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A NOTE ON LEARNING IN A CREDIT ECONOMY

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This paper introduces imperfect knowledge and learning behavior of economic agents into the Kiyotaki and Moore model and studies the interaction of agents' collateral price beliefs, collateral constraint, and aggregate economic activity over the business cycle. It establishes the E-stability condition and the convergence of the real time learning process. In addition, it shows that learning strengthens the role of collateral constraints in aggregate fluctuations.

Keywords: Learning, Collateral Constraint, Aggregate Fluctuations

1. INTRODUCTION

This paper studies the interaction of agents' collateral price beliefs, credit constraint, and aggregate economic activity over the business cycle. An influential contribution is Kiyotaki and Moore (1997; henceforth KM), which studies the role of credit constraints as an amplification mechanism generating large and persistent fluctuations in output and asset prices. This paper incorporates imperfect knowledge about the structure of the economy and learning about collateral prices by economic agents into KM, showing that learning amplifies and propagates shocks to the economy.

The mechanism of the learning model is as follows. A surprise increase in collateral prices, due say to a positive productivity shock, makes agents optimistic about future collateral prices. The optimism enhances borrowing capacity, which in turn boosts borrowers' collateral demand and collateral prices. The realized collateral prices reinforce agents' optimism and lead to further optimism. Associated with agents' shifting expectations and credit expansions/contractions are nonfundamental fluctuations of collateral prices and self-reinforcing rises and falls of aggregate activities.

A related paper is Assenza and Berardi (2009; henceforth AB), replacing the assumption of rational expectations (RE) in the basic version of KM by adaptive learning. The paper shows that the "optimality" conditions in AB are in fact either

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suboptimal or infeasible. This has implications for the learning dynamics of the model.

2. THE MODEL

The model setup is the same as that of AB, presented in their Section 2, except that shocks to the farmer's productivity are added. The AB model without the productivity shocks is exactly the basic version of KM, except that there are some notational differences, the gatherers' production function is assigned a specific functional form $G(K_t^G) = 2\sqrt{K_t^G}$, and RE is replaced by adaptive learning. Homogenous expectations and learning among borrowers and lenders are assumed.

We reproduce a few major equations from AB. The flow-of-funds constraint of the farmer (or borrower) is

$$q_t \left(K_t^F - K_{t-1}^F\right) + Rb_{t-1} + c_t^F = y_t^F + b_t, \tag{1}$$

where q_t, $K_t^F - K_{t-1}^F$, b_{t-1}, c_t^F, and y_t^F are land (or collateral) price, investment in collateral holding, debt holding, consumption, and production. The production function of the farmer is

$$y_t^F = (a + \epsilon_t + \bar{c})K_{t-1}^F, \tag{2}$$

where $(a + \epsilon_t)K_{t-1}^F$ is the tradable part of production and $\bar{c}K_{t-1}^F$ the nontradable part. ϵ_t is an i.i.d. innovation with zero mean and constant variance.

The farmer's financing constraint is

$$b_t \leq \frac{E_t q_{t+1}}{R} K_t^F. \tag{3}$$

The maximum loan he can get is the discounted and expected liquidation value of the collateral.

2.1. Optimality Conditions

Some assumptions, as in KM, are made to ensure that the return to investment in collateral holding is sufficiently high, so the farmer prefers to borrow up to the maximum and invest in land. His optimal consumption is the nontradable part of his production,

$$c_t^F = \bar{c}K_{t-1}^F, \tag{4}$$

and his optimal borrowing is

$$b_t = \frac{E_t q_{t+1}}{R} K_t^F. \tag{5}$$

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Combining equations (1), (4), (5), the lagged version of equation (5), and the farmer's production function, (2), delivers the farmer's land demand equation as

$$K_t^F = \frac{1}{\mu_t}(a + q_t + \epsilon_t - E_{t-1}q_t)K_{t-1}^F, \tag{6}$$

where $\mu_t = q_t - E_t q_{t+1}/R$ is the down payment required to purchase a unit of collateral.

The gatherer's (or lender's) land demand equation is

$$\mu_t = \frac{G'(K_t^G)}{R}, \tag{7}$$

where G is the production function of the gatherer. Equation (7) says that the present value of marginal product of land equals the user cost of land.

