^{-1}(q)|. \tag{3.4}$$

If p is a prime such that $(p, a) = 1$, then from Lemma 3.5, $\lambda_a^{-1}(p) = -\chi_4(p)/p$ and $\lambda_a^{-1}(p^j) = 0$ for $j \geq 2$. By (3.4), we obtain the estimate

$$|\lambda_a^{-1}(z)| \leq D_a \frac{1}{q} \leq \frac{a^2 D_a}{z}.$$

Hence, $|f_{\lambda_a}(z)| \leq a^2 D_a$ when z is an odd positive number. □

Recall that b is an even number. For a multiplicative character ρ_b modulo b, we put

$$J(\rho_b, b, a, x) = \sum_{n \leq x} \rho_b(n)\eta_a(n).$$

LEMMA 3.8. *When ρ_b is a nontrivial multiplicative character, $J(\rho_b, b, a, x) \ll x \log x$.*

PROOF. Let α be an integer with $1 \leq \alpha \leq b$ and $(\alpha, b) = 1$. Define

$$I(\alpha, b, a, x) := \sum_{\substack{n \leq x \\ n \equiv \alpha \pmod b}} \eta_a(n) = \sum_{\substack{n \leq x \\ n \equiv \alpha \pmod b}} n\lambda_a(n) = \sum_{\substack{n \leq x \\ n \equiv \alpha \pmod b}} n \sum_{d \mid n} \lambda_a^{-1}(d).$$

Using the notation and result of Lemma 3.7, we can continue the above equation as

$$\begin{aligned}
\sum_{\substack{n \leq x \\ n \equiv \alpha \pmod b}} \Big(\sum_{md = n} m f_{\lambda_a}(d) \Big) &= \sum_{d \leq x,\ (d,b)=1} f_{\lambda_a}(d) \sum_{\substack{md \leq x \\ md \equiv \alpha \pmod b}} m \\
&= \sum_{d \leq x,\ (d,b)=1} f_{\lambda_a}(d) \Big(\Big(\sum_{\substack{md \leq x \\ md \equiv 0 \pmod b}} m \Big) + O(x/d) \Big) \\
&= \Big(\sum_{d \leq x,\ (d,b)=1} f_{\lambda_a}(d) \sum_{\substack{m \leq x/d \\ m \equiv 0 \pmod b}} m \Big) + O(x \log x).
\end{aligned} \tag{3.5}$$

With the equation $J(\rho_b, b, a, x) = \sum_{\alpha=1,\ (\alpha,b)=1}^{b} \rho_b(\alpha) \cdot I(\alpha, b, a, x)$ and the vanishing of the character sum $\sum_{\alpha=1,\ (\alpha,b)=1}^{b} \rho_b(\alpha) = 0$, we have (with (3.5))

$$\begin{aligned}
J(\rho_b, b, a, x) &= \Big(\sum_{\alpha=1,\ (\alpha,b)=1}^{b} \rho_b(\alpha) \Big) \Big(\Big(\sum_{d \leq x,\ (d,b)=1} f_{\lambda_a}(d) \sum_{\substack{m \leq x/d \\ m \equiv 0 \pmod b}} m \Big) + O(x \log x) \Big) \\
&= O\Big(\sum_{\alpha=1,\ (\alpha,b)=1}^{b} |\rho_b(\alpha)| x \log x \Big) = O(bx \log x) = O(x \log x).
\end{aligned}$$

Here, we remark that the implicit constants are allowed to depend on b. □

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Define the function $G(\rho_b, b, a, s)$ of the complex variable s by

$$G(\rho_b, b, a, s) = \sum_{n=1}^{\infty} \frac{\rho_b(n)\lambda_a(n)}{n^s}.$$

COROLLARY 3.9. *For a nontrivial character ρ_b, $G(\rho_b, b, a, s)$ is analytic on* $\mathrm{Re}(s) > 0$.

