Math. Proc. Camb. Phil. Soc. (1984), 95, 49
Printed in Great Britain

Non-commutative unique factorization domains

BY A. W. CHATTERS
School of Mathematics, University of Bristol, Bristol BS8 TW1

(*Received* 21 *February* 1983; *revised* 30 *May* 1983)

1. *Introduction*

We introduce a concept of unique factorization for elements in the context of Noetherian rings which are not necessarily commutative. We will call an element p of such a ring R prime if (i) $pR = Rp$, (ii) pR is a height-1 prime ideal of R, and (iii) R/pR is an integral domain. We define a Noetherian U.F.D. to be a Noetherian integral domain R such that every height-1 prime P of R is principal and R/P is a domain, or equivalently every non-zero element of R is of the form cq, where q is a product of prime elements of R and c has no prime factors. Examples include the Noetherian U.F.D.'s of commutative algebra and also the universal enveloping algebras of solvable Lie algebras. The latter class provides a rich supply of genuinely non-commutative examples.

In Section 2 we establish some of the general properties of a Noetherian U.F.D. R. For example, we show that R is integrally closed in an appropriate sense. Also the intersection T of the localizations of R at its height-1 primes coincides with the partial quotient ring of R formed by inverting the elements which have no prime factors (these elements are not always units of R). It follows that T also is a Noetherian U.F.D. and that every one-sided ideal of T is two-sided. In Section 3 we study the special case in which R is the universal enveloping algebra of a solvable Lie algebra L. In this case the partial quotient ring T of R is a principal ideal domain provided only that L is not Abelian.

2. *Noetherian unique factorization domains*

In this section we will define unique factorization domains in a non-commutative setting and will establish some of their basic properties. We shall consider only the case of associative Noetherian integral domains with identity element, where 'Noetherian' means 'left and right Noetherian'.

Definitions and standard notation: Let R be a prime ring. A height-1 prime of R is a prime ideal P of R such that P is minimal among non-zero prime ideals of R. An element p of R is prime if $pR = Rp$ is a height-1 prime of R and R/pR is a domain. If I is an ideal of R then $C(I)$ is the set of elements of R which are regular (i.e. not zero-divisors) modulo I. Throughout this paper we will set $C = \cap C(P)$, where P ranges over the height-1 primes of R, and if there is more than one ring under consideration we will write $C(R)$ rather than just C. Further background material can be found in [1].

We will base our definition of a unique factorization domain on the conditions given in the following result. Its proof could be shortened by quoting standard results, but for the reader's convenience we will give *direct* proofs as far as possible.

PROPOSITION 2·1. *Let R be a prime Noetherian ring with at least one height-1 prime, then the following conditions on R are equivalent*:

https://doi.org/10.1017/S0305004100061296 Published online by Cambridge University Press

(1) *Every height-*1 *prime of* R *is of the form* pR *for some prime element* p *of* R.

(2) R *is a domain and every non-zero element of* R *is of the form* $cp_1p_2\dots p_n$ *for some* $c\in C$ (*as defined above*) *and for some finite sequence* $p_1,\dots,p_n$ *of prime elements of* R.

We follow the convention that 1 *is the product of the empty set of prime elements.*

Proof. Suppose firstly that R satisfies condition (1). Let P be a height-1 prime of R then $P=pR$ for some prime element p. For every positive integer n we have $P^n=p^nR=Rp^n$. We note that p is a regular element of R and that $C(P)=R-P$ because R/pR is a domain (where $R-P$ denotes the set-theoretic complement of P in R). Suppose that $\cap_{n=1}^{\infty}P^n\neq 0$; we shall obtain a contradiction. There is a regular element x of R such that $x\in P^n$ for all n. Thus for each n we have $x=p^nx_n$ for some $x_n\in R$, and each x_n is regular. We have for example that $x=px_1=p^2x_2$ so that $x_1=px_2$. Hence $Rx_1\subseteq Rx_2$. But $R\neq Rp$ and x_2 is regular, so that $Rx_1=Rpx_2\neq Rx_2$. In this way we see that the left ideals Rx_n form an infinite strictly ascending sequence, which is a contradiction.