The land market clearing condition is

$$K_t^G + K_t^F = \overline{K}. \tag{8}$$

Collateral prices and the collateral holding process under RE are determined by equations (6), (7), and (8), given initial conditions. Under learning, solving the collateral prices and collateral holding process requires additionally agents' belief updating equations and initial beliefs, presented later.

2.2. Incorrect Optimality Conditions in AB

AB replace the assumption of RE in the basic version of KM with adaptive learning. This section shows that their "optimality" conditions imply that agents' "optimal" choices are either suboptimal or infeasible.

To compare with AB, the productivity shock ϵ_t is shut down in this subsection. They have a different land demand equation for the farmer, which is their equation (5), reproduced here:

$$K_t^F = \frac{a}{\mu_t}K_{t-1}^F. \tag{9}$$

In contrast to equation (6), equation (9) omits capital gains/losses in land holdings $(q_t - E_{t-1}q_t)K_{t-1}^F$. Under RE equilibrium and deterministic environment, as in the KM model, agents' expectations about future land prices realize themselves because of perfect foresight. The current market value of inherited land from the last period, $q_t K_{t-1}^F$, exactly offsets the debt repayment, $E_{t-1}q_t K_{t-1}^F$. The two collateral demand equations, i.e., equations (6) and (9), coincide.

However, under adaptive learning, every period agents' expectations about future collateral prices may not realize during the learning transition. The farmers may make persistent forecast errors, which produce capital gains/losses in their land holdings and additional variations in their net worth.

Implicit in AB is that the capital gains/losses are completely absorbed by consumption. This can be seen more clearly by deriving the implied consumption rule

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for the farmer in AB. Combining equation (1), (5), the lagged version of equation (5), (7), and (9), the "optimal" consumption of the farmer implied by AB can be derived as

$$c_t^F = \bar{c}K_{t-1}^F + (q_t - E_{t-1}q_t)K_{t-1}^F. \tag{10}$$

The implications of equation (10) are as follows. When there is capital gain in the farmer's collateral holding, i.e., $q_t K_{t-1}^F > E_{t-1}q_t K_{t-1}^F$, he will consume the capital gain instead of investing in collateral holding. He will consume part of the tradable output beyond the nontradable output, which is *suboptimal* relative to investing it, because the return to the latter is higher than consuming. When facing capital loss, i.e., $q_t K_{t-1}^F < E_{t-1}q_t K_{t-1}^F$, borrowers will consume only part of nontradable output. This implies further that they invest part of nontradable output in collateral holdings, which is *infeasible* given that they are perishable and nontradable.

2.3. Belief Updating and the Actual Law of Motion

Appendix A shows that linearizing equations (6), (7), and (8) yields

$$q_t = \gamma_0 + \gamma_1 K_{t-1}^F + \gamma_2 E_t q_{t+1} + \gamma_3 E_{t-1} q_t - \gamma_3 \epsilon_t, \tag{11}$$

$$K_t^F = \gamma_4 + \gamma_5 q_t + \gamma_6 E_t q_{t+1}, \tag{12}$$

where $\gamma_0 = a$, $\gamma_1 = \frac{a^3R^2}{2}$, $\gamma_2 = \frac{\bar{K}a^2R^2+1}{2R}$, $\gamma_3 = -\frac{1}{2}[\bar{K}a^2R^2 - 1]$, $\gamma_4 = \bar{K} - \frac{3}{(aR)^2}$, $\gamma_5 = \frac{2}{a^3R^2}$, and $\gamma_6 = -\frac{2}{(aR)^3}$.

This model economy has the following minimum state variable (MSV) RE solution:

$$q_t = \Psi_0 + \Psi_1 K_{t-1}^F + \Phi_q \epsilon_t, \tag{13}$$

$$K_t^F = \Psi_2 + \Psi_3 K_{t-1}^F + \Phi_K \epsilon_t, \tag{14}$$

where the parameters are to be determined.

