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Immersions with non-zero normal vector fields

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Introduction

Let M be a smooth n-manifold, X be a smooth $(2n-1)$-manifold, and $g: M \to X$ be a map. It was proved in [**6**] that g is always homotopic to an immersion. The set of homotopy classes of monomorphisms from TM into g^*TX, which is denoted by S_g, may be enumerated either by the method of I. M. James and E. Thomas or by the singularity method of U. Koschorke (see [**1**] and references therein). When the natural action of $\pi_1(X^M, g)$ on S_g is trivial, for example, if X is euclidean, the set S_g is in one-to-one correspondence with the set of regular homotopy classes of immersions homotopic to g (see e.g. [**4**]).

Notice that if X is of the form $Y \times \mathbb{R}$, an immersion $f: M \to X$ is regularly homotopic to an immersion of M into Y if and only if f admits a non-zero normal vector field (abbreviated to nvf). It is thus of interest to study immersions homotopic to g, or weakly, monomorphism classes in S_g, that admit nvf's. For g with an orientable stable normal bundle, the existence and enumeration problems of monomorphism classes in S_g that admit nvf's were discussed in [**5**] by using classical obstruction theory. Here we shall adopt the singularity method of Koschorke [**2**] which enables us to handle the non-orientable situations with equal ease. Our approach greatly simplifies the argument in [**5**].

Using his singularity method, Koschorke [**3**] has in fact solved the problem of whether or not the map g is homotopic to an immersion with an nvf for $n > 4$. In this paper we extend results of [**3**, **5**] and then completely enumerate the monomorphism classes in S_g that admit nvf's for the case where M is closed and $n > 4$.

Immersions with non-zero normal vector fields

Let $w_k(g)$ be the kth reduced Stiefel–Whitney class of the stable normal bundle $\tilde{\nu}_g$ of a map $g: M \to X$, where M is a smooth n-manifold and X is a smooth $(2n-1)$-manifold, and let $\mathbb{Z}_g$ be the integer coefficient system twisted by $w_1(g)$. We denote by $\tilde{W}_g$, which lies in $H^{n-1}(M, \mathbb{Z}_g)$ or $H^{n-1}(M, \mathbb{Z}_2)$ for n even or odd respectively, the stable normal Euler class of g, and for $f \in S_g$ denote by $W_f \in H^{n-1}(M, \mathbb{Z}_g)$ the normal Euler class of f. If $f \in S_g$ admits an nvf then clearly $W_f = 0$. Now let

$$S'_g = \{f \in S_g \mid W_f = 0\}.$$

Then it follows by lemma 2 of [**5**] that S'_g is non-empty if and only if $\tilde{W}_g = 0$.

Definition. We say that C^m_g holds if $m \equiv 0\,(4)$ and $w_1(M) = 0$; or if $m \equiv 2\,(4)$ and $w_1(g) = 0$; or if $m \equiv 3\,(4)$ and $w_2(g) = \bar{w}_2(M)$. If M is closed connected we let $K_g \subseteq \mathbb{Z}_2$ be defined by

$$K_g = (w_2(g) + \bar{w}_2(M)) \cup \rho H^{n-2}(M, \mathbb{Z}_g)$$

where ρ is the modulo two reduction.

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We now come to the main result of this paper.

THEOREM 1. *Let M be a closed connected smooth n-manifold, X be a smooth $(2n-1)$-manifold, $n > 4$, and $g: M \to X$ be a map with $\tilde{W}_g = 0$.*

(1) If $K_g \neq 0$ then all of the classes in S'_g admit nvf's.

(2) If $K_g = 0$ and if C_g^n does not hold, then half of the classes in S'_g admit nvf's.

(3) If $K_g = 0$ and if C_g^n holds, then either all or none of the classes in S'_g admit nvf's depending on whether or not a well-defined $O_g \in \mathbb{Z}_2$ vanishes.

Our theorem extends the following result of Koschorke [3].

PROPOSITION. *Let M be a closed connected smooth n-manifold, X be a smooth $(2n-1)$-manifold, $n > 4$, and $g: M \to X$ be a map. If $K_g \neq 0$ or if C_g^n does not hold, then g is homotopic to an immersion with an nvf if and only if $\tilde{W}_g = 0$.*

To deduce the proposition from the original result of Koschorke (see theorem B(i) of [3]), one may notice that

$$w_1(g)\, w_1(M) \cup \rho H^{n-2}(M, \mathbb{Z}_g) = 0. \tag{I}$$

This is an easy consequence of the Wu formula and the formula of R. Greenblatt and H. Samelson on the twisted Bockstein coboundary (see e.g. [1]).

