Student Problems

Students up to the age of 19 are invited to send solutions to either or both of the following problems to Stan Dolan, 126A Harpenden Road, St Albans, Herts., AL3 6BZ.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is, therefore, not necessary to submit solutions to both. Solutions should arrive by September 20th 2017. Please give your School year, the name and address of your School or College, and the name of a teacher through whom the award will be made. Please print your own name clearly! The names of all successful solvers will be published in the November 2017 edition.

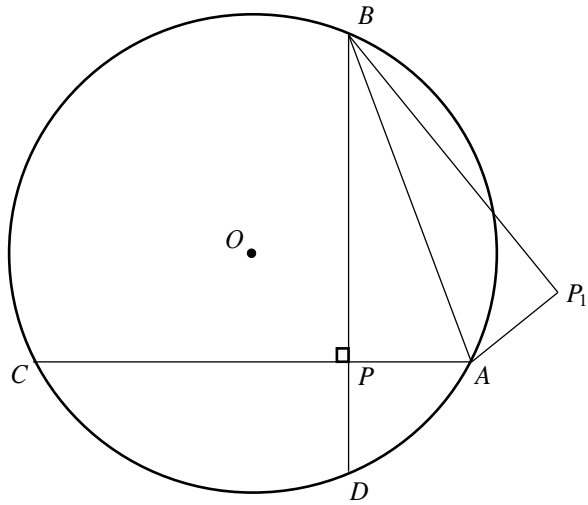
Problem 2017.3 (Gordon Screaton)

(a) Find all prime numbers which (in binary form) are $10101\ldots0101_2$.

(b) Express each composite number $10101\ldots0101_2$ as a non-trivial product of two binary numbers.

Problem 2017.4 (Paul Stephenson)

A cyclic quadrilateral $ABCD$ has circumcentre O and perpendicular diagonals which intersect at P. The reflections of P in each of the sides of $ABCD$ are P_1, P_2, P_3 and P_4. The point P_1 is shown in the diagram.



Prove that P_1, P_2, P_3 and P_4 lie on a circle centre O.

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Solutions to 2017.1 and 2017.2

Problem 2017.1 was solved by Ngo Hang Chan (Charterhouse) and Zhuowen Su (Charterhouse). Problem 2017.2 was solved by Matthew Collins (Bishop Wordsworth's School), Thomas Jackson (Cedars Upper School, Leighton Buzzard) and Yuhka Machino (Millfield School). Both problems were solved by Elliot Gathercole (Cedars Upper School, Leighton Buzzard), Joseph Lazarro (Dulwich College) and Andrew Ng (Dulwich College).

Problem 2017.1 (Anthony C. Robin)

A student is learning to drive and believes that her chance of failing her driving test will decrease with each possible future retest.

She denotes her chance of failing the driving test first time by q and assumes that, if she has to retake her driving test at least n times, then her chance of failing the n th test is $\frac{q}{n}$.

Find the mean and mode of the number of tests she will need to take to pass.

Solution

All the solvers started by first showing that the probability of passing on the n th test is

$$\frac{q^{n-1}}{(n-1)!}\left(1-\frac{q}{n}\right).$$

The mean number of tests is then

$$\begin{aligned}\sum \frac{q^{n-1}(n-q)}{(n-1)!} &= q\sum \frac{q^{n-2}}{(n-2)!} + \sum \frac{q^{n-1}}{(n-1)!} - q\sum \frac{q^{n-1}}{(n-1)!}\\ &= (q+1-q)e^q\\ &= e^q.\end{aligned}$$

Ngo, Elliot, Joseph, Andrew and Zhuowen all had slightly different methods for expressing

$$\sum \frac{q^{n-1}(n-q)}{(n-1)!}$$

in terms of the known series for e^q. Ngo's solution was especially complete and clearly explained.

It is easy algebra to show that, for $0 \leqslant q \leqslant 1$, the probabilities of passing on the 2nd, 3rd, 4th, … tests get progressively smaller.

