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SEPARABLY CLOSED VALUED FIELDS: QUANTIFIER ELIMINATION

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Abstract. It is proved in this article that the theory of separably closed nontrivially valued fields of characteristic $p > 0$ and imperfection degree $e > 0$ ($e \leq \infty$) has quantifier elimination in the language $\mathscr{L}_{p,\mathrm{div}} = \{+, -, \times, 0, 1\} \cup \{\lambda_{n,j}(x; y_1, \ldots, y_n)\}_{0 \leq n < \omega, 0 \leq j < p^n} \cup \{|\}$; in particular, when e is finite, the corresponding theory has quantifier elimination in the language $\mathscr{L} = \{+, -, \times, 0, 1\} \cup \{b_1, \ldots, b_e\} \cup \{\lambda_{e,j}(x; b_1, \ldots, b_e)\}_{0 \leq j < p^e} \cup \{|\}$.

§1. Introduction. It is perhaps not amiss to say that, at least among the researchers who have thought about the model theory of separably closed valued fields, it is part of the folklore that the theory admits quantifier elimination in the language consisting of the language of rings, symbols for the p-coordinate functions (namely the so-called λ-functions), and a division predicate for the valuations, or in some variant of this language. However, the proof seems to be more subtle and interesting than one would expect.

Separably closed (valued) fields of finite imperfection degree in general are considered much easier to understand than those of infinite imperfection degree. Here we give a unified proof of the quantifier elimination results for both the finite imperfection degree case and the infinite imperfection degree case, using relative p-coordinate functions introduced in [12]. We remark that, in his Ph.D. thesis [6], the author wrote up a proof of the quantifier elimination result for the finite imperfection degree case, but there were gaps in his proof, and the language was not minimal (requiring an extra constant symbol). The remarks about the infinite imperfection case in said thesis were also not accurate. We have obtained better results in this paper.

The model-theoretic study of separably closed fields was first investigated by Eršov [4]. Later, it was shown that separably closed fields are stable fields by Macintyre, Shelah and Wood [14]. It was even conjectured that they are the only infinite stable fields. The subsequent investigation of model theory of separably closed fields was notably also used in Hrushovski's proof of the Mordell–Lang Conjecture for function fields in positive characteristic (see [7]).

To the best of the author's knowledge, Delon was the first to start the investigation of the model theory of separably closed valued fields, which was done in a more general framework in terms of Kaplansky separably algebraically maximal valued

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fields, see [2]. It in fact follows from results in [2] that $\mathrm{SCVF}_{p,e}$ (see definition below) is complete and that the theory is model-complete in a slightly weaker language (than ours). It is entirely possible that one might be able to prove a quantifier elimination result using her methods and results, however, that is not what we will pursue here (neither is it obvious to the author how to do it). In fact, it does follow from the proof of Theorem 6.1 in a recent preprint of F.V. Kuhlmann and K. Pal, [8], that when the imperfection degree e is finite, our $\mathrm{SCVF}_{p,e}$ below admits quantifier elimination in a larger language, namely our $\mathscr{L}_{p,\mathrm{div}}$ adjoint by a constant symbol for an element which has a nonzero value (which is exactly what was shown in [6]); but our proof here works when the imperfection degree e is infinite and shows that $\mathscr{L}_{p,\mathrm{div}}$ is enough.

Apart from the work mentioned above, recently more investigations related to separably closed valued fields are being carried out. For example, [1] examines separably closed fields in a different context; Martin Hils and Moshe Kamensky are preparing a preprint on results related to elimination of imaginaries for separably closed valued fields, using the so-called geometric sorts. The author also hopes to write a subsequent paper on *immediate expansions* (namely that the theory of separably closed valued fields is an immediate expansion of the theory of separably closed fields). We hope that our work will contribute to the subsequent and on-going model-theoretic investigations on separably closed valued fields.

It is well-known (see [11] and Page 83 of [10]) that the theory of algebraically closed nontrivially valued fields admits quantifier elimination in $\mathscr{L}_{\mathrm{div}}$ described below, so in this article, we only consider separably closed fields that are not algebraically closed (characteristic nonzero and imperfect). Without doubt, in the case where the imperfection degree is finite, a nicer language could be chosen and the proof will be slightly (if not much) easier; see below.

NOTATION 1.1. *The letter p always denotes a (positive) prime integer. The letter e always denotes an element in the set $(\mathbf{N}\backslash\{0\})\cup\{\infty\}$.*

NOTATION 1.2. *If (K,V) is a valued field, the valuation map corresponding to the valuation ring V is denoted by its lower case v, the value group vK, the residue field Kv, and the residue map $\mathrm{res}_v : K \to Kv\cup\{\infty\}$.*

NOTATION 1.3. *For a field K, the n-th Cartesian power of K is denoted by $\Pi^n K$, and the subset of n-th powers of K is denoted by K^n.*

NOTATION 1.4. *The symbol $\mathscr{L}_{\mathrm{r}}$ always denotes the language of rings $\{+,-,\times,0,1\}$. The symbol $\mathscr{L}_{\mathrm{div}}$ denotes the language of valued fields (rings) $\mathscr{L}_{\mathrm{r}}\cup\{|\}$, where '$|$' is the usual (cf. [13]) binary division predicate associated to a valuation; namely if the valuation is v, then $x \mid y$ if and only if $v(x)\le v(y)$.*

Recall that on a field K of characteristic p, there is a notion of p-independence. A subset X of K is said to be *p-independent* if for any proper subset X' of X, $K^p(X')$ is a proper subfield of $K^p(X)$. Given $X\subseteq K$, a *p-monomial* in X is a term of the form $\prod_{x\in X} x^{i(x)}$, where $i : X \to p = \{0,1,\ldots,p-1\}$ is a function with finite support. $X\subseteq K$ is p-independent if and only if all the p-monomials in X are K^p-linearly independent. A *p-basis* of K is a p-independent subset X of K such that $K = K^p(X)$. The *imperfection degree* of K is the cardinality of a p-basis of K. If the imperfection degree e of K is not a finite number, we usually write '$e=\infty$'.

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Regarding the p-independence relation, we refer the reader to Section 2, for a more thorough review of relevant facts.

DEFINITION 1.5. Let $\mathscr{L}_p$ denote the following first-order language (*of relative p-coordinate functions*) (cf. [12])

$$\mathscr{L}_p := \mathscr{L}_{\mathrm{r}} \cup \{\lambda_{n,j}(\cdot)\}_{0\leq n<\omega, 0\leq j<p^n},$$

where each $\lambda_{n,j}(\cdot)$ is an $(n+1)$-ary function symbol, which is usually going to be written as $\lambda_{n,j}(x; y_1, \ldots, y_n)$ if the input is the tuple $(x, y_1, \ldots, y_n)$; we call the entries $y_1, \ldots, y_n$ *basis entries*. Let $\mathscr{L}_{p,\mathrm{div}}$ be $\mathscr{L}_p \cup \mathscr{L}_{\mathrm{div}}$.