Our next result is the following theorem which, together with Theorem 1 above, completely enumerates the classes in S_g that admit nvf's for the case where M is closed and $n > 4$.

THEOREM 2. *Let M be a closed connected smooth n-manifold, X be a smooth $(2n-1)$-manifold, $n \geqslant 4$, and $g: M \to X$ be a map with $\tilde{W}_g = 0$. Let G be $H^{n-1}(M, \mathbb{Z}_2)$ or $H^{n-1}(M, \mathbb{Z}_g) \oplus \mathbb{Z}_2$ for n even or odd respectively, and let $G' = \{x \in G \mid 2x = 0\}$.*

(1) If $K_g \neq 0$ or if C_g^{n+1} does not hold, then $(S_g, S'_g) \leftrightarrow (G, G')$.

(2) If $K_g = 0$ and if C_g^{n+1} holds, then $(S_g, S'_g) \leftrightarrow (G \oplus \mathbb{Z}_2, G' \oplus \mathbb{Z}_2)$.

Now we consider the special case where X is euclidean. Let $I'(M)$ be the set of regular homotopy classes of immersions of M into $\mathbb{R}^{2n-1}$ that admit nvf's. $I'(M)$ is non-empty if and only if M can be immersed into $\mathbb{R}^{2n-2}$ and the latter problem has been completely solved by Mahowald [7] and Olk [8] for $n > 4$. Now let p and q be the dimensions of the $\mathbb{Z}_2$-vector spaces $H^{n-1}(M, \mathbb{Z}_2)$ and $\{x \in H^{n-1}(M, \mathbb{Z}_M) \mid 2x = 0\}$ respectively where $\mathbb{Z}_M$ is the integer coefficient system twisted by $w_1(M)$. Then Theorems 1 and 2 have the following consequence.

COROLLARY. *Let M be a closed connected smooth n-manifold, $n > 4$, and let $I'(M)$ be non-empty. Then the cardinality of $I'(M)$ is 2^r. Here according as $n \equiv 0, 1, 2,$ or $3\,(4)$ respectively, r takes the values*

(1) $w_1(M) = 0 : p,\ q+1,\ p+1,$ or $q+2$,

(2) $w_1(M) \neq 0 : p-1, q, p,$ or $q+1$.

Notes. The result of [5] is incorrect when $n \equiv 0\,(4)$, $g^* w_1(X) = w_1(M) \neq 0$, and $g^* \bar{w}_2(X) \cup \rho H^{n-2}(M, \mathbb{Z}) = 0$ because in this case, according to Theorem 1 above, not all, but half of the classes in S'_g admit nvf's.

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If $n = 2$, 3 or if M is open, then every class in S'_g admits an nvf (see [5]). So the remaining case is $n = 4$. This will be treated elsewhere.

Proofs of Theorems 1 *and* 2

Throughout this section let M be a closed connected smooth n-manifold, X be a smooth $(2n-1)$-manifold, and $g: M \to X$ be a map. For brevity we shall always omit obvious pull-backs of vector bundles and cohomology classes. We refer the reader to [2] for definitions and properties of normal bordism groups and singularity obstructions (see also [3]).

We start with a brief description of the proof of the Proposition which is essentially contained in [3]. Let $\dim \tilde{\nu}_g = n-1+l, l \geqslant 0$, and let

$$\tilde{\omega}(g) \in \Omega_1(M_\infty, \tilde{\phi}), \quad \tilde{\phi} = \lambda \otimes g^*TX - \lambda - TM - \lambda \otimes TM,$$

be the singularity obstruction for finding a monomorphism $\mathbb{R}^{l+1} \to \tilde{\nu}_g$, where $M_\infty = M \times \mathbb{P}_\infty$ and λ is the canonical line bundle over infinite-dimensional real projective space $\mathbb{P}_\infty$. If $n > 4$ then $\tilde{\omega}(g) = 0$ if and only if there is a monomorphism $\mathbb{R}^{l+1} \to \tilde{\nu}_g$, or equivalently, g is homotopic to an immersion with an nvf (see [2, 3]).

Denote by μ the natural Hurewicz homomorphism which takes a normal bordism class into a homology class. Then it follows from the discussion in section 5 of [2] that $\tilde{W}_g = 0$ if and only if $\mu(\tilde{\omega}(g)) = 0$ (see also [3]). By theorem 9·3 of [2] we have the following exact sequence:

$$H_2(M_\infty, \mathbb{Z}_{\tilde{\phi}}) \xrightarrow{\tilde{\sigma}} \mathbb{Z}_2 \xrightarrow{\tilde{\delta}} \Omega_1(M_\infty, \tilde{\phi}) \xrightarrow{\mu} H_1(M_\infty, \mathbb{Z}_{\tilde{\phi}}) \longrightarrow 0 \tag{II}$$

where $\mathbb{Z}_{\tilde{\phi}}$ is the integer coefficient system twisted by $w_1(\tilde{\phi})$ and $\tilde{\sigma}$ is defined by $\tilde{\sigma}(x) = \omega_2(\tilde{\phi}) \cap \rho(x)$. It follows that μ in (II) fails to be injective exactly when $\tilde{\sigma} = 0$. Now the Proposition follows immediately from the following result which, by virtue of (I), is essentially obtained in [3].