Now let $c\in C(P)$. We shall show by induction that $c\in C(P^n)$ for every positive integer n. Suppose that $c\in C(P^n)$ and that $r\in R$ with $cr\in p^{n+1}R$. We have $cr\in P^n$ so that $r=sp^n$ for some $s\in R$. Thus $csp^n\in Rp^{n+1}$ so that $cs\in Rp$. Hence $s\in Rp$, i.e. $r\in Rp^{n+1}$. It follows by symmetry that $c\in C(P^{n+1})$ as required. Therefore $C(P)\subseteq C(P^n)$ for all n. Also $\cap_{n=1}^{\infty}P^n=0$. Therefore $C(P)\subseteq C(0)$, i.e. the elements of $C(P)$ are regular elements of R.

Let a be a non-zero element of P, then there is a positive integer n such that $a\in P^n$ and $a\notin P^{n+1}$. Thus $a=p^nb$ for some $b\in C(P)$ and it follows that a is regular. Therefore R is a domain because every non-zero element of R belongs to either P or $C(P)$ and so is regular.

Let x be a non-zero element of R. Because only a finite number of prime ideals of R are minimal over RxR, only a finite number of height-1 primes of R contain x. Therefore there are prime elements $p_1,\dots,p_n$ such that $x=cp_1p_2\dots p_n$ for some $c\in C$. Thus R satisfies condition (2).

Conversely suppose that R satisfies condition (2) and let P be a height-1 prime of R. Let x be a non-zero element of P then $x=cp_1p_2\dots p_n$ for some $c\in C$ and prime elements p_i. Because $c\notin P$ and each p_iR is a two-sided ideal it follows that $p_i\in P$ for some i and that $P=p_iR$.

Definition. A Noetherian unique factorization domain ('Noetherian U.F.D.' for short) is a Noetherian integral domain which has at least one height-1 prime and which satisfies the equivalent conditions of Proposition 2·1.

Remarks 2.2. (*a*) If every Noetherian ring satisfies the descending chain condition for prime ideals (which as far as we know is still an open problem), then the restriction that R contains at least one height-1 prime would automatically be satisfied by any Noetherian ring R which is not simple.

(*b*) In this paper we shall deal only with left and right Noetherian rings. To define 'U.F.D.' in greater generality it would probably be advisable to base the definition on condition (2) of Proposition 2·1.

(*c*) Condition (2) of Proposition 2·1 looks like the usual definition of a U.F.D. in commutative algebra except for the presence of the elements of C, i.e. elements which have no prime factors. This immediately raises the question: Are the elements of

https://doi.org/10.1017/S0305004100061296 Published online by Cambridge University Press

C units? We shall see in Section 3 that there are Noetherian U.F.D.s in which not all the elements of C are units. In a subsequent paper we shall show that the elements of C are units of R whenever R is a prime Noetherian ring which satisfies a polynomial identity. Here we shall do only the commutative case.

Proposition 2·3. *Let R be a commutative Noetherian domain; then the elements of C are units of R.*

Proof. Let $c \in C$; then c belongs to no height-1 primes of R. It follows from the principal ideal theorem then there are no primes minimal over c, so that $cR = R$. ∎

COROLLARY 2.4. *Let R be a commutative Noetherian domain which is a* U.F.D. *in the classical sense; then R is also a* U.F.D. *according to the above definition.*

Proof. This is immediate from Proposition 2·3 and the fact that irreducible elements of R are prime. ∎

The remainder of this section is devoted to establishing some of the basic properties of Noetherian U.F.D.s.

Proposition 2·5. *Let R be a Noetherian* U.F.D.*; then R satisfies the Ore condition with respect to C.*

Proof. Let $a \in R$ and $c \in C$. By Goldie's theorem we have $ax = cb$ for some non-zero elements x and b of R. By Proposition 2·1 we have $x = dp_1p_2 \dots p_n$ for some $d \in C$, where each p_i is a prime element of R. Thus $cb = adp_1p_2 \dots p_n \in Rp_n$ with $c \in C(Rp_n)$. Therefore $b = b_np_n$ for some $b_n \in R$. Thus $adp_1p_2 \dots p_{n-1} = cb_n$. Repeating this process a finite number of times gives $ad = cb_1$ for some $b_1 \in R$. ∎

Example 2.6. The following ring R is a Noetherian domain which does not satisfy the Ore condition with respect to C. Let R be the ring given in example 7·3 of [6]; then R is a hereditary Noetherian domain with precisely one non-trivial ideal P, and R/P is a field. We have $C = C(P)$, and because $P = P^2$ the classical localization of R at P cannot exist. Therefore R does not satisfy the Ore condition with respect to C. ∎

THEOREM 2.7. *Let R be a Noetherian* U.F.D. *and let T be the partial quotient ring of R with respect to C, then:*

(1) *T is a Noetherian* U.F.D.