DEFINITION 1.6. Let $\mathrm{SCF}_{p,e}$ be the theory of separably closed fields of positive characteristic p and imperfection degree e, consisting of the following collection of statements in $\mathscr{L}_p$:

(1) a model is a field of characteristic p;
(2) a model has imperfection degree e;
(3) a model is separably closed;
(4) the $\lambda_{n,j}$ are the *relative p-coordinate functions*, satisfying the following axioms:
   (a) if $y_1, \ldots, y_n$ are p-independent and there exist elements s_j such that

$$x = \sum_{j\in p^n} s_j^p \prod_{i=1}^{n} y_i^{j(i)},$$

   then $\lambda_{n,j}(x; y_1, \ldots, y_n) = s_j$; otherwise $\lambda_{n,j}(x; y_1, \ldots, y_n) = 0$. Accordingly, $\lambda_{0,0}(x) = x^{1/p}$ if $x \in K^p$, and $\lambda_{0,0}(x) = 0$ if $x \notin K^p$.

DEFINITION 1.7. Let $\mathrm{SCVF}_{p,e}$ be the theory of separably closed nontrivially valued fields of positive characteristic p and imperfection degree e, consisting of the following collection of statements in $\mathscr{L}_{p,\mathrm{div}}$:

(1) all the axioms of $\mathrm{SCF}_{p,e}$;
(2) '$|$' is the division predicate corresponding to a valuation on the underlying field; that is, if the valuation is denoted by v, then '$a|b$' if and only if '$v(a) \leq v(b)$';
(3) the model is nontrivially valued, for instance '$\exists x(x \nmid 1)$'.

The main result of this article is

THEOREM 1.8. $\mathrm{SCVF}_{p,e}$ $(1 < p < \infty, 1 \leq e \leq \infty)$ *has quantifier elimination in* $\mathscr{L}_{p,\mathrm{div}}$.

One can, of course, discuss separably closed valued fields using multi-sorted languages. However, for the clarity of the presentation and owing to the preference of the author, in this article we only focus on the one-sorted case; the multi-sorted results can be done as expected.

NOTATION 1.9. *For the convenience of exposition, we introduced* (*see* [14]) *the* predicates of p-indepndence Q_n; *for each* $n > 0$:

$$Q_n(y_1, \ldots, y_n) \Leftrightarrow y_1, \ldots, y_n \textit{ are p-independent}.$$

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Notice that in fact they are equivalent to quantifier-free $\mathscr{L}_p$-formulas (modulo, say, $\mathrm{SCF}_{p,e}$*),*

$$Q_n(y_1, \ldots, y_n) \Leftrightarrow \lambda_{n,0}(1; y_1, \ldots, y_n) = 1,$$

assuming the 0-th p-coordinate is the p-coordinate corresponding to the coefficient of the p-monomial 1*. We define Q_0 to be the true sentence $\lambda_{0,0}(1) = 1$.*

§2. **Relative p-bases.** Recall that a field extension L over K has a *separating transcendence basis* T, if L is a separable algebraic extension over $K(T)$ and T is a subset of L which is algebraically independent over K.

It is quite well-known that, related to separating transcendence bases, there is a criterion called Mac Lane's Criterion, which says that every finitely generated separable field extension admits a separating transcendence basis; see [9], or below. In the same article [9], he also introduced the notion of relative p-independence, and proved some useful properties of it, related to separating transcendence bases. We give a quick review of some results that are needed in our proof.

Definition 2.1. Suppose L/K is a field extension of characteristic p. A subset X of L is *relatively p-independent* in L/K if $L^p(K, X')$ is a proper subfield of $L^p(K, X)$ whenever X' is a proper subset of X. A *relative p-basis* B for L/K is a relatively p-independent subset of L such that $L = L^p(K, B)$.

Every extension L/K has a relative p-basis, and any two relative p-bases have the same cardinality; every relatively p-independent subset of L for L/K can be enlarged into a relative p-basis for L/K. When K is the prime field $\mathbf{F}_p$, we get the usual notion of *p-independence*.

Fact 2.2 (Theorem 7 of [9]). *Suppose that L/K is a field extension of characteristic p, then the following are equivalent*:

(1) *every set $X \subset K$ p-independent in K is p-independent in L;*
(2) *there is a p-basis B of K which is p-independent in L;*
(3) *$K^p(S) = K \cap L^p(S)$ for every finite set $S \subset K$;*
(4) *$K'(K^p) = K \cap K'(L^p)$ for every subfield $K' \subset K$.*

In fact, any of the above equivalent conditions can be taken to be the definition of L/K being *separable*.

Fact 2.3 (Theorem 11 of [9]). *The elements of any relative p-basis B of a characteristic-p field extension L/K which is separable are algebraically independent over K.*

Fact 2.4 (Theorem 12 of [9]). *If the characteristic-p field extension L/K is separable, and if B and C are disjoint subsets of L with $C \subset K$, then any two of the following statements imply the third*:

(1) *B is a relative p-basis of L/K;*
(2) *C is a p-basis for K;*
(3) *$B \cup C$ is a p-basis for L.*

Fact 2.5 (Theorem 15 of [9]). *If the characteristic-p field extension L/K is separable and finitely generated, then L has a separating transcendence basis over K. Under the same assumption, T of L is a separating transcendence basis for L/K if and only if it is a relative p-basis for L/K.*

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§3. Relative λ-resolvents. What is more on a general separably closed field than on an algebraically closed field is that the former has a nontrivial p-independence relation. In order to understand the $\mathscr{L}_{\mathrm{div}}$-structure of a separably closed valued field, it is necessary to understand the interaction of the valuation and the p-independence relation. On the face of it, this seems to be a hard problem, because the formulas could easily get very complex. But we will see that the complexity of a quantifier-free $\mathscr{L}_{p,\mathrm{div}}$-formula on a separably closed valued field is in fact not too much higher the complexity of a quantifier-free $\mathscr{L}_{\mathrm{div}}$-formula over the same field. Indeed, in the following, we will see that, given a quantifier-free $\mathscr{L}_{p,\mathrm{div}}$-formula Φ satisfied by a tuple $\vec{a}$ in a model $\mathcal{K}$ of $\mathrm{SCVF}_{p,e}$, $\Phi(\vec{a})$ is equivalent to a formula of the form $Q \wedge \phi$ in a tuple of different but related parameters, where Q is a predicate of p-independence and ϕ is a quantifier-free $\mathscr{L}_{\mathrm{div}}$-formula. Furthermore, this formula $Q \wedge \phi$ only depends on the 'p-independence type' of $\vec{a}$. In the case where the imperfection degree of the field K is finite, we could even replace $Q \wedge \phi$ by a quantifier-free $\mathscr{L}_{\mathrm{div}}$-formula φ. We may call a formula of the form $Q \wedge \phi$ a 'resolution' of Φ, since it 'resolves' Φ into something more susceptible to being studied. This motivates our definition of a 'relative λ-resolvent' below.