LEMMA 1. *$\tilde{\sigma} = 0$ if and only if $K_g = 0$ and C^n_g holds.*

We now extend the argument above to prove Theorem 1. Given a monomorphism $f \in S_g$, we denote by ν_f its normal bundle. Let

$$\omega(f) \in \Omega_1(M, \phi), \quad \phi = g^*TX - 2TM,$$

be the singularity obstruction for finding a non-zero cross-section of ν_f. Then it follows from [2] that $\omega(f) = 0$ if and only if f admits an nvf provided that $n > 4$. Now consider the following exact Koschorke sequence:

$$H_2(M, \mathbb{Z}_\phi) \xrightarrow{\sigma} \mathbb{Z}_2 \xrightarrow{\delta} \Omega_1(M, \phi) \xrightarrow{\mu} H_1(M, \mathbb{Z}_\phi) \longrightarrow 0 \tag{III}$$

where $\sigma(x) = w_2(\phi) \cap \rho(x)$. Since $w_2(\phi) = w_2(g) + \bar{w}_2(M) + w_1(g)\, w_1(M)$, it follows from (I) that $\operatorname{Im} \sigma = K_g$. By definition it is not difficult to see that $\mu(\omega(f))$ is the Poincaré dual of W_f (see [2, 3]). So $f \in S'_g$ if and only if $\mu(\omega(f)) = 0$. Now suppose that $K_g \neq 0$. Then μ in (III) is isomorphic and so all of the classes in S'_g admit nvf's. This proves Theorem 1(1).

In the case of $K_g = 0$ we can define for each $f \in S'_g$ an invariant $O_f \in \mathbb{Z}_2$ by $\delta(O_f) = \omega(f)$. It is clear that $f \in S'_g$ admits an nvf if and only if $O_f = 0$. If we assume

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further that C_g^n holds, then by Lemma 1 we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_2 & \xrightarrow{\delta} & \Omega_1(M,\phi) & \xrightarrow{\mu} & H_1(M,\mathbb{Z}_\phi) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \\ 0 & \longrightarrow & \mathbb{Z}_2 & \xrightarrow{\tilde{\delta}} & \Omega_1(M_\infty,\tilde{\phi}) & \xrightarrow{\mu} & H_1(M_\infty,\mathbb{Z}_{\tilde{\phi}}) & \longrightarrow & 0. \end{array}$$

Here i_* is induced by the natural inclusion $i: M \to M_\infty$. By definition $i_*(\omega(f)) = \tilde{\omega}(g)$ for all $f \in S_g$. It follows that O_f depends only on g and so one may denote it by O_g. In this case, if g is homotopic to an immersion with an nvf then all of the classes in S_g' admit nvf's. This proves Theorem 1(3).

To prove Theorem 1(2) we need some preparations. By [4] one may identify S_g with the set of homotopy classes of monomorphisms from $\mathbb{R}^l$ into $\tilde{\nu}_g$ naturally, provided that $n > 2$. Given $f_0, f \in S_g$, we denote by

$$d_l(f_0, f) \in \Omega_1(M_\infty, \tilde{\phi} + \lambda)$$

the singularity difference of the corresponding monomorphisms from $\mathbb{R}^l$ into $\tilde{\nu}_g$ (see [2]). Now consider the following exact Thom–Gysin sequence of Dax and Salomonsen (see proposition 9·20 of [2] for its definition):

$$\Omega_1(M_\infty, \tilde{\phi} + \lambda) \xrightarrow{d} \Omega_1(M,\phi) \xrightarrow{i_*} \Omega_1(M_\infty, \tilde{\phi}).$$

Then it follows from theorem 19·6 of [2] that

$$\pm d(d_l(f_0, f)) = \omega(f) - \omega(f_0).$$

This immediately leads to the following

LEMMA 2. *Let $f_0, f \in S_g$ and $x = d_l(f_0, f)$.*
(1) *If $n \geqslant 4$ then $\pm \mu d(x)$ is the Poincaré dual of $W_f - W_{f_0}$.*
(2) *If $n > 4$ and f_0 admits an nvf, then f admits an nvf if and only if $d(x) = 0$.*

