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Explicit half-range expansions for Sturm–Liouville operators

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Boundary value problems for linear transport equations sometimes require the explicit construction of solutions when boundary conditions are prescribed only on parts of the boundary and thus necessitate the construction of half-range expansions. In contrast to the standard eigenfunction expansion, a half-range expansion of a given function must be reconstructed just over half the domain, using just half of the eigenfunctions. The difficulty of such expansions arises because the eigenfunctions are not orthogonal, though they are complete, over half the domain, and there is no obvious method of obtaining the expansion coefficients. Here we use complex variable techniques to find explicit formulas for the coefficients of half-range expansions for regular, negative definite Sturm–Liouville operators. We prove that the half-range expansion formula is unique, and find the corresponding half-range Green functions.

1 Introduction

Many scientific fields contain a common paradigm in which steady solutions $p(x, z)$ obey equations of the form

$$v(z)p_x = \mathscr{L}p, \tag{1.1}$$

where $\mathscr{L}$ is negative definite in z, and $v(z)$ takes both positive and negative values. Examples of such equations include the Fokker-Planck equation [1], two–way diffusion equations [2, 3, 4], linear transport equations such as the Boltzmann equation for electrons and holes in semiconductors [5, 6], and the radiation and neutron transport equations [7]. The unusual mathematical nature of this paradigm comes from the two signs of v.

For example, consider (1.1) in the region $0 < x < \infty$, requiring $p(x, z)$ to be bounded as $x \to \infty$. Boundary conditions at $x = 0$ need to be posed where, and only where, $v(z)$ is positive [7, 8]:

$$p(0, z) = \begin{cases} f(z) & \text{for values of } z \text{ where } v(z) > 0\,, \\ \text{unknown} & \text{for values of } z \text{ where } v(z) < 0\,. \end{cases} \tag{1.2}$$

To outline the half-range expansion problem, suppose we solve (1.1, 1.2) by separating the variables and looking for solutions of the form $p = e^{\lambda x}H(z)$. Because $v(z)$ has both signs, the eigenvalue problem $\mathscr{L}H = \lambda vH$ has an infinite number of both negative ($\lambda = -\mu_k^-$) and positive ($\lambda = \mu_k^+$) eigenvalues with corresponding eigenfunctions $H_{+k}(z)$ and $H_{-k}(z)$, respectively. Let us assume that the eigenvalues are simple, real and non-zero.

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Then expanding $p(x,z)$ in these eigenfunctions, and satisfying the growth condition and the boundary condition at $x=0$, gives

$$p(0,z) \sim \sum_k c_k H_{-k}(z) = f(z) \text{ for values of } z \text{ where } v(z) > 0. \tag{1.3}$$

Finding the coefficients c_k that satisfy (1.3) is the half–range expansion problem.

In this paper we consider half-range expansions for a class of regular, negative-definite Sturm–Liouville operators $\mathscr{L}$ acting only on the variable z and independent of the variable x. There are many theoretical results for this problem, but these results do not provide explicit expressions for the coefficients c_k. We know that if the coefficients of $\mathscr{L}$ satisfy some regularity conditions, then half of the eigenfunctions $H_k(z)$ are complete for z where $v(z) > 0$ [9]. Moreover, half-range expansions exist and are unique for Sturm–Liouville operators [10, 11, 12, 13, 14, 15]. Here, we use complex variable techniques to derive explicit formulas for the coefficient c_k; these expansions are motivated by and extend previous formal constructions [4, 17, 18, 19, 20]. We also extend these previous results by proving that such half-range expansions converge for all z, and converge to $f(z)$ over the correct subdomain, and are unique.

The class of operators we consider are relevant to various applications. For example, half-range expansions are used in Hagan & Ockendon [4] to study counter-current reactors. Ideas similar to the ones developed here are used in Sobehart & Hagan [16] to resolve the kinetic boundary layers in semiconductors. Here, we restrict our analysis to operators with simple eigenvalues. Without any loss of generality we assume for computational simplicity that zero is not an eigenvalue of the operator $\mathscr{L}$. An important example of an operator with a double eigenvalue at zero is the Fokker-Planck operator. Our techniques can be extended to cover this case; for a specific example studying the boundary layers in a stochastic process, see Kłosek & Hagan [17, 18]. A case of an indefinite Sturm–Liouville operator in (1.1) with a discrete spectrum has been solved elsewhere [19, 20, 21, 22]. The method of Hagan *et al.* [19], combined with singular perturbation techniques, was used to analyse equations in the form (1.1) and to solve a class of noise activation problems [17, 18].