Under learning,[1] agents' perceived law of motion (PLM) is

$$q_t = \phi_0 + \phi_1 K_{t-1}^F + \eta_{q,t}, \tag{15}$$

$$K_t^F = \theta_0 + \theta_1 K_{t-1}^F + \eta_{K,t}, \tag{16}$$

where η's are regression residuals. Because beliefs about θ_0 and θ_1 are irrelevant to agents' decisions, I focus on the evolution of beliefs about ϕ_0 and ϕ_1. They are revised according to

$$\phi_t = \phi_{t-1} + g_t R_t^{-1} x_{t-1}(q_{t-1} - \phi_{t-1}' x_{t-1}), \tag{17}$$

$$R_t = R_{t-1} + g_t(x_{t-1}x_{t-1}' - R_{t-1}), \tag{18}$$

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where $x_t \equiv (1 \quad K^F_{t-1})'$ and $\phi_t \equiv (\phi_{0,t} \quad \phi_{1,t})'$. g_t is the gain parameter governing the speed of belief adjustments. As is standard in the literature, two gain sequences are considered: a decreasing sequence, e.g., $g_t = 1/t$, is used to study the convergence of the learning process and a constant sequence, $g_t = g > 0$, is used for numerical simulation. The latter implies that agents discount old data and give relatively greater weight to recent data.

Substituting the conditional expectations $E_t q_{t+1} = \phi_{0,t} + \phi_{1,t} K^F_t$ and $E_{t-1} q_t = \phi_{0,t-1} + \phi_{1,t-1} K^F_{t-1}$ into the model equations (11) and (12) delivers the actual law of motion (ALM) for collateral prices and collateral holdings under learning,

$$q_t = T_1(\phi_{0,t-1}, \phi_{0,t}, \phi_{1,t}) + T_2(\phi_{1,t-1}, \phi_{1,t}) K^F_{t-1} + T_3(\phi_{1,t}) \epsilon_t, \tag{19}$$

$$K^F_t = \frac{\gamma_4 + \gamma_6 \phi_{0,t}}{1 - \gamma_6 \phi_{1,t}} + \frac{\gamma_5}{1 - \gamma_6 \phi_{1,t}} q_t, \tag{20}$$

where T_1, T_2, and T_3 are presented in Appendix B.

3. LEARNING DYNAMICS

An unexpected positive impulse to farmers' productivity is considered to illustrate the learning dynamics assuming agents' initial beliefs are at or different from the RE belief. In addition, this section conducts E-stability analysis and studies the convergence of the real-time learning process.

3.1. Parameterization

Numerical simulations are performed with the following parameterization. The farmer's productivity is normalized at one, i.e., $a = 1$. Following KM, the parameter $\bar{K}$ is set so that farmers hold two-thirds of the collateral in the steady state. The gross interest rate is set to 1.06. The standard deviation of the productivity shock is set to 0.00712. The functional form of the gatherer's production function implies that the collateral intensity equals 0.5. The gain parameter is set to 0.02. This is consistent with the literature, which uses gain parameters ranging from 0.007 to 0.05—see, for example, Branch and Evans (2006) and Milani (2007).

3.2. Dynamic Response

Figure 1 displays impulse response functions (IRFs) of land prices and the farmer's land holdings for 10 years under RE, learning, and AB's[2] "optimality" conditions to a 1% positive impulse to farmers' productivity, assuming agents' initial belief is at the RE value.[3]

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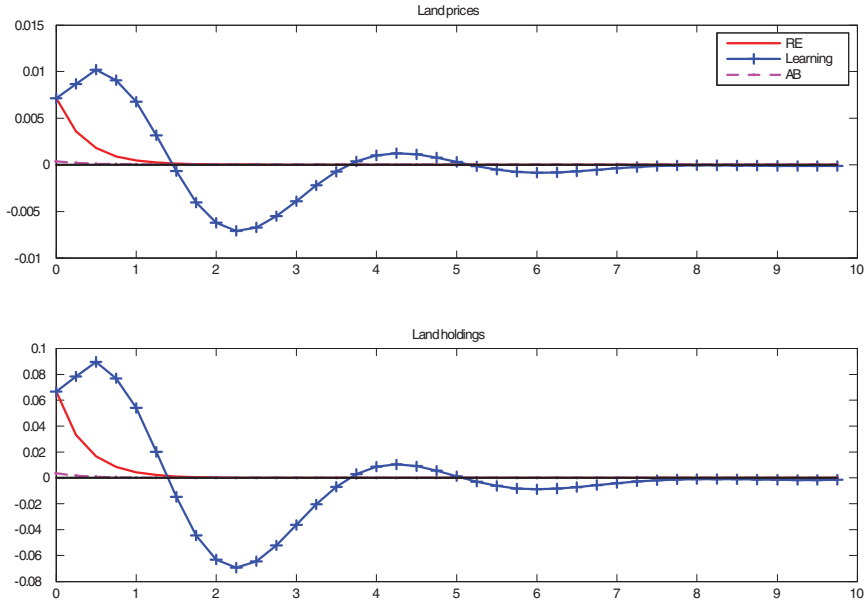


FIGURE 1. Response to 1% positive productivity impulse.