Now suppose that $K_g = 0$ and C_g^n does not hold. Then we have a commutative diagram of horizontal and vertical exact sequences in which δ is injective:

$$\begin{array}{ccccc} & & \mathbb{Z}_2 & & \\ & & \downarrow{\scriptstyle \delta} & & \\ \Omega_1(M_\infty, \tilde{\phi}+\lambda) & \xrightarrow{d} & \Omega_1(M,\phi) & \xrightarrow{i_*} & \Omega_1(M_\infty,\tilde{\phi}) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \cong} \\ H_1(M_\infty, \mathbb{Z}_{\tilde{\phi}+\lambda}) & \xrightarrow{d'} & H_1(M,\mathbb{Z}_\phi) & \xrightarrow{i_*} & H_1(M_\infty,\mathbb{Z}_{\tilde{\phi}}). \end{array} \qquad \text{(IV)}$$

Here the lower horizontal sequence is the Thom–Gysin sequence of homology groups whose definition can be found in [8]. Fix an $f_0 \in S_g$ and let $n \geqslant 4$. Then it follows from [2] that S_g may be identified with $\Omega_1(M_\infty, \tilde{\phi} + \lambda)$ via the correspondence $f \leftrightarrow d_l(f_0, f)$. Notice that f_0 can be chosen to admit an nvf by the Proposition. It follows from Lemma 2(1) and the diagram (IV) that S_g' then corresponds to $\ker \mu d$, and so there must exist a homomorphism $k: S_g' \to \mathbb{Z}_2$ so that $\delta k(f) = d(d_l(f_0, f))$ for all

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$f \in S'_g$. Now Lemma 2(2) implies that $f \in S'_g$ admits an nvf if and only if $k(f) = 0$. Thus to prove Theorem 1(2) it suffices to show that k is surjective.

Let $1 \in \mathbb{Z}_2$ be the generator. Then $i_* \delta(1) = 0$ and hence there is a monomorphism $f \in S_g$ so that $d(d_1(f_0, f)) = \delta(1)$. It is obvious that $f \in S'_g$ and $k(f) = 1$. This completes the proof of Theorem 1.

Proof of Theorem 2. We consider the following commutative diagram with exact row:

$$\begin{array}{ccccccc} H_2(M_\infty, \mathbb{Z}_{\tilde{\phi}+\lambda}) & \xrightarrow{\sigma'} & \mathbb{Z}_2 & \longrightarrow & \Omega_1(M_\infty, \tilde{\phi}+\lambda) & \xrightarrow{\mu} & H_1(M_\infty, \mathbb{Z}_{\tilde{\phi}+\lambda}) \\ & & & & \downarrow d & & \downarrow d' \\ & & & & \Omega_1(M, \phi) & \xrightarrow{\mu} & H_1(M, \mathbb{Z}_{g^*TX}) \end{array} \qquad \text{(V)}$$

where $\sigma'(x) = w_2(\tilde{\phi}+\lambda) \cap \rho(x)$. Let $x \in H_1(M_\infty, \mathbb{Z}_{\tilde{\phi}+\lambda})$ be defined by a map $h: \mathbb{S}^1 \to M_\infty$ and an isomorphism between $T\mathbb{S}^1$ and the orientation bundle of $h^*(\tilde{\phi}+\lambda)$. Then $d'(x) \in H_1(M, \mathbb{Z}_{g^*TX})$ is given by the natural map $S(h^*\lambda) \to S(\lambda) \simeq M$ between sphere bundles together with an obvious bundle isomorphism (see [**2**, **8**]). Since

$$G \cong H_1(M_\infty, \mathbb{Z}_{\tilde{\phi}+\lambda}) \cong \begin{cases} H_1(M, \mathbb{Z}_2) & \text{for } n \text{ even} \\ H_1(M, \mathbb{Z}_{g^*TX}) \oplus H_1(\mathbb{P}_\infty, \mathbb{Z}) & \text{for } n \text{ odd} \end{cases}$$

by the Künneth formula, it follows that $d' = 0$ for n even and $d'(x, y) = 2x$ for n odd. Now let us fix an $f_0 \in S'_g$ and identify S_g with $\Omega = \Omega_1(M_\infty, \tilde{\phi}+\lambda)$ via the correspondence $f \leftrightarrow d_1(f_0, f)$. Then it is readily seen from the diagram (V) and the above discussion that

$$(S_g, S'_g) \leftrightarrow (\Omega, \operatorname{Ker} d'\mu) \leftrightarrow (\operatorname{Coker} \sigma' \oplus G, \operatorname{Coker} \sigma' \oplus G').$$

Now what remains is to evaluate σ'. This can be achieved by an argument similar to that in [**1**] or [**3**]. The details are omitted here.

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