In this paper, we solve the canonical free-space problem

$$\left.\begin{array}{l} v p_x = \mathscr{L} p + \delta(x) v(z) f(z) \quad \text{for} -\infty < x < \infty, \\ p(x,z) \quad \text{bounded as} \quad x \to \pm\infty. \end{array}\right\} \tag{1.4}$$

Here the source term $\delta(x)v(z)f(z)$ is interpreted as $v(z)[p(0^+,z) - p(0^-,z)] = v(z)f(z)$. The jump at $x=0$ implies that $p(x,z)$ is the steady state density in free space resulting from a steady source characterized by the rate $v(z)f(z)$ at $x=0$.

The second problem is the canonical half-range problem

$$\left.\begin{array}{l} v p_x = \mathscr{L} p \quad \text{for } 0 < x < \infty, \\ p(0,z) = \left\{ \begin{array}{ll} f(z) & \text{where } v(z) > 0, \\ \text{unknown} & \text{where } v(z) < 0, \end{array} \right. \\ p(x,z) \text{ bounded as } x \to +\infty. \end{array}\right\} \tag{1.5}$$

It yields the density resulting from the steady flux $v(z)f(z)$ entering the region $0 < x < \infty$ across the boundary at $x=0$.

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The last canonical problem is the reverse half-range problem

$$\left.\begin{array}{l} vp_x = \mathscr{L}p \quad \text{for} \quad -\infty < x < 0, \\ p(0,z) = \left\{ \begin{array}{ll} f(z) & \text{where } v(z) < 0, \\ \text{unknown} & \text{where } v(z) > 0, \end{array} \right. \\ p(x,z) \ \text{ bounded as } \ x \to -\infty. \end{array} \right\} \tag{1.6}$$

In § 2 we state the assumptions on the operator $\mathscr{L}$ and introduce the notation. In § 3 we state theorems on solutions to canonical problems (1.4–1.6). In § 4 and § 5 we prove the theorems of § 3 using estimates and lemmas derived in § 6–8.

2 Assumptions and notation

2.1 Operator $\mathscr{L}$

We consider the regular, negative-definite Sturm–Liouville operator $\mathscr{L}$ over an interval $(-1, 1)$

$$\mathscr{L}p = (ap_z)_z + cp, \quad -1 < z < 1, \tag{2.1}$$

with the boundary conditions

$$B^- p \equiv -a(-1)p_z(-1) + \alpha p(-1) = 0 \quad \text{at} \quad z = -1, \tag{2.2}$$

$$B^+ p \equiv a(1)p_z(1) + \beta p(1) = 0 \quad \text{at} \quad z = 1, \tag{2.3}$$

where α and β are real constants. To avoid needless difficulties, we require the coefficients $a(z)$, $c(z)$, and $v(z)$ to be real and smooth, $a(z) \in C^3$, $c(z) \in C^0$, $v(z) \in C^3$ for $-1 \leqslant z \leqslant 1$, and $a(z)$ to be strictly positive: $a(z) \geqslant \delta > 0$ for $-1 \leqslant z \leqslant 1$ for some constant δ. For simplicity, we assume that $v(z) < 0$ for $-1 \leqslant z < 0$, $v(z) > 0$ for $0 < z \leqslant 1$, with $v'(0) > 0$. We assume that $\mathscr{L}$ together with the boundary conditions (2.2–2.3) is negative–definite in the standard inner product, $\langle p, q \rangle = \int_{-1}^{1} p^*(z)q(z)dz$.

2.2 Eigenproblem

Separation of variables in (1.1) leads to the eigenvalue problem

$$\mathscr{L}H = \lambda v(z)H, \tag{2.4}$$

$$B^- H = 0 \quad B^+ H = 0. \tag{2.5}$$

With the assumptions on $\mathscr{L}$ stated in § 2.1, the eigenvalues of the problem (2.4–2.5) are real, simple and non-zero, with accumulation points at $\pm\infty$ [23].