PROOF. Let $\delta > 0$ be fixed. By Lemma 3.8 and summation by parts,

$$\sum_{n\le x} \frac{\rho_b(n)\lambda_a(n)}{n^{\delta}} = \sum_{n\le x} \rho_b(n)\eta_a(n)n^{-1-\delta} \ll 1.$$

In particular, by [1, Theorem 11.8], the Dirichlet series $G(\rho_b, b, a, s)$ converges for $\mathrm{Re}(s) > 0$ and, by [1, Theorem 11.12], it is analytic in that domain. □

We need to show that $G(\rho_b, b, a, 1)$ does not vanish whenever ρ_b is a nontrivial character modulo b. With the help of Corollary 3.6, we see that for $s > 1$, $G(\rho_b, b, a, s)$ has the Euler product

$$G(\rho_b, b, a, s) = \prod_{p \text{ an odd prime}} \left(1 + \sum_{d=1}^{\infty} \frac{\rho_b(p^d)\lambda_a(p^d)}{p^{ds}}\right). \tag{3.6}$$

Put

$$G_p(\rho_b, b, a, s) := 1 + \sum_{d=1}^{\infty} \frac{\rho_b(p^d)\lambda_a(p^d)}{p^{ds}}.$$

LEMMA 3.10. *When p is an odd prime and ρ_b is a multiplicative character modulo b, then $G_p(\rho_b, b, a, 1) \neq 0$. Furthermore, when $s \ge 1$ and ρ_b is a real character, then $G_p(\rho_b, b, a, s) > 0$.*

PROOF. Using Lemmas 3.3, 3.4 and 3.5, we bound

$$\begin{aligned} |G_p(\rho_b, b, a, 1)| &> 1 - \sum_{d=1}^{\infty} \frac{1 + d(1 - 1/p)}{p^d} \\ &= 1 - \frac{1}{p-1} - \left(1 - \frac{1}{p}\right)\frac{p}{(p-1)^2} = 1 - \frac{2}{p-1}, \end{aligned}$$

so that $|G_p(\rho_b, b, a, 1)| > 0$ for all odd primes. The proof for positivity of $G_p(\rho_b, b, a, s)$ (for real characters) on the region $s \ge 1$ follows similarly. □

Put

$$G_p^*(\rho_b, b, a, s) := \sum_{d=2}^{\infty} \frac{\rho_b(p^d)\lambda_a(p^d)}{p^{ds}}.$$

Using Lemmas 3.3, 3.4 and 3.5, for $s > 1$,

$$|G_p^*(\rho_b, b, a, s)| < \sum_{d=2}^{\infty} \frac{d(1 - 1/p)}{p^{ds}} \le \frac{8}{p^2}. \tag{3.7}$$

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Let χ_0 be the trivial character modulo b. Using Lemmas 3.3, 3.4 and 3.5, we see that $G(\chi_0, b, a, s)$ is analytic on $\mathrm{Re}(s) > 1$.

LEMMA 3.11. *If $s > 1$, then $0 < G(\chi_0, b, a, s) \le K_a \cdot \zeta(s)$ for some constant $K_a > 0$.*

PROOF. From Corollary 3.6, $0 \le \lambda_a(n) \le K_a$. We now write

$$G(\chi_0, b, a, s) = \sum_{n=1}^{\infty} \frac{\chi_0(n)\lambda_a(n)}{n^s} \le K_a \cdot \zeta(s).$$

Clearly, $G(\chi_0, b, a, s)$ is positive whenever $s > 1$. □

Put

$$H(b, a, s) := \prod_{\rho_b \pmod b} G(\rho_b, b, a, s),$$

where the product is taken over all characters modulo b. Note that $H(b, a, s)$ is analytic for $\mathrm{Re}(s) > 1$.

LEMMA 3.12. *We have $\lim_{s\to 1^+} |H(b, a, s)| = \infty$.*

PROOF. Let

$$H_p(b, a, s) := \prod_{\rho_b \pmod b} G_p(\rho_b, b, a, s).$$

When $s > 0$ is real, then $H_p(b, a, s)$ is real-valued. We expand $H_p(b, a, s)$ as

$$\begin{aligned} H_p(b, a, s) &= \prod_{\rho_b \pmod b} \left(1 + \frac{\rho_b(p)\lambda_a(p)}{p^s} + G_p^*(\rho_b, b, a, s)\right) \\ &= 1 + \frac{\theta_b(p)\lambda_a(p)}{p^s} + H_p^*(b, a, s), \end{aligned} \tag{3.8}$$

where $\theta_b(p) = b$ if $p \equiv 1 \pmod b$ and 0 otherwise. By (3.7), $H_p^*(b, a, s)$ is real-valued for $s > 1$ and satisfies