Definition 3.1. Suppose that $\mathcal{K} \models \mathrm{SCVF}_{p,e}$. Given a tuple $\vec{a} \in \Pi^n K$ $(n \geq 1)$, and a quantifier-free $\mathscr{L}_{p,\mathrm{div}}$-formula $\Phi(\vec{x})$ with $\mathcal{K} \models \Phi(\vec{a})$, a *relative λ-resolvent* of $\vec{a}$ with respect to $\Phi(\vec{x})$ is a tuple $\vec{c} \in \Pi^m K$ for some $m \geq 1$ such that there is some k with $0 \leq k \leq m$ and the following hold

(1) there exist n $\mathscr{L}_{\mathrm{r}}$-terms $t_1(x_1, \ldots, x_m), \ldots, t_n(x_1, \ldots, x_m)$ in m free variables, such that for all $1 \leq i \leq n$, $a_i = t_i(\vec{c})$;
(2) $\mathcal{K} \models Q_k(c_1, \ldots, c_k)$;
(3) suppose that $\mathcal{K}' \models \mathrm{SCVF}_{p,e}$ and $\vec{c'} \in \Pi^m K'$ such that $\vec{c}$ and $\vec{c'}$ are $\mathscr{L}_{\mathrm{div}}$-isomorphic, then $\mathcal{K}' \models Q_k(c'_1, \ldots, c'_k) \to \Phi(t_1(\vec{c'}), \ldots, t_n(\vec{c'}))$.

Theorem 3.2. *Suppose that $\mathcal{K}$ is a model of $SCVF_{p,e}$, $\vec{a}$ a tuple in $\prod^n K$, and $\Phi(\vec{x})$ a quantifier-free $\mathscr{L}_{p,\mathrm{div}}$-formula with $\mathcal{K} \models \Phi(\vec{a})$. Then there exists a relative λ-resolvent of $\vec{a}$ with respect to Φ.*

Proof. First, if all the $\mathscr{L}_p$-terms occurring in Φ are $\mathscr{L}_{\mathrm{r}}$-terms already, then clearly $k = 0$ and $\vec{c} = \vec{a}$ is a relative λ-resolvent of $\vec{a}$.

We may assume that $\Phi(\vec{x})$ is a quantifier-free $\mathscr{L}_{p,\mathrm{div}}$-formula involving symbols of relative p-coordinate function. We prove by induction on the number of relative p-coordinate functions occurring in the expression of Φ. We may assume that $\Phi(\vec{x})$ is written as $\Psi\left(\vec{\lambda}(\vec{t}(\vec{x})), \vec{x}\right)$, where Ψ is a quantifier-free $\mathscr{L}_{p,\mathrm{div}}$-formula having less symbols of relative p-coordinate function than Φ, and $\vec{\lambda}$ is a tuple of symbols of relative p-coordinate function and $\vec{t}(\vec{x})$ is a tuple of $\mathscr{L}_{\mathrm{r}}$-terms in free variables $\vec{x}$.

Introducing a new tuple of free variables $\vec{z}$, we have

$$\mathrm{SCVF}_{p,e} \models \left\{\exists \vec{z}\left[\vec{z} = \vec{\lambda}\left(\vec{t}(\vec{x})\right)\right] \wedge \Psi\left(\vec{z}, \vec{x}\right)\right\} \leftrightarrow \Phi(\vec{x}). \tag{3.1}$$

Let $\{a_{i_1}, \ldots, a_{i_s}\}$ be a maximal p-independent (in $\mathcal{K}$) subset of the set $\{a_i\}_i$.

If $s = 0$, then it means that all the a_i are in K^p; let $\vec{c} = (c_i)_i$ be the tuple $(a_i^{1/p})_i$, then there is a tuple of $\mathscr{L}_{\mathrm{r}}$-terms $\vec{s}(\vec{x})$ such that $\lambda(\vec{t}(\vec{a}))$ is $\vec{s}(\vec{c})$. This means that

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if $\vec{c'}$ is a tuple in $\mathcal{K}_2 \models \mathrm{SCVF}_{p,e}$ such that $\vec{c'}$ and $\vec{c}$ are $\mathscr{L}_{\mathrm{div}}$-isomorphic, then

$$\mathcal{K}_2 \models \Psi\left(\vec{s}(\vec{c'}), \overrightarrow{c_i'^{p}}\right) \to \Phi\left(\overrightarrow{c_i'^{p}}\right).$$

By the induction hypothesis, there is a relative λ-resolvent $\vec{d'}$ of $\vec{c_i}$ with respect to $\Psi(\vec{s}(\vec{y}), \overrightarrow{y^p})$ and an integer k satisfying the three conditions in Definition 3.1. It follows that $\vec{d'}$ is a λ-resolvent of $\vec{a}$ with respect to Φ.

So we may assume that $s > 0$. Let $(b_1, \ldots, b_s)$ be $(a_{i_1}, \ldots, a_{i_s})$, and let $(b_{s+1}, \ldots)$ be the tuple of all the relative p-coordinates of the rest of a_i with respect to $(b_1, \ldots, b_s)$. It is then clear that there is a (canonical) tuple of $\mathscr{L}_{\mathrm{r}}$-terms $(T_i)_i$ such that $\vec{a} = (T_i(\vec{b}))_i$. Without loss of generality, we may assume that the length of $\vec{z}$ is just 1. Then Equation (3.1) becomes

$$\mathrm{SCVF}_{p,e} \models \left\{\exists z \left[z = \lambda_{r,j}\left(t_0(\vec{x}); t_1(\vec{x}), \ldots, t_r(\vec{x})\right)\right] \wedge \Psi(z, \vec{x})\right\} \leftrightarrow \Phi(\vec{x}), \quad (3.2)$$

for some $r \geq 0$, $j \in p^r$.

Let $\chi(\vec{x})$ be the formula

$$Q_s(x_{i_1}, \ldots, x_{i_s}) \wedge (\wedge_{l \notin \{i_1, \ldots, i_s\}} \neg Q_{s+1}(x_l, x_{i_1}, \ldots, x_{i_s}));$$

Let the variables $y_{l,l'}$ be $\lambda_{s,l'}(x_l; x_{i_1}, \ldots, x_{i_s})$ for all $l' \in p^l$ and $l \notin \{i_1, \ldots, i_s\}$, and let $y_1, \ldots, y_s$ be $x_{i_1}, \ldots, x_{i_s}$. Clearly $\vec{x}$ is a tuple of $\mathscr{L}_{\mathrm{r}}$-terms in $y_1, \ldots, y_s$ and $\{y_{l,l'}\}_{l,l'}$; we may assume that $x_i = T_i(y_1, \ldots, y_s, (y_{l,l'})_{l,l'})$.

If $r > s$, then clearly

$$\mathrm{SCVF}_{p,e} \models \chi(\vec{x}) \to z = \lambda_{r,j}\left(t_0(\vec{x}); t_1(\vec{x}), \ldots, t_r(\vec{x})\right) = 0,$$

because $t_i(\vec{x}) \in \mathbf{F}_p(y_{l,l'})^p_{l,l'}(y_1, \ldots, y_s)$ for all $0 \leq i \leq r$.