Let the eigenvalues be $\lambda = \pm\mu_k^{\pm}$ with $\ldots < -\mu_2^- < -\mu_1^- < -\mu_0^- < 0 < \mu_0^+ < \mu_1^+ < \mu_2^+ < \ldots$ and the corresponding eigenfunctions $H_{\pm k}(z)$. The eigenfunctions corresponding to distinct eigenvalues are orthogonal in the sense that

$$\langle vH_{-j}, H_{-k} \rangle = 0, \ \langle vH_{+j}, H_{+k} \rangle = 0 \text{ for all } j \neq k, \ \langle vH_{-j}, H_{+k} \rangle = 0, \text{ for all } j, k, \tag{2.6}$$

and

$$\langle vH_{-j}, H_{-j} \rangle > 0, \quad \langle vH_{+j}, H_{+j} \rangle < 0 \quad \text{for } j = 0, 1, 2, \ldots .$$

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In §6 we show that the eigenvalues are distributed according to

$$\mu_k^- = \left(k+\frac{1}{4}\right)^2 \pi^2/I_R^2 + O(1), \quad \mu_k^+ = \left(k+\frac{1}{4}\right)^2 \pi^2/I_L^2 + O(1) \ \text{ as } k \to \infty\,, \tag{2.7}$$

where

$$I_R = \int_0^1 \sqrt{v/a}\,\mathrm{d}z, \qquad I_L = \int_{-1}^0 \sqrt{-v/a}\,\mathrm{d}z.$$

(Later we need the term $O(k)$ in the asymptotic expansion (2.7), beyond the standard result $\mu_k^{\pm} = O(k^2)$).

We define the functions $H^L(\lambda, z)$ and $H^k(\lambda, z)$ as the unique solutions to $\mathscr{L}H = \lambda v(z)H$, satisfying the initial conditions

$$H^L(\lambda,-1) = 1, \quad a(-1)H_z^L(\lambda,-1) = \alpha,$$

$$H^R(\lambda,1) = 1, \quad -a(1)H_z^R(\lambda,1) = \beta,$$

respectively. Hence H^L and H^R satisfy the left and the right boundary conditions, $B^-H^L = 0$ and $B^+H^R = 0$, respectively. At any eigenvalue λ, both $H^L(\lambda, z)$ and $H^R(\lambda, z)$ satisfy both boundary conditions. The eigenfunctions can be represented by either $H^L(\lambda, z)$ or $H^R(\lambda, z)$; they differ only by a normalization constant, so we write

$$H_{-k}^L(z) \equiv H^L(-\mu_k^-, z), \quad H_{+k}^L(z) \equiv H^L(\mu_k^+, z), \quad k = 0, 1, 2, \ldots,$$
$$H_{-k}^R(z) \equiv H^R(-\mu_k^-, z), \quad H_{+k}^R(z) \equiv H^R(\mu_k^+, z), \quad k = 0, 1, 2, \ldots,$$

and use whichever representation is more convenient. When normalization is irrelevant, we often omit the superscripts and write the eigenfunctions as $H_{\pm k}(z)$, as in (2.6).

2.3 Wronskian, functions $\mathscr{N}(\lambda)$ and $\mathscr{M}(\lambda)$

We define the Wronskian $W(\lambda) = a(z)H_z^L(\lambda, z)H^R(\lambda, z) - a(z)H^L(\lambda, z)H_z^R(\lambda, z)$. Direct calculations yield $W'(\lambda) = B^-H_\lambda^R = B^+H_\lambda^L$, so that we have

$$W'(\lambda) = \langle vH^L(\lambda, z),\ H^R(\lambda, z)\rangle \neq 0 \text{ at } \lambda = \pm\mu_k^{\pm}, \quad k = 0, 1, 2, \ldots\,. \tag{2.8}$$

Consequently, the Wronskian $W(\lambda)$ is an entire function with a simple zero at each eigenvalue and no other zeroes. In §6 we study (*cf.* 6.21, 6.22) asymptotic expansions of $W(\lambda)$ for $|\lambda| \to \infty$, and use it to locate the eigenvalues $\pm\mu_k^{\pm}$ as given in (2.7) for $k \gg 1$, by studying zeros of $W(\lambda)$.