$$H_p^*(b, a, s) \le \frac{Q_b}{p^2} \tag{3.9}$$

for some constant $Q_b > 0$. Using Lemma 3.10, we see that $H_p(b, a, 1) \ne 0$. Using Corollary 3.6, we can find a positive number $T_b \ge 3$ such that

$$\left|\frac{\theta_b(p)\lambda_a(p)}{p^s} + \frac{Q_b}{p^2}\right| \le \frac{1}{2},$$

whenever $p > T_b$. We will make use of the following elementary inequality:

$$\log(1 + x) \ge x - 2x^2 \quad \text{for } |x| \le \tfrac{1}{2}. \tag{3.10}$$

We now write

$$H(b, a, s) = \prod_{p \le T_b} H_p(b, a, s) \prod_{p > T_b} H_p(b, a, s), \tag{3.11}$$

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where, by Lemma 3.10,

$$\prod_{p\le T_b} H_p(b,a,1)\neq 0. \tag{3.12}$$

For $s>1$, by (3.8), (3.9), (3.10), Lemma 3.5 and Corollary 3.6,

$$\begin{aligned}\log\Big(\prod_{p>T_b} H_p(b,a,s)\Big) &= \sum_{p>T_b}\log\Big(1+\frac{\theta_b(p)\lambda_a(p)}{p^s}+H_p^*(b,a,s)\Big)\\ &\ge \sum_{p>T_b}\frac{\theta_b(p)\lambda_a(p)}{p^s}+H_p^*(b,a,s)-2\Big(\frac{\theta_b(p)\lambda_a(p)}{p^s}+H_p^*(b,a,s)\Big)^2\\ &\ge \Big(\sum_{\substack{p>T_b\\ p\equiv 1 \pmod b\\ p\nmid b}}\frac{b(1-p^{-1})}{p^s}\Big)-\sum_{p>T_b}\Big(\frac{Q_b}{p^2}+2\Big(\frac{bK_a}{p^s}+\frac{Q_b}{p^2}\Big)^2\Big).\end{aligned}$$

Since $\sum_{p\equiv 1 \pmod b} 1/p=\infty$ and $\sum_{p\equiv 1 \pmod b} 1/p^2<\infty$,

$$\lim_{s\to 1^+}\prod_{p>T_b} H_p(b,a,s)=\infty. \tag{3.13}$$

Equations (3.11), (3.12) and (3.13) prove the result. □

LEMMA 3.13. *We have* $G(\rho_b,b,a,1)\neq 0$.

PROOF. It is clear that $G(\chi_0,b,a,1)=\infty$. Now, assume that $\rho_b\neq\chi_0$. Suppose for the sake of contradiction that $G(\rho_b,b,a,1)=0$. Since $\zeta(s)$ has a simple pole at $s=1$, Lemmas 3.11 and 3.9 imply that $H(b,a,s)$ is bounded near $s=1$. However, this contradicts Lemma 3.12. □

Recall

$$\beta(\psi,a)=\sum_{d=1}^{\infty}\psi(d)\frac{\eta_a(d)}{d^2}=\sum_{d=1}^{\infty}\psi(d)\frac{\lambda_a(d)}{d}=G(\psi,b,a,1), \tag{3.14}$$

so that by Lemma 3.13, it is necessary that

$$\beta(\psi,a)\neq 0. \tag{3.15}$$

From (3.6), on the ray $s>1$,

$$G(\psi,b,a,s)=\prod_{p \text{ an odd prime}}\Big(1+\sum_{d=1}^{\infty}\frac{\psi(p^d)\lambda_a(p^d)}{p^{ds}}\Big)=\prod_{p \text{ an odd prime}} G_p(\psi,b,a,s)$$

and, from Lemma 3.10, each entry in this infinite product is positive. Hence, $G(\psi,b,a,s)\ge 0$ when $s>1$. By Corollary 3.9, $G(\psi,b,a,1)\ge 0$ and, by (3.14) and (3.15), we have $\beta(\psi,a)>0$.