If $1 \leq r \leq s$, then the formula $z = \lambda_{r,j}(t_0(\vec{x}); t_1(\vec{x}), \ldots, t_r(\vec{x}))$ is equivalent (modulo $\mathrm{SCVF}_{p,e}$) to

$$\left\{Q_r(t_1(\vec{x}), \ldots, t_r(\vec{x})) \wedge \neg Q_{r+1}(t_0(\vec{x}), \ldots, t_r(\vec{x})) \wedge \left[z = \lambda_{r,j}(t_0(\vec{x}); t_1(\vec{x}), \ldots, t_r(\vec{x}))\right]\right\} \bigvee$$
$$\bigvee \left[(\neg Q_r(t_1(\vec{x}), \ldots, t_r(\vec{x})) \vee Q_{r+1}(t_0(\vec{x}), \ldots, t_r(\vec{x}))) \wedge (z = 0)\right];$$

if $r = 0$ (then $j = 0$), then it is equivalent (modulo $\mathrm{SCVF}_{p,e}$) to

$$\left[Q_1(t_0(x)) \wedge (z = 0)\right] \bigvee \left[\neg Q_1(t_0(x)) \wedge (z^p = t_0(x))\right].$$

For each α, δ such that $0 \leq \delta \leq r$ and $0 \leq \alpha \leq r - \delta$, we have $\mathrm{SCVF}_{p,e} \models Q_{\delta+1}(t_\alpha(\vec{x}), \ldots, t_{\alpha+\delta}(\vec{x}))$ if and only if the matrix

$$\left[\lambda_{s,l}\left(\prod_{i=\alpha}^{\alpha+\delta} t_i(\vec{x})^{q(i)}; y_1, \ldots, y_s\right)\right]_{l \in p^s, q \in p^{\delta+1}}$$

has rank $p^{\delta+1}$, which can be expressed using a quantifier-free $\mathscr{L}_{\mathrm{r}}$-formulas in $y_1, \ldots, y_s$ and $\{y_{l,l'}\}_{l,l'}$ saying that there is a nonzero $p^{\delta+1} \times p^{\delta+1}$-minor. Let $A^\delta_\alpha(y_1, \ldots, y_s, (y_{l,l'})_{l,l'})$ be this quantifier-free $\mathscr{L}_{\mathrm{r}}$-formula.

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If $1 \leq r \leq s$, and $\mathrm{SCVF}_{p,e} \models \chi(\vec{x}) \wedge Q_r(t_1(\vec{x}), \ldots, t_r(\vec{x})) \wedge \neg Q_{r+1}(t_0(\vec{x}), \ldots, t_r(\vec{x}))$, then the formula $z = \lambda_{r,j}(t_0(\vec{x}); t_1(\vec{x}), \ldots, t_r(\vec{x}))$ is equivalent to a quantifier-free $\mathscr{L}_{\mathrm{r}}$-formula in $z, y_1, \ldots, y_s$ and $\{y_{l,l'}\}_{l,l'}$ as follows:

$$\begin{aligned} t_0(\vec{x}) &= \left[\prod_{i=1}^{r} t_i(\vec{x})^{q(i)}\right]_{q \in p^r} \cdot \left[\lambda_{r,q}(t_0(\vec{x}); t_1(\vec{x}), \ldots, t_r(\vec{x}))^p\right]_q^{\mathrm{T}} \\ &= \left[\prod_{i=1}^{s} y_i^{u(i)}\right]_{u \in p^s} \cdot \left[\lambda_{s,u}\left(\prod_{i=1}^{r} t_i(\vec{x})^{q(i)}; y_1, \ldots, y_s\right)^p\right]_{u \in p^s, q \in p^r} \\ &\quad \times \left[\lambda_{r,q}(t_0(\vec{x}); t_1(\vec{x}), \ldots, t_r(\vec{x}))^p\right]_q^{\mathrm{T}}, \end{aligned}$$

therefore,

$$\begin{aligned} \left[\lambda_{s,u}\left(t_0(\vec{x}); y_1, \ldots, y_s\right)\right]_{u \in p^s}^{\mathrm{T}} &= \left[\lambda_{s,u}\left(\prod_{i=1}^{r} t_i(\vec{x})^{q(i)}; y_1, \ldots, y_s\right)\right]_{u \in p^s, q \in p^r} \\ &\quad \times \left[\lambda_{r,q}(t_0(\vec{x}); t_1(\vec{x}), \ldots, t_r(\vec{x}))\right]_q^{\mathrm{T}}. \end{aligned}$$

Under our assumption, the coefficient matrix, denoted by C, on the right-hand side has rank p^r. Denote by $C_{\mu_1,\ldots,\mu_{p^r}}$ the $p^r \times p^r$-submatrix of C consisting of the rows of number $\mu_1, \ldots, \mu_{p^r}$, and denote by $\left(C^*_{\mu_1,\ldots,\mu_{p^r}}\right)_j$ the j-th row of the adjugate of $C_{\mu_1,\ldots,\mu_{p^r}}$. Then the formula $z = \lambda_{r,j}(t_0(\vec{x}); t_1(\vec{x}), \ldots, t_r(\vec{x}))$ is equivalent to the quantifier-free $\mathscr{L}_{\mathrm{r}}$-formula $\beta_j(z, y_1, \ldots, y_s, (y_{l,l'})_{l,l'})$ defined as

$$\begin{aligned} \bigvee_{\mu_1,\ldots,\mu_{p^r}} \Big\{ &\det\left(C_{\mu_1,\ldots,\mu_{p^r}}\right) \neq 0 \wedge \left(C^*_{\mu_1,\ldots,\mu_{p^r}}\right)_j \cdot \left[\lambda_{s,u}\left(t_0(\vec{x}), y_1, \ldots, y_s\right)\right]_{u=\mu_1,\ldots,\mu_{p^r}}^{\mathrm{T}} \\ &= z \det\left(C_{\mu_1,\ldots,\mu_{p^r}}\right) \Big\}. \end{aligned}$$

To conclude, if $r > s$,

$$\mathrm{SCVF}_{p,e} \models \chi(\vec{x}) \rightarrow \left[\Psi(0, \vec{x}) \leftrightarrow \Phi(\vec{x})\right];$$

if $r = 0$,

$$\begin{aligned} \mathrm{SCVF}_{p,e} \models \chi(\vec{x}) \rightarrow \Big\{ \Big[&\left[A_0^0(y_1, \ldots, y_s, (y_{l,l'})_{l,l'}) \wedge (z = 0)\right] \\ &\bigvee \left[\neg A_0^0(y_1, \ldots, y_s, (y_{l,l'})_{l,l'}) \wedge (z^p = t_0(\vec{x}))\right] \Big] \wedge \Psi(z, \vec{x}) \rightarrow \Phi(\vec{x}) \Big\}; \end{aligned}$$

if $1 \leq r \leq s$, then letting $\vec{y}$ be $(y_1, \ldots, y_s, (y_{l,l'})_{l,l'})$,

$$\begin{aligned} \mathrm{SCVF}_{p,e} \models \chi(\vec{x}) \rightarrow \Big\{ \Big\{ &\left[A_1^{r-1}(\vec{y}) \wedge \neg A_0^r(\vec{y}) \wedge \beta_j(z, \vec{y})\right] \\ &\bigvee \left[(\neg A_1^{r-1}(\vec{y}) \vee A_0^r(\vec{y})) \wedge (z = 0)\right] \Big\} \bigwedge \Psi(z, \vec{x}) \rightarrow \Phi(\vec{x}) \Big\}. \end{aligned}$$