The RE dynamics is discussed in KM and briefly reviewed here. Following a positive impulse to the farmers' productivity, their net worth and demand for collateral rise. Collateral transfers from gatherers to farmers. User cost of collateral rises and so does collateral price. Rising current collateral holdings enhance the farmers' ability to invest in the following periods, which induces a persistent stay of the farmers' collateral holdings above the steady state. This reinforces the response of collateral prices when agents anticipate future user costs to be above the steady state. However, after the impact period, rising user cost chokes off further rises in prices and quantities, so they converge persistently and monotonically back to the steady state.

Under learning, agents have correct forecast functions for collateral prices initially, so responses under learning are identical to those under RE in the impact period. Afterwards, a positive surprise in collateral prices induces an upward belief revision and optimism about future prices relative to RE. The optimistic expectations induce credit expansion. Equation (6) says that farmers can increase their collateral holdings when the capital gain is higher than the down payment (relative to its steady state value a), i.e., $q_t - E_{t-1}q_t > \mu_t - a$. This boosts collateral prices further. The realized collateral price reinforces agents' initial optimism and leads to further optimism. The learning model generates prolonged periods of expansion of prices and quantities. The peak responses of land prices and land holdings of farmers under learning are 43.4% and 35.5% higher than under RE, respectively.[4]

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The reversal of prices and quantities relates to the convergence of the learning process discussed in the next section. At some point the negative effect of excessive debt repayments dominates, so that the collateral price falls short of agents' forecast. This sets a self-reinforcing decline of prices and quantities in motion. Agents become pessimistic about future collateral prices, which tightens credit and contracts borrowers' collateral demand and collateral prices. The realized prices reinforce agents' pessimism and lead to further downward adjustments of beliefs. Collateral prices and quantities oscillate around the steady state for many periods and then converge to the steady state.[5]

Oscillatory dynamics has recently been found to be a prominent feature of fiscal policy changes in the standard real business cycle model with adaptive learning; see Mitra et al. (2013) and the references therein.

3.3. E-Stability and Real-Time Learning

Appendix B shows that the E-stability of the REE (11)–(12) requires

$$\bar{K}a^2R^2 > \frac{2-R}{R}.$$

Following KM, the gatherers' production function G is assumed to satisfy $G' > 0, G'' < 0, G'(0) > aR > G'(\overline{K})$,[6] which ensures that both farmers and gatherers are producing and holding (positive amounts of) collateral in the steady state. Given that $G(K_t^G) = 2(K_t^G)^{1/2}$, farmers' collateral holding in the steady state being positive implies that $K^F = \overline{K} - (aR)^{-2} > 0$. Appendix B also shows that $\bar{K}a^2R^2 > 1$ ensures the stationarity of the RE solution. The following proposition summarizes the E-stability result.

PROPOSITION 1. *For all admissible parameters in* $\Theta = \{(a, R, \bar{K}) : \bar{K}a^2R^2 > 1, R > 1\}$, *the REE for the economy represented by equations* (11) *and* (12) *is E-stable.*

Next I consider real-time learning of the REE (13)–(14). The model is mixed dating and Appendix C uses stochastic approximation theory to prove the convergence of least-squares learning.

PROPOSITION 2. *The REE (13)–(14) is stable under least-squares learning if the model parameters satisfy the E-stability conditions given in Proposition 1.*

4. CONCLUSION

The paper shows that interaction of agents' price beliefs, price realizations, and credit limits generates interesting and realistic learning dynamics. Overoptimism

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is believed to have contributed significantly to the build-up of the U.S. housing boom in the 2000s and associated macro dynamics, such as rising household debt and consumption boom. As an application, the current model may be extended to analyze the role of the interaction of agents' beliefs and credit constraints in producing the price boom and associated aggregate fluctuations during the housing cycle.[7]

NOTES

1. Recent work studying this type of learning models includes Branch et al. (2013) and Bullard et al. (2010).

2. The IRFs under AB are much dampened. The working paper version of the paper (available at the author's Website) also studies an extension of the model with heterogenous learning and voluntary default by borrowers.