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We now use these results in (3.2) to obtain an asymptotic formula. By Lemma 3.8,

$$J(x) = \sum_{j=1,\,\psi(j-a)=1}^{b} \frac{\pi\cdot\eta_j(b)}{2b^2} \sum_{1\le d<\sqrt{x}} \frac{\psi(d)\eta_a(d)}{d^2}(x-d^2) + O(x^{5/6}\cdot\log^{19}x)$$
$$= \sum_{j=1,\,\psi(j-a)=1}^{b} \frac{\pi\cdot\eta_j(b)}{2b^2} \sum_{1\le d<\sqrt{x}} \frac{\psi(d)\eta_a(d)}{d^2}(x) + O(x^{5/6}\cdot\log^{19}x).$$

From Lemma 3.8 and summation by parts, $\sum_{d\ge\sqrt{x}} \psi(d)\eta_a(d)/d^2 \ll x^{-1/2+\varepsilon}$. Thus,

$$J(x) = \left(\beta(\psi,a) \sum_{j=1,\,\psi(j-a)=1}^{b} \frac{\pi\eta_j(b)}{2b^2}\right)x + O(x^{5/6}\cdot\log^{19}x),$$

which finishes the proof.

4. Proof of Theorem 1.7

Suppose that $a = n^2 - 3m^2$ for some integers n and m. Under this assumption, for each integer s, we have $s^2 + 3m^2 \in \triangle$ and $s^2 + 3m^2 + a = s^2 + n^2 \in \square_2$. Hence, Theorem 1.7 holds in this case.

Moving to the more nontrivial situation, we choose integers $l_1, l_2 \in \{0,1\}$ so that $l_1^2 - 3l_2^2 - a \equiv 1 \pmod 2$, and require integers v and d to satisfy $v \equiv l_1 \pmod 2$ and $d \equiv l_2 \pmod 2$. If (c,d,u,v) is an integer solution to

$$c^2 + 3d^2 + a = u^2 + v^2, \tag{4.1}$$

then $(c+u)(c-u) = v^2 - 3d^2 - a$, so that there exists a divisor $g \mid (v^2 - 3d^2 - a)$ and a simultaneous system of equations satisfying

$$c + u = g, \quad c - u = \frac{v^2 - 3d^2 - a}{g}.$$

Thus, the solutions (c,d,u,v) to (4.1) come from the set

$$\left\{\left(\frac{1}{2}\left(g + \frac{v^2-3d^2-a}{g}\right), d, \frac{1}{2}\left(g - \frac{v^2-3d^2-a}{g}\right), v\right) : g,v,d \in \mathbb{Z},\ g \ne 0\right\}.$$

At $g = 1$, we obtain a subfamily of solutions to (4.1) given by

$$(c,d,u,v) = \left(\frac{v^2-3d^2-a+1}{2}, d, \frac{-v^2+3d^2+a+1}{2}, v\right).$$

Consequently,

$$\left(\frac{v^2-3d^2-a+1}{2}\right)^2 + 3d^2 \in S(\triangle, \square_2, a). \tag{4.2}$$

Let

$$f(v,d) = \left(\frac{v^2-3d^2-a+1}{2}\right)^2 + 3d^2$$

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be a complex valued function. Let $q(x)$ be the largest real number such that $f(0, q(x)) = x$ (which exists for sufficiently large x). Observe that

$$x^{1/4} \ll q(x) \ll x^{1/4}. \tag{4.3}$$

Define $Q(x)$ to be the smallest integer d, $d \equiv l_2 \pmod 2$, such that $f(0, d) > x$ and $d \geq q(x)$. Note that $q(x) \leq Q(x) \leq q(x) + 2$ for sufficiently large x. Setting $Q^*(x) = Q(x) + 2$, we deduce that

$$x^{1/4} \ll Q^*(x) \ll x^{1/4}. \tag{4.4}$$

Put $E(x) = f(0, Q^*(x)) - x$. Differentiating $r(y) := f(0, y)$ as a function of y gives

$$r'(y) = 9y^3 + (3 + 3a)y.$$

Hence, writing $E(x) = f(0, Q^*(x)) - x$ as $r(Q^*(x)) - r(q(x))$, and applying the mean value theorem (together with (4.3) and $2 \leq Q^*(x) - q(x) \leq 4$) yields

$$x^{3/4} \ll E(x) \ll x^{3/4}, \tag{4.5}$$

so that for sufficiently large x,

$$(3 \cdot Q^*(x)^2 + a - 1)^2 - 4 \cdot E(x) \geq 0. \tag{4.6}$$

For a given $x \geq 1$, let

$$\begin{aligned} g_x(w) &:= \left(w + \frac{-3 \cdot Q^*(x)^2 - a + 1}{2}\right)^2 + 3 \cdot Q^*(x)^2 \\ &= w^2 - (-3 \cdot Q^*(x)^2 - a + 1)w + f(0, Q^*(x)), \end{aligned}$$

so that $g_x(v^2/2) = f(v, Q^*(x))$. Now, solve the equation $g_x(w) = x$ for $w \in \mathbb{C}$ giving the solutions

$$w = \frac{3 \cdot Q^*(x)^2 - a + 1 \pm \sqrt{(3 \cdot Q^*(x)^2 - a + 1)^2 - 4(f(0, Q^*(x)) - x)}}{2}.$$