Let $z_0 \in K$ be the value of z in K. Therefore, as long as $(\vec{b'}, z_0') \in \mathcal{K}_2 \models \mathrm{SCVF}_{p,e}$ with $(\vec{b}, z_0)$ and $(\vec{b'}, z_0')$ being $\mathscr{L}_{\mathrm{div}}$-isomorphic and $\mathcal{K}_2 \models Q_s(y_1, \ldots, y_s)$, then, denoting the corresponding tuple $(T_i(\vec{b'}))_i$ by $\vec{x'}$, we have

$$\mathcal{K}_2 \models \chi(\vec{x'}),$$

and if $\mathcal{K}_2 \models \Psi(z_0', \vec{x'})$ then $\mathcal{K}_2 \models \Phi(\vec{x'})$.

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We can then carry on the induction to find a relative λ-resolvent of the tuple $(\vec{b}, z_0)$ with respect to $\Psi(z, \vec{x})$, by repeating the steps above (making sure that the new maximal p-independence subset now includes $b_1, \ldots, b_s$). ⊣

Remark 3.3. From the proof above, we can also see that a specific kind of relative λ-resolvents exist. Using the notation in Definition 3.1, for any fixed p-basis $B := \{b_1, \ldots, b_n, \ldots\}$ of K, we can find a relative λ-resolvent $\vec{c}$ of $\vec{a}$ with respect to Φ such that $\vec{c}$ further more satisfies the property that $\{c_1, \ldots, c_k\} \subseteq B$ (to see this, replace the 'maximal p-independent subset of the set $\{a_i\}_i$' in the proof by some suitable subset of B). We will use this kind of relative λ-resolvents in the proof of the quantifier elimination result.

§4. **Quantifier elimination.** For a valued field (K, V), K naturally becomes a topological field where the topology is generated by sets of the form $\{x \in K | v(x - a) > \delta\}$ with $a \in k$ and $\delta \in vK$. It is easy to see that any such a set is Zariski-dense in $\mathbb{A}_k$; any Zariski-open set is V-open (i.e., open with respect to the topology induced by V).

The following theorem provides a convenient tool for studying the interaction between the p-independence relation and the valuation on a separably closed valued field.

Theorem 4.1 (cf. [6]). *Suppose that K is a separably closed field of positive characteristic p with a valuation ring V. Then for any natural number n and any element $b \in K^\times$, the set bK^{p^n} is V-dense in K.*

Proof. We may assume that v is not trivial and $n > 0$. Then the value group is an ordered abelian group without upper bound. It is then obvious that bK^{p^n} is V-dense at 0.

Suppose that z is an element of $K^\times$, and $\gamma > 0$ is an element of the value group of K; it is enough to show that there exists some $z_\gamma \in K^\times$ such that $v(z - bz_\gamma^{p^n}) > \gamma$.

Case 1. $v(z) \geq 0$.

We can always find some $d \in K^\times$ such that $v(z/b) \geq v(d^{p^n})$. Pick $a \in K^\times$ such that $v(a) > \gamma$, then consider the polynomial $f(X) = bd^{p^n}X^{p^n} - aX - z$. Notice that the formal derivative of f, $f'(X) = -a \neq 0$. Thus f splits over K. In particular since $\infty \neq v(z/(bd^{p^n})) \geq 0$, there exists some root $r \neq 0$ of f with $v(r) \geq 0$. Then $v(z - b(dr)^{p^n}) = v(ar) > \gamma$ and $z_\gamma = dr$ is what we want.

Case 2. $v(z) < 0$.

One can always pick one $g \in K^\times$ such that $v(g^{p^n}z) \geq 0$. Let $\epsilon = v(g^{p^n}) + \gamma$. Then applying Case 1 (to $g^{p^n}z$ and ϵ), one has a $z' \neq 0$ such that $v(g^{p^n}z - bz'^{p^n}) > \epsilon$. Therefore, letting $z_\gamma = z'g^{-1}$, we get $v(z - bz_\gamma^{p^n}) > \gamma$. ⊣

Corollary 4.2. *The set of p-independent tuples are V-dense. That is, suppose that $\vec{a} = (a_1, \ldots, a_n)$ is a point in $\Pi^n K$ with $\mathcal{K} \models \mathrm{SCVF}_{p,e}$, $1 \leq n \leq e$, and $\epsilon \in vK$, then there is some point $\vec{b} \in \Pi^n K$ such that $\mathcal{K} \models Q_n(\vec{b})$ and $v(a_i - b_i) > \epsilon$.*

Proof. Let $\vec{c} \in \Pi^n K$ be p-independent. Then for any tuple $\vec{g} \in \Pi^n K$, with $g_i \neq 0$ for all $1 \leq i \leq n$, $(c_i g_i^p)_i$ is still p-independent. It then immediately follows from Theorem 4.1 that $\vec{g}$ could be chosen so that $v(a_i - c_i g_i^p) > \epsilon$ for all $1 \leq i \leq n$, and then $(b_i)_i = (c_i g_i^p)_i$ is what we want. ⊣

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REMARK 4.3. By the same reasoning, the set of p-dependent tuples are also V-dense. In fact, the set of tuples of any fixed p-independence type is also V-dense.

It also follows from Theorem 4.1 that K is V-dense in K^{alg}, which means K^{alg}/K is an immediate valued-field extension, provided that K is not trivially valued.

The following lemma and its proof are taken almost verbatim from [13].

LEMMA 4.4 (van den Dries). *Suppose that (L, V) is a valued-field extension of (K, W) (so $v|_K = w$) such that $Kw = Lv$, $a_1, \ldots, a_n \in K$, and $x \in L \backslash K$ such that $v(x - a_i) \in vK$ for all $i = 1, \ldots, n$. Then there exists $a \in K$ such that $v(x - a_i) = v(a - a_i)$ for all $i = 1, \ldots, n$.*

PROOF. Any $a \in K$ with $v(a - x) > v(a - a_i)$ for all i has the desired property. We may assume that $v(x - a_1) \geq v(x - a_i)$ for $i = 2, \ldots, n$. Since $v(x - a_1) \in vK$, we can take $b \in K$ such that $v(x - a_1) = v(b)$. So $v((x - a_1)/b) = 0$ and since $Kw = Lv$, there exists some $c \in K$ such that $(x - a_1)/b = c + \epsilon$, $v(c) = 0$ and $v(\epsilon) > 0$. Then $a = a_1 + bc$ works because $x - a = b\epsilon$ and $v(b\epsilon) > v(x - a_i)$. $\dashv$

For the reader's convenience, we recall some frequently used facts about valued field.