3. The pattern of collateral prices and lending under learning is similar to collateral prices and holding here. Responses under different initial beliefs rather than RE beliefs are presented in the working paper version of the paper. The amplification can be enhanced further with different initial beliefs.

4. Note that in the basic version of the KM model, the steady state leverage ratio (or debt/asset ratio) equals $1/R$ and is very large for R close to 1. A small change in collateral prices will generate large fluctuations of collateral holdings. Nevertheless, this figure can be used to illustrate the different dynamics between learning and RE.

5. The IRFs under learning depend continuously on the gain parameter. The larger (smaller) the gain, the stronger (weaker) the amplification and the more (less) likely are oscillatory dynamics.

6. See equation (5) on p. 219 of KM.

7. Kuang (2014) provides such an example.

8. Extended proofs of Propositions 1 and 2 can be found in the working paper version of the paper.

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APPENDIX A: LINEARIZING THE LEARNING ECONOMY

Assuming that the gatherer's production function is Cobb–Douglas, i.e., $G(K_t^G) = 2\sqrt{K_t^G}$, equation (7) leads to $K_t^G = (R\mu_t)^{-2}$. The steady state of the system is $a = \mu$, $q = \frac{aR}{R-1}$, $K^G = (aR)^{-2}$- and $K^F = \bar{K} - (aR)^{-2}$. I proceed to linearize the system around its steady state. Rearranging equation (6),

$$K_{t-1}^F = \frac{\bar{K}\left(q_t - \frac{1}{R}E_t q_{t+1}\right)^2 - \frac{1}{R^2}}{(a + q_t - E_{t-1}q_t + \epsilon_t)\left(q_t - \frac{1}{R}E_t q_{t+1}\right)}. \tag{A.1}$$

Linearizing the right-hand side of equation (A.1),

$$K_{t-1}^F \simeq \bar{K} - \frac{1}{(aR)^2} + \frac{2}{a^3R^2}\left(q_t - \frac{aR}{R-1}\right) - \frac{1}{aR}\left[\bar{K} + \frac{1}{(aR)^2}\right]\left(E_t q_{t+1} - \frac{aR}{R-1}\right)$$
$$+ \frac{1}{a}\left[\bar{K} - \frac{1}{a^2R^2}\right]\left(E_{t-1}q_t - \frac{aR}{R-1}\right) + \frac{1}{a}\left[\bar{K} - \frac{1}{a^2R^2}\right]\epsilon_t.$$

Rearranging this equation yields equation (11). Combining $K_t^G = \bar{K} - K_t^F$ and equation (7) leads to

$$K_t^F \simeq \bar{K} - \frac{1}{(aR^2)} + \frac{2R}{(aR)^3}\left(q_t - \frac{aR}{R-1}\right) - \frac{2}{(aR)^3}\left(E_t q_{t+1} - \frac{aR}{R-1}\right).$$

Collecting the coefficients in this equation delivers equation (12).

APPENDIX B: PROOF OF PROPOSITION 1

Substituting the conditional expectations into equations (11)–(12) yields the ALM for the farmer's land demand,

$$K_t^F = \frac{\gamma_4 + \gamma_6\phi_0}{1 - \gamma_6\phi_1} + \frac{\gamma_5}{1 - \gamma_6\phi_1}q_t, \tag{B.1}$$

and the ALM for collateral prices [using equation (B.1)],

$$q_t = T_1(\phi_0, \phi_1) + T_2(\phi_0, \phi_1)K_{t-1}^F + T_3(\phi_1)\epsilon_t, \tag{B.2}$$

where

$$T_1(\phi_0, \phi_1) = \frac{(\gamma_0 + \gamma_2\phi_0 + \gamma_3\phi_0) + \gamma_2\phi_1\frac{\gamma_4 + \gamma_6\phi_0}{1-\gamma_6\phi_1}}{1 - \gamma_2\phi_1\frac{\gamma_5}{1-\gamma_6\phi_1}}, \tag{B.3}$$

$$T_2(\phi_0, \phi_1) = \frac{\gamma_1 + \gamma_3\phi_1}{1 - \gamma_2\phi_1\frac{\gamma_5}{1-\gamma_6\phi_1}}, \tag{B.4}$$

and

$$T_3(\phi_1) = \frac{-\gamma_3}{1 - \gamma_2\phi_{1,t}\frac{\gamma_5}{1-\gamma_6\phi_{1,t}}}. \tag{B.5}$$