For x sufficiently large so as to satisfy (4.6) and $3 \cdot Q^*(x)^2 - a + 1 \geq 1$ (the second inequality holds for large x because of (4.4)), let

$$w_*(x) = \frac{3 \cdot Q^*(x)^2 - a + 1 - \sqrt{(3 \cdot Q^*(x)^2 - a + 1)^2 - 4 \cdot E(x)}}{2}.$$

Then,

$$g_x(w_*(x)) = x \quad \text{and} \quad f(\sqrt{2w_*(x)}, Q^*(x)) = x.$$

We now write

$$w_*(x) = \frac{1}{2} \cdot \frac{4 \cdot E(x)}{3 \cdot Q^*(x)^2 - a + 1 + \sqrt{(3 \cdot Q^*(x)^2 - a + 1)^2 - 4 \cdot E(x)}}.$$

By (4.4) and (4.5), we have $x^{1/4} \ll w_*(x) \ll x^{1/4}$ and

$$x^{1/8} \ll \sqrt{2w_*(x)} \ll x^{1/8}. \tag{4.7}$$

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For x large so that $\sqrt{2w_*(x)} \geq 3$, let $v_*(x)$ be the largest number less than $\sqrt{2w_*(x)}$ that satisfies $v_*(x) \equiv l_1 \bmod 2$. From (4.7),

$$x^{1/8} \ll v_*(x) \ll x^{1/8}. \tag{4.8}$$

By the element-set relation (4.2),

$$\left(\frac{v_*(x)^2 - 3 \cdot Q^*(x)^2 - a + 1}{2}\right)^2 + 3 \cdot Q^*(x)^2 \in S(\triangle, \square_2, a). \tag{4.9}$$

It remains to show that the quantity mentioned above is sufficiently close to x. Let $h_x(y) := f(y, Q^*(x))$ be a function in the variable $y \in \mathbb{R}$.

LEMMA 4.1. *For sufficiently large x, $h_x(\cdot)$ is decreasing on the interval $[0, Q^*(x))$.*

PROOF. Differentiating $h_x(y)$ in the variable y,

$$h_x'(y) = 4y \cdot \left(\frac{y^2 - 3 \cdot Q^*(x)^2 - a + 1}{2}\right), \tag{4.10}$$

which is less than 0 when $y \in [0, Q^*(x))$, provided x is sufficiently large. □

For x sufficiently large,

$$\left(\left(\frac{v_*(x)^2 - 3 \cdot Q^*(x)^2 - a + 1}{2}\right)^2 + 3 \cdot Q^*(x)^2\right) - x = h_x(v_*(x)) - h_x(\sqrt{2w_*(x)}) > 0. \tag{4.11}$$

Furthermore, by the mean value theorem,

$$\begin{aligned}\left|\left(\left(\frac{v_*(x)^2 - 3 \cdot Q^*(x)^2 - a + 1}{2}\right)^2 + 3 \cdot Q^*(x)^2\right) - x\right| &= |h_x(v_*(x)) - h_x(\sqrt{2w_*(x)})| \\ &= |h_x'(z(x))|(\sqrt{2w_*(x)} - v_*(x)),\end{aligned} \tag{4.12}$$

where $z(x)$ is a quantity in the interval $[v_*(x), \sqrt{2w_*(x)}]$. With (4.12), (4.10), (4.4), (4.8) and the inequality $|v_*(x) - \sqrt{2w_*(x)}| \leq 2$, we obtain

$$\left|\left(\left(\frac{v_*(x)^2 - 3 \cdot Q^*(x)^2 - a + 1}{2}\right)^2 + 3 \cdot Q^*(x)^2\right) - x\right| \ll x^{5/8}. \tag{4.13}$$

The theorem follows from (4.13), (4.11) and the element-set relation (4.9).

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