FACT 4.5 (Corollary 2.2.2 of [3]). *Suppose that (K, V) is a valued field. Then there is exactly one extension W of V to the rational function field $K(X)$ such that $w(X) = 0$ and $\text{res}_w(X)$ is transcendental over Kv. Furthermore, for this w, $K(X)w = (Kv)(\text{res}_w(X))$ and $w(K(X)) = vK$.*

FACT 4.6 (Corollary 2.2.3 of [3]). *Suppose that (K, V) is a valued field, vK an ordered subgroup of Γ, with $\gamma \in \Gamma$ having the property that if $n \in \mathbf{Z}$ satisfies $n\gamma \in vK$ then $n = 0$. Then there is exactly one valuation w on the rational function field $K(X)$ extending v, with $w(X) = \gamma$. Furthermore, for this w, $K(X)w = Kv$ and $w(K(X)) = (vK) \oplus \mathbf{Z}\gamma$ with the ordering induced from Γ.*

FACT 4.7 (Theorem 3.2.15 of [3]). *Suppose that L/K is a normal field extension, V is a valuation of K, W_1 and W_2 are valuation rings in L extending V. Then there exists $\sigma \in \text{Aut}(L/K)$ with $\sigma(W_1) = \sigma(W_2)$.*

Recall that the rational rank $\text{rr}(G)$ of an ordered abelian group G is the maximal number of elements in G that are linearly independent over $\mathbf{Z}$.

FACT 4.8 (Theorem 3.4.3 of [3]). *Suppose that L/K is a field extension, v is a valuation on K and w is an extension of v to L. Then*

$$\text{tr.deg}((Lw)/(Kv)) + \text{rr}((wL)/(vK)) \leq \text{tr.deg}(L/K).$$

The following is an easy exercise taken from [5] (which remains to be an easy exercise for the reader).

FACT 4.9 (Exercise 2.4 in [5]). *Let v be a valuation on the rational function field $K(X)$ which is trivial on K. Suppose for all $f \in K[X]$, $v(f) \geq 0$ and there is a $g \in K[X]$ with the smallest degree such that $v(g) > 0$. Then v is isomorphic to the valuation induced by the irreducible polynomial $g(X)$.*

The following extension lemma is central to the proof of quantifier elimination.

LEMMA 4.10 (Cf. [6]). *Suppose that $\mathcal{K}_1 := (K_1, V_1)$ and $\mathcal{K}_2 := (K_2, V_2)$ are two models of $\text{SCVF}_{p,e}$ and $\mathcal{K}_2$ is $|K_1|^+$-saturated. Suppose that $f : F \to \mathcal{K}_2$ is an $\mathscr{L}_{\text{div}}$-embedding, and F is a separably closed subfield of K_1.*

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Then for any $a \in K_1$ which is transcendental over F, there is an $\mathscr{L}_{\mathrm{div}}$-embedding $\tilde{f} : F(a)^{\mathrm{sep}} \to \mathcal{K}_2$ extending f.

PROOF. We may assume that $v_1(a) \geq 0$ (otherwise do the argument for a^{-1}). We remark that F might be trivially valued, but should it not be, the residue field Fv_1 would be algebraically closed and the value group v_1F divisible. v_1F is divisible in any case, be it trivial or not. There are four cases.

CASE 0. $F(a)v_1 \neq Fv_1$ and $v_1(F(a)) \neq v_1F$.

According to Fact 4.8, if $F(a)v_1/Fv_1$ is transcendental, then $v_1(F(a)) = v_1F$. Thus this case is only possible if $F(a)v_1/Fv_1$ is algebraic, in which case, Fv_1 cannot be algebraically closed, thus F is trivially valued.

In this case, there are two polynomials $g(X), h(X) \in F[X]$ such that $h(a) \neq 0$ and $\mathrm{res}_{v_1}(g(a)/h(a))$ is not in Fv_1. In fact we can choose the polynomials such that a is separable over $F(g(a)/h(a))$. In other words, we can find $G(X), H(X) \in F[X]$ with the properties that $H(a) \neq 0$, $\mathrm{res}_{v_1}(G(a)/H(a)) = \mathrm{res}_{v_1}(g(a)/h(a))$ and a is separable over $F(G(a)/H(a))$. To see this, consider two polynomial with undetermined coefficients A, B, C and D (in F):

$$G(X) := g(X) + AX^q + B,$$
$$H(X) := h(X) + CX^q + D,$$

where $q > \max\{\deg g, \deg h\}$ and $q \neq p$. The element a is a root of the polynomial

$$\frac{G(a)}{H(a)}H(X) - G(X) \in F(G(a)/H(a))[X].$$

It is enough to make sure that $G(a)H(X) - H(a)G(X)$ has a as a simple root, it being equivalent to saying that the formal derivative $G(a)H'(X) - H(a)G'(X)$ does not vanish at a, that is

$$G(a)H'(a) - H(a)G'(a) \neq 0. \tag{4.1}$$

One checks that the above inequality (4.1) is a nontrivial equation in variables A, B, C and D. Thus it has a solution in any V-open subset in Π^4F. In particular, there is a solution of Inequality (4.1) such that $\mathrm{res}_{v_1}(g(a)/h(a)) = \mathrm{res}_{v_1}(G(a)/H(a))$.

Thus we may assume that $\mathrm{res}_{v_1}(g(a)/h(a))$ is not in Fv_1 and a is separable over $F(g(a)/h(a))$. Let $t = g(a)/h(a)$. Then there is a smallest n such that $\mathrm{res}_{v_1}(t)^{p^n} = \mathrm{res}_{v_1}(s)$ for some $s \in F$. Since K_2v_2 is algebraically closed, there is some $b \in K_2$ such that $\mathrm{res}_{v_2}(b)^{p^n} = \mathrm{res}_{v_2}(f(s))$ with n being the smallest exponent of p satisfying this property. b can be then also chosen to be transcendental over $f(F)$, by the saturation of $\mathcal{K}_2$.

Then by Fact 4.9, the map $\bar{f} : F(t) \to f(F)(b)$ is a valued field isomorphism, since $t^{p^n} - s$ and $b^{p^n} - f(s)$ are polynomials of the smallest degree such that their valuations are positive, in their respective function fields (the trivial valued field F is Henselian, thus certain types of factorizations in the residue field can be lifted back to the field). Thus, one can extend f to an $\mathscr{L}_{\mathrm{div}}$-embedding $\tilde{f} : F(a)^{\mathrm{sep}} = F(t)^{\mathrm{sep}} \to \mathcal{K}_2$ by Fact 4.7.

CASE 1. $F(a)v_1 \neq Fv_1$ and $v_1(F(a)) = v_1F$.