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Combining (B.1) and (B.2) yields

$$K_t^F = \frac{\gamma_4 + \gamma_6\phi_0}{1-\gamma_6\phi_1} + \frac{\gamma_5}{1-\gamma_6\phi_1}T_1(\phi_0,\phi_1) + \frac{\gamma_5 T_2(\phi_0,\phi_1)}{1-\gamma_6\phi_1}K_{t-1}^F + \frac{\gamma_5 T_3(\phi_1)}{1-\gamma_6\phi_1}\epsilon_t. \quad \textbf{(B.6)}$$

At the fixed point, we have $\phi_1 = T_2(\phi_0,\phi_1) = \Psi_1$, which yields $\Psi_1 = \frac{a^3R^2}{2}/(\frac{\bar{K}a^2R^2+1}{2} - \frac{1}{R})$. Note that ϕ_0 does not appear in T_2. So only two partial derivatives, $\frac{\partial T_2}{\partial \phi_1}|_{\phi_0=\Psi_0,\phi_1=\Psi_1}$ and $\frac{\partial T_1}{\phi_0}|_{\phi_0=\Psi_0,\phi_1=\Psi_1}$, matter for the E-stability conditions. The partial derivative $\frac{\partial T_2}{\partial \phi_1}|_{\phi_0=\Psi_0,\phi_1=\Psi_1} < 1$ is needed to ensure E-stability and is equivalent to $\bar{K}a^2R^2 > \frac{2-R}{R}$. The other partial derivative, $\frac{\partial T_1}{\phi_0}|_{\phi_0=\Psi_0,\phi_1=\Psi_1} < 1$, is also needed to ensure E-stability and is equivalent to $\bar{K}a^2R^2 > \frac{2-R}{R}$. To sum up, the E-stability for the MSV RE equilibrium requires that $\bar{K}a^2R^2 > \frac{2-R}{R}$.

The stationarity of the borrower's collateral holding process (B.6) at REE requires $\left|\frac{\gamma_5\Psi_1}{1-\gamma_6\Psi_1}\right| < 1$, which is equivalent to $\bar{K}a^2R^2 > 1$.[8]

APPENDIX C: PROOF OF PROPOSITION 2

The proof follows Adam et al. (2006) with mixed dating. As the parameters in the evolution of farmers' collateral holdings, i.e., θ_0 and θ_1, do not appear in the ALM under learning, the proof focuses on the evolution of agents' beliefs about the parameters in the PLM for collateral prices. They are updated according to

$$\begin{pmatrix}\phi_{0,t}\\ \phi_{1,t}\end{pmatrix} = \begin{pmatrix}\phi_{0,t-1}\\ \phi_{1,t-1}\end{pmatrix} + g_t S_{t-1}^{-1}\begin{pmatrix}1\\ K_{t-2}^F\end{pmatrix}(q_{t-1} - \phi_{0,t-1} - \phi_{1,t-1}K_{t-2}^F) \quad \textbf{(C.1)}$$

and

$$\begin{pmatrix}\phi_{0,t-1}\\ \phi_{1,t-1}\end{pmatrix} = \begin{pmatrix}\phi_{0,t-2}\\ \phi_{1,t-2}\end{pmatrix} + g_t\begin{pmatrix}\frac{g_{t-1}}{g_t}\end{pmatrix} S_{t-1}^{-1}\begin{pmatrix}1\\ K_{t-3}^F\end{pmatrix}(q_{t-2} - \phi_{0,t-2} - \phi_{1,t-2}K_{t-3}^F), \quad \textbf{(C.2)}$$

for the periods t and $t-1$.

A timing change is made, i.e., $S_t = R_{t+1}$, so the moment matrix evolves according to

$$S_t = S_{t-1} + g_t\left(\frac{g_{t+1}}{g_t}\right)\left(x_t x_t' - R_{t-1}\right) \quad \textbf{(C.3)}$$

and

$$S_{t-1} = S_{t-2} + g_t\left(x_{t-1}x_{t-1}' - R_{t-1}\right), \quad \textbf{(C.4)}$$

where $g_t = 1/t$ and $x_t = (1 \quad K_{t-1}^F)'$.