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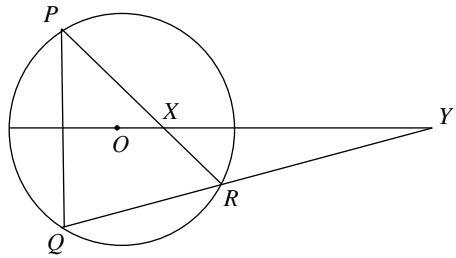
The probabilities of passing on the 1st and 2nd tests are $1 - q$ and $q\left(1 - \frac{1}{2}q\right)$, respectively. Then

$$1 - q = q\left(1 - \tfrac{1}{2}\right)q \Leftrightarrow q^2 - 4q + 2 = 0$$

$$\Leftrightarrow q = 2 - \sqrt{2}.$$

The modal number of tests is 1 for $q < 2 - \sqrt{2}$, 2 for $q > 2 - \sqrt{2}$ and both 1 and 2 for $q = 2 - \sqrt{2}$.

Problem 2017.2 (Geoff Strickland)



Let X be a fixed point on a diameter of a circle centre O and let PQ be a variable chord perpendicular to the diameter. The line PX meets the circle again at R and QR meets the extended diameter at Y.

Prove that the position of Y is independent of the position of PQ.

Solution

Matthew produced an interesting but complicated solution based upon cyclic quadrilateral $OQRX$ whereas Joseph and Andrew were successful in solving the problem using relatively involved coordinate geometry.

The Get Stoked Problem Solving Group of Mountain Lakes High School, New Jersey, gave a reasonably short coordinate geometry proof. They first chose points $P\left(a, \sqrt{r^2 - a^2}\right)$, $Q\left(-a, \sqrt{r^2 - a^2}\right)$ and $X\,(b,\ 0)$. The x-coordinate of the intersection of the line PX with the circle is then given by a quadratic equation. Since this equation has known solution $x = a$, (for point Q), the second root (for point R) can be found by synthetic division. Finally, the equation of QR can be determined and, putting $y = 0$, the x-coordinate of point Y is found to be $\dfrac{r^2}{b}$, independent of a.

Thomas, Elliot and Yuhka used similar triangles. Based upon Elliot's solution, we have:

$$2\angle OQR + \angle QOR = 180° \qquad \text{(Angles in a triangle)}$$

$$\angle QOR = 2\angle QPR \qquad \text{(Angle at centre)}$$

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Therefore

$$\angle OQY = 90° - \angle QPR.$$

So
$$\angle OQR = \angle OXP$$

$$\angle QOY = \angle XOP. \qquad \text{(Symmetry)}$$

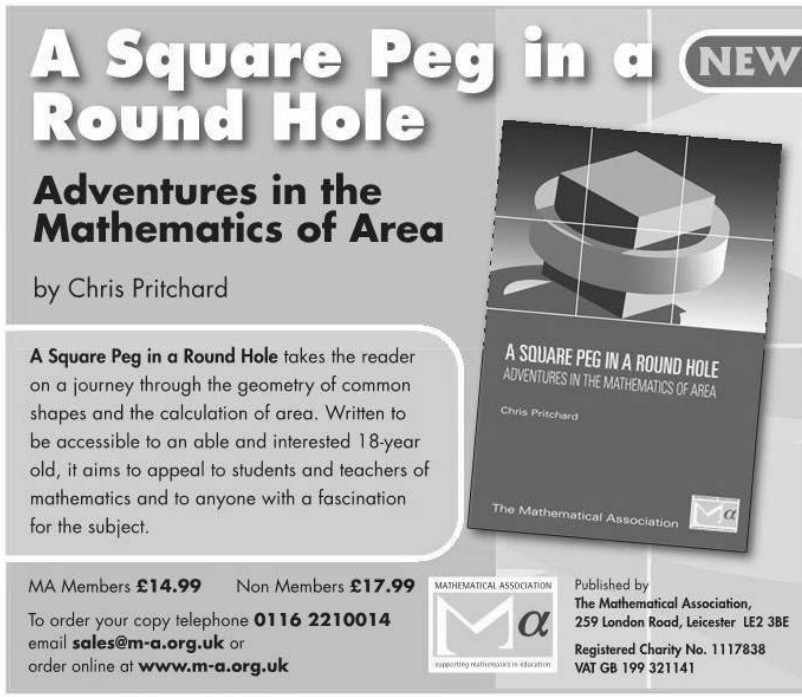
Therefore triangles QOY and XOP are similar and so

$$\frac{OQ}{OY} = \frac{OX}{OP}.$$

Then $OX \times OY$ is equal to the square of the radius, independent of the position of PQ.

The first prize of £25 is awarded to Elliot Gathercole. The second prize of £20 is awarded to Ngo Hang Chan.

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