(2) *The elements of $C(T)$ are units of T.*

(3) *Every one-sided ideal of T is two-sided.*

(4) *$AB = BA$ for all ideals A and B of T.*

Proof. Let p be a prime element of R. Because $C \subseteq C(pR)$ it is easy to show directly or by quoting standard results that $pT = Tp$ and T/pT is a domain. Also the height-1 primes of T are precisely the extensions to T of the height-1 primes of R. It follows that prime elements of R are also prime as elements of T and that every height-1 prime of T is of the form pT for some prime element p of R. This proves (1).

Let $t \in C(T)$ then $t = ac^{-1}$ for some $a \in R$ and $c \in C(R)$. Thus $a = tc$, where c is a unit of T so that $a \in C(T)$. Therefore $a \in C(R)$ so that a, and hence also t, is a unit of T.

Let x be a non-zero element of T, then by (2) and Proposition 2·1 we have $x = up_1p_2 \dots p_n$ for some unit u of T and prime elements p_i of T. It follows easily that $Tx = xT$ and that every one-sided ideal of T is two-sided. Also the problem of showing that the multiplication of ideals of T is commutative reduces to that of showing that

https://doi.org/10.1017/S0305004100061296 Published online by Cambridge University Press

if p and q are prime elements of T then $pTqT = qTpT$. But if $pT \neq qT$ then $pTqT = pT \cap qT = qTpT$. |

Because the concept of a maximal order will not be used in Section 3, we will give a form of the definition which is convenient for the present purposes and refer to [4] for further details.

Definition. Let R be a prime Noetherian ring with quotient ring Q; then R is a maximal order if whenever I is a non-zero ideal of R and $q \in Q$ with $qI \subseteq I$ or $Iq \subseteq I$ then $q \in R$.

A commutative Noetherian domain is a maximal order if and only if it is integrally closed.

Proposition 2·8. *Let R be a Noetherian* U.F.D. *and let P be a height-1 prime of R; then the classical localization R_P of R at P exists and is a maximal order.*

Proof. We have $P = pR$ for some prime element p of R and also $C(P) = R - P$. Let $a \in R$ and $c \in C(P)$. By Goldie's theorem we have $ax = cy$ for some non-zero elements x and y of R. As in the proof of Proposition 2·1, there is a positive integer n such that $x = dp^n$ for some $d \in C(P)$. Also $c \in C(Rp^n)$ and $cy = ax = adp^n$ so that $y = bp^n$ for some $b \in R$. Therefore $ad = cb$ and R satisfies the Ore condition with respect to $C(P)$.

For convenience we will use S rather than R_P to denote the partial quotient ring of R with respect to $C(P)$. The Jacobson radical of S is pS and S is a local Noetherian domain. Every proper ideal of S is of the form p^nS for some positive integer n (see, for example, the last part of the proof of theorem 12·8 of [1]). It follows immediately that S is a maximal order. |

Theorem 2.9 *Let R be a Noetherian* U.F.D. *and let T be the partial quotient ring of R with respect to C; then T is a maximal order and $T = \cap R_P$, where P ranges over the height-1 primes of R.*

Proof. Set $U = \cap R_P$, then $T \subseteq U$ because $C \subseteq C(P)$ for every height-1 prime P of R. The height-1 primes of T are of the form PT, where P is a height-1 prime of R. Also $R_P = T_{PT}$ and $U = \cap T_{PT}$.

Let $u \in U$ then $xu \in T$ for some non-zero element x of T. By Proposition 2·1 and Theorem 2·7 we may without loss of generality suppose that $x = p_1p_2 \dots p_n$, where the p_i are prime elements of T. Thus $p_1p_2 \dots p_nu \in T$ and $p_2p_3 \dots p_nu \in U$. Hence $p_2p_3 \dots p_nuc \in T$ for some $c \in C(p_1T)$ (we are using the fact that U is contained in the localization of T at p_1T). Because $p_1p_2 \dots p_nuc \in p_1T$ we have $p_1p_2 \dots p_nu \in p_1T$ so that $p_2p_3 \dots p_nu \in T$. Continuing this process gives $u \in T$ and $T = U$.