We define the functions $\mathscr{N}(\lambda)$ and $\mathscr{M}(\lambda)$ as

$$\mathscr{N}(\lambda) = \prod_{k=0}^{\infty}\left(1 + \frac{\lambda}{\mu_k^-}\right)\,, \tag{2.9}$$

$$\mathscr{M}(\lambda) = \prod_{k=0}^{\infty}\left(1 - \frac{\lambda}{\mu_k^+}\right). \tag{2.10}$$

Since the eigenfunctions are distributed according to (2.7) both products (2.9) and (2.10) converge uniformly in every bounded domain in the complex λ–plane. Therefore $\mathscr{N}(\lambda)$ and $\mathscr{M}(\lambda)$ are entire functions with simple zeros at $\lambda = \mu_k^-$, and $\lambda = \mu_k^+$. Consequently

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$E(\lambda) = \mathscr{N}(\lambda)\mathscr{M}(\lambda)/W(\lambda)$ is an entire function. In §6 we show that $E(\lambda) = const.$ Since $\mathscr{N}(0) = \mathscr{M}(0) = 1$ we rewrite (2.8) as

$$W'(-\mu_k^-) = \langle vH_{-k}^L, H_{-k}^R\rangle = W(0)\mathscr{N}'(-\mu_k^-)\mathscr{M}(-\mu_k^-), \tag{2.11}$$

$$W'(\mu_k^+) = \langle vH_{+k}^L, H_{+k}^R\rangle = W(0)\mathscr{N}(\mu_k^+)\mathscr{M}'(\mu_k^+). \tag{2.12}$$

2.4 Spaces $\mathscr{B}$ and $L^2[v]$

We define the space $\mathscr{B}$ as follows:

$$\mathscr{B} = \{\text{all functions } p(z) \text{ in } C^2 \text{ for } -1 \leqslant z \leqslant 1 \text{ with } B^-p = B^+p = 0\},$$

and the space $L^2[v]$, which we define as all functions $f(z)$ which are integrable in the sense that

$$\| f \|_v^2 = \langle |v|f, f\rangle = \int_{-1}^{1} |v(z)|f^*(z)f(z)\mathrm{d}z$$

exists and is finite. Then $L^2[v]$ is a metric space with norm $\| f \|_v = \langle |v|f, f\rangle^{1/2}$. This metric space is complete, since $f(z)$ belonging to $L^2[v]$ is equivalent to $|v|^{1/2}f(z)$ being in L^2 [25]. Standard arguments also show that the functions in $\mathscr{B}$ are dense in the metric space $L^2[v]$.

3 Results

Theorem 3.1 (Full-range expansions) *Assume that $\mathscr{L}$ satisfies the assumptions in* §2.1. *Then for any $f(z)$ in $L^2[v]$*

$$p(x,z) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} \frac{\langle vH_{-k}, f\rangle}{\langle vH_{-k}, H_{-k}\rangle} H_{-k}(z)\mathrm{e}^{-\mu_k^- x} & \text{for } x > 0 \\ \\ \displaystyle -\sum_{k=0}^{\infty} \frac{\langle vH_{+k}, f\rangle}{\langle vH_{+k}, H_{+k}\rangle} H_{+k}(z)\mathrm{e}^{\mu_k^+ x} & \text{for } x < 0 \end{cases} \tag{3.1}$$

is a strong solution of the free-space problem (1.4), *and there is no other solution* (*weak or strong*) *in $L^2[v]$. The full range expansion*

$$f(z) = \sum_{k=0}^{\infty} \frac{\langle vH_{+k}, f\rangle}{\langle vH_{+k}, H_{+k}\rangle} H_{+k}(z) + \sum_{k=0}^{\infty} \frac{\langle vH_{-k}, f\rangle}{\langle vH_{-k}, H_{-k}\rangle} H_{-k}(z) \text{ for } -1 \leqslant z \leqslant 1 \tag{3.2}$$

converges in $L^2[v]$. In addition, if $f(z)$ is in $\mathscr{B}$, then the series converges absolutely and uniformly to $f(z)$ for $-1 \leqslant z \leqslant 1$.

To formulate the half-range expansion theorem we define the half-range adjoints

$$\Psi_{-k}(z) = H_{-k}^L(z) + \mathscr{M}(-\mu_k^-)\sum_{j=0}^{\infty} \frac{H_{+j}^L(z)}{(\mu_j^+ + \mu_k^-)\mathscr{M}'(\mu_j^+)} \quad \text{for } -1 \leqslant z \leqslant 1. \tag{3.3}$$

where $\mathscr{M}$ is given by (2.10).