If F is trivially valued, then $v_1(F(a)) = v_1F = 0$, so $F(a)$ is also trivially valued. For each $g(X) \in F[X]$, one can find some $b' \in f(F)$ such that $v_2(f(g)(b')) = 0$

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(as long as b' is not a root of $f(g)(X)$). Then by the saturation of $\mathcal{K}$, there is some $b \in K_2$ such that for all $g(x) \in f(F)[X]$, $v_2(g(b)) = 0$; b is automatically transcendental over $f(F)$, then one can extend f into a valued-field isomorphism from $F(a)$ to $f(F)(b)$ by sending a to b. Thus there is no problem of extending the map f to $F(a)^{\text{sep}}$, by Fact 4.7. So we may assume that F is nontrivially valued.

As done in Case 0 above, we may assume that a is separable over $F(g(a)/h(a))$, and $\text{res}_{v_1}(g(a)/h(a))$ is transcendental over Fv_1. Since $\mathcal{K}_2$ is $|\mathcal{K}_1|^+$-saturated, there exists some $b \in K_2$ with $\text{res}_{v_2}(b) \in (K_2)v_2$ such that $\text{res}_{v_2}(b)$ is transcendental over $f(F)v_2$. So b is transcendental over $f(F)$ as well. It is also clear that $v_1(g(a)/h(a)) = 0$ and $v_2(b) = 0$. Then by Fact 4.5, f extends to a valued field embedding which is also an $\mathscr{L}_{\text{div}}$-embedding $f_1 : F(g(a)/h(a)) \to \mathcal{K}_2$ sending $g(a)/h(a)$ to b. But by Fact 4.7, f_1 clearly has an extension to the separable closure $\tilde{f} : F(a)^{\text{sep}} = F(g(a)/h(a))^{\text{sep}} \to \mathcal{K}_2$.

Case 2. $F(a)v_1 = Fv_1$ and $v_1(F(a)) \neq v_1F$.

Since $F(a)$ is the quotient field of $F[a]$ and $v_1(F(a)) \neq v_1F$, there is some $g(a) \in F[a]$ such that $v(g(a)) \notin v_1F$. Because F is separably closed, there are $c', c_i \in F$ and natural numbers n_i such that $g(a) = c' \prod_i (a^{p^{n_i}} - c_i)$. This implies that there is some factor $a^{p^{n_i}} - c_i$ with $v_1(a^{p^{n_i}} - c_i) \notin v_1F$; let n be this n_i and c be this c_i. Then there is some $k \in K_2$ such that $v_2(k) \notin v_2f(F)$ and $v_2(k)$ realizes the quantifier-free type of $v_1(a^{p^n} - c)$ over $v_1(F)$ (identified with $v_2(f(F))$) in the language of ordered abelian groups (the theory of divisible ordered abelian groups has quantifier elimination in this language). Furthermore, by the V_2-denseness of $K_2^{p^n}$ in K_2, there is some $b \in K_2$ such that b^{p^n} and $k + f(c)$ are sufficiently close, whence $v_2(b^{p^n} - f(c)) = v_2(k + f(c) - f(c)) = v_2(k)$. Since $\mathcal{K}_2$ is sufficiently saturated, we may also assume that b is transcendental over $f(F)$. Then, by Fact 4.6, it is clear that f can be extended to a valued-field isomorphism $\bar{f} : F(a^{p^n} - c) \to f(F)(b^{p^n} - f(c))$. But then $F(a)$ is a purely transcendental extension of $F(a^{p^n} - c)$ and $f(F)(b)$ is also a purely transcendental extension of $f(F)(b^{p^n} - f(c))$. Thus $\bar{f}$ extends to a valued-field isomorphism from $F(a)$ to $f(F)(b)$, which in turn extends to a valued-field isomorphism $\tilde{f} : F(a)^{\text{sep}} \to f(F)(b)^{\text{sep}} \subseteq K_2$ by Fact 4.7.

Case 3. $F(a)v_1 = Fv_1$ and $v_1(F(a)) = v_1F$.

First we show that we can extend f to an $\mathscr{L}_{\text{div}}$-embedding on $F(a)$.

Because $\mathcal{K}_2$ is sufficiently saturated, it is enough to show that for any $g(X), h(X) \in F[X]$ such that $g(a)|h(a)$ we have a transcendental element $b \in K_2$ over $f(F)$ such that $f(g)(b)|f(h)(b)$. Furthermore, one can factorize g into $d \prod_j (X^{p^{n_j}} - d_j)$ with $d, d_j \in F$ (and accordingly for $h(X)$), since F is separably closed. Therefore, it is enough to show that for any finitely many element $c_1, \ldots, c_n \in F$ and any integers $\alpha_1, \ldots, \alpha_n$, if $v_1(a^{p^{\alpha_i}} - c_i) = v_1(k_i) \neq \infty$, $k_i \in F$ for $1 \leq i \leq n$, then there is some b transcendental over $f(F)$ such that $v_2(b^{p^{\alpha_i}} - f(c_i)) = v_2(f(k_i))$, for $1 \leq i \leq n$.

We may assume that $\alpha_1 \geq \alpha_i$ for all i. By Lemma 4.4 we can find some $w \in F$ such that for all i,

$$\infty \neq v_1(a^{p^{\alpha_1}} - c_i^{p^{\alpha_1 - \alpha_i}}) = v_1(w - c_i^{p^{\alpha_1 - \alpha_i}}). \tag{4.2}$$

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By Theorem 4.1 (or its proof), we can find infinitely many different $t \in F$ such that $v_1(w - t^{p^{\alpha_1}})$ are sufficiently large and hence, for all i

$$v_1(a^{p^{\alpha_1}} - c_i^{p^{\alpha_1-\alpha_i}}) = v_1(t^{p^{\alpha_1}} - c_i^{p^{\alpha_1-\alpha_i}}),$$

namely,

$$\infty \neq v_1(k_i) = v_1(a^{p^{\alpha_i}} - c_i) = v_1(t^{p^{\alpha_i}} - c_i).$$

Thus in K_2, we have, for all $1 \leq i \leq n$,

$$\infty \neq v_2(f(k_i)) = v_2(f(t)^{p^{\alpha_i}} - f(c_i)).$$

This is true for infinitely many different $t \in F$, so by the saturation of $\mathcal{K}_2$, there is a b which is transcendental over $f(F)$ such that for all $1 \leq i \leq n$,

$$\infty \neq v_2(f(k_i)) = v_2(b^{p^{\alpha_i}} - f(c_i)).$$

Therefore, f can be extended to an $\mathcal{L}_{\text{div}}$-embedding on $F(a)$. By Fact 4.7 again, we get an $\mathcal{L}_{\text{div}}$-embedding $\tilde{f} : F(a)^{\text{sep}} \to \mathcal{K}_2$ extending f to include the separable closure of $F(a)$. ⊣

Lemma 4.10 could be improved to preserve the p-independence relation.