Let I be a non-zero ideal of T and let y be an element of the quotient ring of T such that $yI \subseteq I$. Let Q be any height-1 prime of T; then $yIT_Q \subseteq IT_Q$. Also T_Q is a maximal order (Theorem 2·7 and Proposition 2·8) and IT_Q is a non-zero ideal of T_Q. Therefore $y \in T_Q$ for all such Q so that $y \in T$. |

THEOREM 2.10. *Let R be a Noetherian* U.F.D. *such that every non-zero prime ideal of R contains a height-1 prime; then R is a maximal order.*

Proof. Let D be the multiplicatively closed set generated by the prime elements of R then $dR = Rd$ for all $d \in D$. Hence R satisfies the Ore condition with respect to D. Let S be the partial quotient ring of R with respect to D and let I be a non-zero ideal of R. There are prime ideals $P_1, \dots, P_n$ of R such that $P_1P_2 \dots P_n \subseteq I$ and $I \subseteq P_i$ for all i.

https://doi.org/10.1017/S0305004100061296 Published online by Cambridge University Press

Because each P_i contains a prime element we have $I \cap D \neq \varnothing$, so that $IS = S$. In fact S is a simple ring. Let q be an element of the quotient ring of R such that $qI \subseteq I$ then $qIS \subseteq IS$, i.e. $qS \subseteq S$, i.e. $q \in S$. Let T be the partial quotient ring of R with respect to C then $qIT \subseteq IT$ and IT is an ideal of T. Therefore $q \in T$ by Theorem 2·9.

We shall complete the proof by showing that $S \cap T = R$. Let $u \in S \cap T$; then there are prime elements $p_1, \ldots, p_n$ of R such that $p_1 p_2 \ldots p_n u \in R$. Also $p_2 p_3 \ldots p_n u \in T$ so that $p_2 p_3 \ldots p_n uc \in R$ for some $c \in C$. Thus $p_1 p_2 \ldots p_n uc \in p_1 R$ with $c \in C(p_1 R)$ so that $p_1 p_2 \ldots p_n u \in p_1 R$. Hence $p_2 p_3 \ldots p_n u \in R$ and repeating this process gives $u \in R$ as required. ∎

Example 2.11. Let D be the division algebra of quaternions over the rational numbers with basis $1, i, j, k$ as usual and set $R = D[x]$. We shall show that R is not a Noetherian U.F.D. Let I be an ideal of R which contains $x+i$ then

$$(x+i)j - j(x+i) = 2k \in I$$

so that $I = R$. Suppose that R is a Noetherian U.F.D.; then $x+i \in C$ and similarly $x-i \in C$. But $(x+i)(x-i) = x^2+1 \notin C$ because x^2+1 is a central non-unit of R. Thus we have a contradiction. In fact $(x^2+1)R$ is a height-1 prime of R and $R/(x^2+1)R$ is not a domain. Strictly speaking D is not a Noetherian U.F.D., but if we set $S = H[x]$, where H is the localization of the ring of Hurwitz quaternions at the maximal ideal containing 2, then H is a Noetherian U.F.D. but S is not. ∎

3. *Enveloping algebras of solvable Lie algebras*

Let R be the universal enveloping algebra of a solvable Lie algebra L. Then R is a Noetherian U.F.D. (see Proposition 3·1), and R is not commutative provided L is not Abelian. Thus we have a wide class of non-commutative examples to which the results of Section 2 apply. Let T be the partial quotient ring of R with respect to C. One might expect that the structure of T would depend on such things as the dimension of L or whether L is nilpotent, but in fact T looks surprisingly like a commutative principal ideal domain provided only that L is not Abelian. If L is Abelian then R is a commutative polynomial algebra and $T = R$. For general material on universal enveloping algebras we refer to [2]. We will use $xy - yx$ rather than $[x, y]$ to denote the Lie product of x and y.

PROPOSITION 3·1. *Let $R = U(L)$ be the universal enveloping algebra of a finite-dimensional solvable Lie algebra L over the field k of complex numbers, then R is a Noetherian* U.F.D.

Proof. It is well known that R is a Noetherian domain. By Dixmier's theorem ([2], theorem 3·7·2), every prime factor ring of R is a domain. Also R satisfies the descending chain condition for prime ideals because, for example, the Gelfand–Kirillov dimension of R is finite. Let P be a height-1 prime of R; then P contains a non-zero element of the semi-centre ([2], proposition 4·4·1). It follows from [5] that P contains an element p of the semi-centre such that p is irreducible as an element of R, and that $P = pR = Rp$. ∎

Example 3·2. Let $R = U(L)$, where L is the 2-dimensional solvable Lie algebra with basis x, y, where $xy - yx = y$. Then y is a prime element of R and yR is the only height-1 prime of R. Thus $C = C(yR)$. The partial quotient ring of R with respect to C coincides

https://doi.org/10.1017/S0305004100061296 Published online by Cambridge University Press

with the localization of R at yR which is a local principal left and right ideal domain. We have $x \in C$ and x is not a unit of R. ∎

The proof we shall give of the following theorem is due to J. T. Stafford and is shorter and clearer than that given in the first version of this paper.