To write the entire system as a stochastic approximation algorithm (SRA), I define $\alpha_t = (\phi_{0,t}, \phi_{1,t}, \phi_{0,t-1}, \phi_{1,t-1})'$,

$$B_t = \begin{pmatrix}\alpha_t\\ \text{vec } S_t\\ \text{vec } S_{t-1}\end{pmatrix}, \text{ and } X_t = \begin{pmatrix}K_{t-1}^F\\ K_{t-2}^F\\ K_{t-3}^F\\ 1\end{pmatrix}.$$

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Parameter updating equations (C.1) and (C.2) gives

$$\alpha_t = \alpha_{t-1} + g_t Q(t, \alpha_{t-1}, X_t), \tag{C.5}$$

where the function $Q(t, \alpha_{t-1}, X_t)$ is defined by (C.1)–(C.4). The ALM for farmers' collateral holdings can be expressed in terms of general functional notation as $K_t^F = C_1(\alpha_t) + C_2(\alpha_t)K_{t-1}^F + C_3(\alpha_t)\epsilon_t$.

The state vector X_t evolves according to

$$\begin{pmatrix} K_{t-1}^F \\ K_{t-2}^F \\ K_{t-3}^F \\ 1 \end{pmatrix} = \begin{pmatrix} C_2(\alpha_{t-1}) & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} K_{t-2}^F \\ K_{t-3}^F \\ K_{t-4}^F \\ 1 \end{pmatrix} + \begin{pmatrix} C_1(\alpha_{t-1}) & C_3(\alpha_{t-1}) \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \epsilon_{t-1} \end{pmatrix}$$

or

$$X_t = A(\alpha_{t-1})X_{t-1} + B(\alpha_{t-1})v_t, \tag{C.6}$$

where $v_t = (1\ \epsilon_{t-1})'$.

The system (C.5) and (C.6) is a standard form for SRAs. A variant of standard arguments shows that the convergence is governed by the stability of the ODEs

$$\frac{d}{d\tau}\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_0^1 \\ \phi_1^1 \end{pmatrix} = \begin{pmatrix} T_1(\alpha) - \phi_0 \\ T_2(\alpha) - \phi_1 \\ T_1(\alpha) - \phi_0^1 \\ T_2(\alpha) - \phi_1^1 \end{pmatrix}, \tag{C.7}$$

where the temporary notation of variables with/without the superscript refers to the t and $t-1$ dating in system (C.1)–(C.4)

Denote by α^* the fixed point or the RE belief. The Jacobian matrix of (C.7) evaluated at the fixed point α^* is

$$\begin{pmatrix} \frac{\partial T_1}{\partial \phi_0} & \frac{\partial T_1}{\partial \phi_1} & \frac{\partial T_1}{\partial \phi_0^1} & 0 \\ 0 & \frac{\partial T_2}{\partial \phi_1} & 0 & \frac{\partial T_2}{\partial \phi_1^1} \\ \frac{\partial T_1}{\partial \phi_0} & \frac{\partial T_1}{\partial \phi_1} & \frac{\partial T_1}{\partial \phi_0^1} & 0 \\ 0 & \frac{\partial T_2}{\partial \phi_1} & 0 & \frac{\partial T_1}{\partial \phi_1^1} \end{pmatrix}_{|\alpha=\alpha^*}.$$

Two eigenvalues of the matrix are zero and the remaining two are $\left(\frac{\partial T_2}{\partial \phi_1} + \frac{\partial T_1}{\partial \phi_1^1}\right)|_{\alpha=\alpha^*}$ and $\left(\frac{\partial T_2}{\partial \phi_0} + \frac{\partial T_1}{\partial \phi_0^1}\right)|_{\alpha=\alpha^*}$. Note that $\left(\frac{\partial T_2}{\partial \phi_1} + \frac{\partial T_1}{\partial \phi_1^1}\right)|_{\alpha=\alpha^*} < 1$ and $\left(\frac{\partial T_2}{\partial \phi_0} + \frac{\partial T_1}{\partial \phi_0^1}\right)|_{\alpha=\alpha^*} < 1$ are exactly the E-stability conditions; hence we have Proposition 2.

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