LEMMA 4.11. *Suppose that $\mathcal{K}_1 := (K_1, V_1)$ and $\mathcal{K}_2 := (K_2, V_2)$ are two models of* $\text{SCVF}_{p,e}$ *and $\mathcal{K}_2$ is $|K_1|^+$-saturated. Suppose that $f : F \to \mathcal{K}_2$ is an $\mathcal{L}_{\text{div}}$-embedding, and F is a separably closed subfield of K_1. Suppose that for some $n \geq 0$, $a_1, \ldots, a_n \in F$ satisfy that $\mathcal{K}_1 \models Q_n(a_1, \ldots, a_n)$ and $\mathcal{K}_2 \models Q_n(f(a_1), \ldots, f(a_n))$.*

Then for any $a \in K_1$ transcendental over F and $\mathcal{K}_1 \models Q_{n+1}(a_1, \ldots, a_n, a)$, there is an $\mathcal{L}_{\text{div}}$-embedding $\tilde{f} : F(a)^{\text{sep}} \to \mathcal{K}_2$ extending f, such that $\mathcal{K}_2 \models Q_{n+1}(f(a_1), \ldots, f(a_n), f(a))$.

PROOF. Let $p(x/F)$ be the quantifier-free $\mathcal{L}_{\text{div}}$-type of a over F, it is enough to show that $\mathcal{K}_2$ realizes $p(x/f(F))$ and $Q_{n+1}(f(a_1), \ldots, f(a_n), x)$. To achieve this, it is enough to show that for any formula (which is quantifier-free in $\mathcal{L}_{\text{div}}$) $\phi(x, \vec{c})$ in $p(x/F)$, there is some $b \in K_2$ such that $\mathcal{K}_2 \models \phi(b, f(\vec{c}))$ and $\mathcal{K}_2 \models Q_{n+1}(f(a_1), \ldots, f(a_n), b)$.

By Lemma 4.10, there is some $b' \in K_2$, transcendental over $f(F)$, such that $\mathcal{K}_2 \models \phi(b', f(\vec{c}))$. But as long as $v(b - b')$ is large enough, $\mathcal{K}_2 \models \phi(b, f(\vec{c}))$. We may thus use Corollary 4.2 (or its proof) to guarantee that at the same time $\mathcal{K}_2 \models Q_{n+1}(f(a_1), \ldots, f(a_n), b)$. ⊣

THEOREM 4.12. $\text{SCVF}_{p,e}$ *admits quantifier elimination.*

PROOF. Suppose that $\mathcal{K}_1 := (K_1, V_1)$ and $\mathcal{K}_2 := (K_2, V_2)$ are two models of $\text{SCVF}_{p,e}$ and $\mathcal{K}_2$ is $|K_1|^+$-saturated. Suppose that $\mathcal{F} := (F, v_1)$ is an $\mathcal{L}_{p,\text{div}}$-substructure of $\mathcal{K}_1$ and $f : \mathcal{F} \to \mathcal{K}_2$ is an $\mathcal{L}_{p,\text{div}}$-embedding. We need to show that for any $a \in K_1 \backslash F$, we can extend f to an $\mathcal{L}_{p,\text{div}}$-embedding of an $\mathcal{L}_{p,\text{div}}$-substructure containing $F(a)$. We may assume that F is already a field (if $x \in F$ is not a p-th power, then $\lambda_{1,j}(x; x^{p+1})$ is $1/x$, since we have the equality $x = (1/x)^p x^{p+1}$; thus the only elements without inverse are p-th powers in F, whence there is only one way to extend the $\mathcal{L}_{p,\text{div}}$-embedding to the quotient field, which one easily checks to preserve the $\mathcal{L}_{p,\text{div}}$-substructure). Furthermore, we may assume that F is separably closed, since the $\mathcal{L}_p$-structure on F^{sep} is entirely determined by the $\mathcal{L}_{\text{r}}$-structure on F (if y is separably algebraic over F, then $F(y)/F(y^p)$ is separable and purely

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inseparably at the same time, hence they are the same, thus $y = \sum_i a_i (y^p)^i$, $a_i \in F$) and by Fact 4.7. It is enough to make sure that the quantifier-free $\mathscr{L}_{p,\mathrm{div}}$-type of a over F can be realized in $\mathcal{K}_2$ (replacing the corresponding parameters $\vec{s}$ in F by $f(\vec{s})$ in $f(F)$). It is then enough to make sure that for any quantifier-free $\mathscr{L}_{p,\mathrm{div}}$-formula $\Phi(a, \vec{c})$ and $\vec{c} \in K_1$, such that $\mathcal{K}_1 \models \Phi(a, \vec{c})$, there is some $d \in K_2$ such that $\mathcal{K}_2 \models \Phi(d, f(\vec{c}))$.

Since K_1 is separable over F (because $\mathcal{F}$ is an $\mathscr{L}_{p,\mathrm{div}}$-substructure), we can extend a p-basis of F to a p-basis B of K_1. By Remark 3.3, there is a relative λ-resolvent $\vec{b}$ of $(a, \vec{c})$ with respect to Φ such that $\{b_1, \ldots, b_k\} \subseteq B$ (k as in Definition 3.1). We may assume that none of $b_1, \ldots, b_k$ is in F; then it follows that they are also algebraically independent over F, by Fact 2.4 and Fact 2.3. Since $F(\vec{b})$ is separably generated, and $\{b_1, \ldots, b_k\}$ is relatively p-independent in K_1/F, $\{b_1, \ldots, b_k\}$ can be enlarged into a separating transcendence basis (or equivalently, a relative p-basis, by Fact 2.5) for $F(\vec{b})/F$, say $\{b_1, \ldots, b_k, b_{k+1}, \ldots, b_s\}$. By iterating Lemma 4.10 and Lemma 4.11, we can find an $\mathscr{L}_{\mathrm{div}}$-embedding $\tilde{f}$ from $F(b_1)^{\mathrm{sep}}(b_2)^{\mathrm{sep}} \cdots (b_s)^{\mathrm{sep}} \supset F(\vec{b})$ to K_2 extending f such that at the same time $\mathcal{K}_2 \models Q_n(f(b_1), \ldots, f(b_k))$. By the definition of a relative λ-resolvent, $\mathcal{K}_2 \models \Phi(d, f(\vec{c}))$ for some d being an $\mathscr{L}_{\mathrm{r}}$-term in terms of elements in $f(F(\vec{b}))$. And we are done. ⊣

COROLLARY 4.13 (cf. [6]). *When $e < \infty$, the theory of separably closed nontrivially valued fields of characteristic $p > 0$ imperfection degree $e > 0$ has quantifier elimination in the language $\mathscr{L} := \mathscr{L}_{\mathrm{r}} \cup \{b_1, \ldots, b_e\} \cup \{\lambda_{e,j}(x; b_1, \ldots, b_e)\}_{0 \leq j < p^e} \cup \{|\}$, where $\{b_1, \ldots, b_e\}$ is interpreted to be a p-basis.*

PROOF. One could change the proof for the general case a little bit to get the result.

Another way to see it is to observe that all the atomic $\mathscr{L}_{p,\mathrm{div}}$-formulas can be expressed using quantifier-free $\mathscr{L}$-formulas, using techniques similar to those used in Section 3 (instead of introducing new variables, one has to deal with rational functions, but the problem is still solvable). ⊣

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