THEOREM 3·3. *Let $R = U(L)$ be the universal enveloping algebra of a finite-dimensional non-Abelian Lie algebra L over the field k of complex numbers. Suppose that R is a Noetherian* U.F.D. *and let T be the partial quotient ring of R with respect to C, then $T = \cap R_P$, where P ranges over the height-1 primes of R and every one-sided ideal of T is two-sided and principal.*

Proof. In view of what has been proved already, it is enough to show that the non-zero prime ideals of T are maximal, i.e. that every prime ideal P of R with height at least 2 satisfies $P \cap C \neq \varnothing$. We note that if $g \in L$ and p is a prime element of R then $gp - pg = ap$ for some $a \in R$, and it follows easily by considering the natural grading on R that in fact $a \in k$.

Let P be a prime ideal of R with height $(P) \geqslant 2$. We wish to show that $P \cap C \neq \varnothing$ and this is trivial if P contains only one height-1 prime of R. Thus we may suppose that there are prime elements p and q of R with $p, q \in P$ and $pR \neq qR$. For all $g \in L$ we have $gp - pg = a(g)\,p$ and $gq - qg = b(g)\,q$ for some $a(g), b(g) \in k$. Firstly we suppose that there exists $g \in L$ with $a(g) \neq b(g)$. Set $c = p + q$ then $c \in P$. Suppose that $c \notin C$; then there is a height-1 prime Q of R such that $c \in Q$. Also $gc - cg = a(g)\,p + b(g)\,q \in Q$. Therefore $(a(g) - b(g))\,p \in Q$, so that $p \in Q$. Hence $Q = pR$. Thus $q = c - p \in pR$, which is a contradiction.

Now suppose that $a(g) = b(g)$ for all $g \in L$. We fix elements $g, h \in L$ such that $gh - hg \neq 0$. For each $w \in k$ set $c_w = p + q(h + w)$; then $c_w \in P$. We suppose that $P \cap C = \varnothing$, so that for each w there is a height-1 prime Q_w of R such that $c_w \in Q_w$. We have $gc_w - c_w g = a(g)\,p + b(g)\,q(h + w) + q(gh - hg) = a(g)\,c_w + q(gh - hg) \in Q_w$, so that $q(gh - hg) \in Q_w$. But $q \notin Q_w$ for otherwise $Q_w = pR = qR$. Therefore $gh - hg \in Q_w$ for all w. Because $gh - hg$ belongs to only a finite number of height-1 primes of R, there are distinct elements x and y of k such that $Q_x = Q_y$. Thus $c_x - c_y = q(x - y) \in Q_x$, which is a contradiction. ∎

Example 3·4. Let $R = U(L)$, where L is a solvable non-Abelian Lie algebra, then, by 3·1 and 3·3, the partial quotient ring T of R with respect to C is a principal ideal domain. Let S be the simple over-ring of R defined in the proof of Theorem 2·10, then S is not always hereditary (S is formed by inverting the prime elements of R). If L is the direct sum of two copies of the 2-dimensional solvable Lie algebra of Example 3·2 then in this case S is not hereditary because the Krull dimension of S is at least 2. ∎

We wish to thank K. A. Brown, C. R. Hajarnavis, D. A. Jordan and J. T. Stafford for their encouragement and provision of information.

REFERENCES

[1] A. W. CHATTERS and C. R. HAJARNAVIS. *Rings with Chain Conditions* (Pitman, 1980).
[2] J. DIXMIER. *Enveloping Algebras* (North Holland, 1977).
[3] D. JORDAN. Noetherian Ore extensions and Jacobson rings. *J. London Math. Soc.* (2) **10** (1975), 281–291.
[4] G. MAURY and J. RAYNAUD. *Ordres maximaux au sens de K. Asano*. Lecture Notes in Math. vol. 808. (Springer-Verlag, 1980).
[5] C. MOEGLIN. Factorialité dans les algèbres enveloppantes. *C. R. Acad. Sci., Paris* **282** (1976), 1269–1272.
https://doi.org/10.1017/S0305004100061296 Published online by Cambridge University Press
[6] J. C. ROBSON. Idealizers and hereditary Noetherian prime rings. *J. Algebra* **22** (1972), 45–81.