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Lemma 3.2 (Properties of the half-range adjoints) *Assume that $\mathscr{L}$ satisfies the assumptions in* § 2.1. *Then for each k,* (3.3) *converges absolutely and uniformly to a function $\Psi_{-k}(z)$ which is continuous for $-1 \leqslant z \leqslant 1$ and satisfies $\Psi_{-k}(z) = 0$ for $-1 \leqslant z \leqslant 0$. Moreover, $\Psi_{-k}(z)$ is bi-orthogonal to the eigenfunctions $H_{-k}(z)$ in the sense that $\langle v\Psi_{-k}, H_{-j}\rangle = 0$ for all $j \neq k$, $\langle v\Psi_{-k}, H_{-k}\rangle = \langle vH_{-k}, H_{-k}\rangle \neq 0$ for all k.*

With Lemma 3.2, in § 5 we prove

Theorem 3.3 (Half–range expansions) *Assume that $\mathscr{L}$ satisfies the assumptions of* § 2.1, *and that $f(z)$ is in $L^2[v]$. Then the sum*

$$p_0(z) = \sum_{k=0}^{\infty} \frac{\langle v\Psi_{-k}, f\rangle}{\langle v\Psi_{-k}, H_{-k}\rangle} H_{-k}(z) \tag{3.4}$$

converges in $L^2[v]$ over the domain $-1 \leqslant z \leqslant 1$ to a function $p_0(z)$ in $L^2[v]$ which satisfies $p_0(z) = f(z)$ almost everywhere in $0 \leqslant z \leqslant 1$. Moreover,

$$p(x,z) = \sum_{k=0}^{\infty} \frac{\langle v\Psi_{-k}, f\rangle}{\langle v\Psi_{-k}, H_{-k}\rangle} H_{-k}(z)\mathrm{e}^{-\mu_k^- x} \tag{3.5}$$

is the unique strong solution to the canonical half-range problem (1.5).

Analogously we construct the solution to the reverse half-range problem (1.6).

Lemma 3.4 (Reverse half-range adjoints) *Assume that $\mathscr{L}$ satisfies the assumptions in* § 2.1. *Then for each k,*

$$\Psi_{+k}(z) \equiv H_{+k}^R(z) - \mathscr{N}(\mu_k^+) \sum_{j=0}^{\infty} \frac{H_{-j}^R(z)}{(\mu_k^+ + \mu_j^-)\mathscr{N}'(-\mu_j^-)}$$

converges absolutely and uniformly to a function which is continuous for $-1 \leqslant z \leqslant 1$ and satisfies $\Psi_{+k}(z) = 0$ for $0 \leqslant z \leqslant 1$. Moreover, $\Psi_{+k}(z)$ is bi-orthogonal to the eigenfunctions $H_{+k}(z)$ in the sense that $\langle v\Psi_{+k}, H_{+j}\rangle = 0$ for all $j \neq k$, $\langle v\Psi_{+k}, H_{+k}\rangle = \langle vH_{+k}^R, H_{+k}\rangle \neq 0$ for all k.

Theorem 3.5 (Reverse half-range expansions) *Assume that $\mathscr{L}$ satisfies the assumptions in* § 2.1 *and that $f(z)$ is in $L^2[v]$. Then*

$$p_0 = \sum_{k=0}^{\infty} \frac{\langle v\Psi_{+k}, f\rangle}{\langle v\Psi_{+k}, H_{+k}\rangle} H_{+k}(z) \tag{3.6}$$

converges in $L^2[v]$ over the domain $-1 \leqslant z \leqslant 1$ to a function $p_0(z)$ in $L^2[v]$ which satisfies $p_0(z) = f(z)$ almost everywhere in $-1 \leqslant z \leqslant 0$. Moreover,

$$p(x,z) = \sum_{k=0}^{\infty} \frac{\langle v\Psi_{+k}, f\rangle}{\langle v\Psi_{+k}, H_{+k}\rangle} H_{+k}(z)\mathrm{e}^{\mu_k^+ x}$$

is a strong solution of the canonical reverse half-range problem (1.6), *and there is no other*

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solution (weak or strong) of (1.6) *in* $L^2[v]$. *In addition, if* $f(z)$ *is in* $\mathscr{B}$, *then* (3.6) *converges absolutely and uniformly to a continuous function* $p_0(z)$ *for* $-1 \leqslant z \leqslant 1$.

(For formal arguments leading to the construction of the half–range expansions in Theorems 3.3 and 3.5, see elsewhere [4, 19].)

4 Full range expansion

We show that the function $p(x,z)$ defined as

$$p(x,z) \equiv \int_{-1}^{1} g(x,z,y)v(y)f(y)\mathrm{d}y, \tag{4.1}$$

where

$$g(x,z,y) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} \frac{H_{-k}(y)H_{-k}(z)}{\langle vH_{-k}, H_{-k}\rangle} \mathrm{e}^{-\mu_k^- x} & \text{for } x > 0 \\ \\ \displaystyle -\sum_{k=0}^{\infty} \frac{H_{+k}(y)H_{+k}(z)}{\langle vH_{+k}, H_{+k}\rangle} \mathrm{e}^{\mu_k^+ x} & \text{for } x < 0 \end{cases} \tag{4.2}$$

is the solution to the canonical problem (1.4). In § 6 we prove uniform bounds for the eigenfunctions and their derivatives (cf. 6.19, 6.20). These bounds imply that the series in (4.2) for any $x_0 > 0$ converges absolutely and uniformly and so do its derivatives $\partial_z^i \partial_y^j \partial_x^k g(x,z,y)$ for $i = 0,1,2$; $j = 0,1,2$; and any integer $k \geqslant 0$. Consequently, $g(x,z,y)$ is in $C_z^2 \cap C_y^2 \cap C_x^\infty$ for all $x \neq 0$, and the series in (4.2) can be differentiated term-by-term. In particular,

$$vg_x = \mathscr{L}g, \quad B^- g = 0, \quad B^+ g = 0 \quad \text{for } x \neq 0. \tag{4.3}$$

Suppose we restrict $f(z)$ to be in $L^2[v]$. Then (4.1) implies that $p(x,z)$ is in $C_z^2 \cap C_x^\infty$ for all $x \neq 0$, and we can differentiate under the integral sign to obtain the problem for p as in (4.3) with g replaced by p. The uniform convergence of (4.2) allows us to integrate term-by-term. Finally, if we differentiate the series in (4.2), the bounds (6.19) and (6.20) prove that for any $x_0 > 0$, the series for $\partial_z^i \partial_x^j p(x,z)$ converges uniformly when $|x| > x_0$, when $i = 0,1,2$ and j is any non-negative integer.

To prove that $p(x,z)$ is a solution of (1.4) it remains to show that the limits of $p(x,z)$ exist as $x \to 0^+$ and $x \to 0^-$, with

$$p(0^+,z) - p(0^-,z) = f(z). \tag{4.4}$$

First we prove (4.4) when $f(z)$ is in $\mathscr{B}$, and then extend it to all of $L^2[v]$. For functions $f(z)$ in $\mathscr{B}$ we can integrate by parts to obtain

$$\frac{\langle vH_{-k}, f\rangle}{\langle vH_{-k}, H_{-k}\rangle} H_{-k}(z) = -\frac{\langle vH_{-k}, \mathscr{L}f\rangle}{\mu_k^- \langle vH_{-k}, H_{-k}\rangle} H_{-k}(z) = -\frac{\langle vH_{-k}^L, \mathscr{L}f\rangle}{\mu_k^- \langle vH_{-k}^L, H_{-k}^R\rangle} H_{-k}^R(z).$$

Letting $F = \max_{-1\leqslant z\leqslant 1} |\mathscr{L}f|$, and using the bounds (6.19, 6.20) shows that for some constant b,

$$\left| \frac{\langle vH_{-k}, f\rangle}{\langle vH_{-k}, H_{-k}\rangle} H_{-k}(z) \mathrm{e}^{-\mu_k^- x} \right| \leqslant bF(\mu_k^-)^{-5/6} \mathrm{e}^{-\mu_k^- x}, \quad -1 \leqslant z \leqslant 1$$

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for all k. Therefore, the series in (4.1) with (4.2) converges uniformly for all $x \geqslant 0$, and thus describes a continuous function with

$$p(0^+, z) = \sum_{k=0}^{\infty} \frac{\langle vH_{-k}, f\rangle}{\langle vH_{-k}, H_{-k}\rangle} H_{-k}(z). \tag{4.5}$$

Similarly, the series

$$p(0^-, z) = -\sum_{k=0}^{\infty} \frac{\langle vH_{+k}, f\rangle}{\langle vH_{+k}, H_{+k}\rangle} H_{+k}(z) \tag{4.6}$$

converges uniformly and describes a continuous function $p(0^-, z)$ for $x \leqslant 0$. So the limits exist and

$$p(0^+, z) - p(0^-, z) = \sum_{k=0}^{\infty} \frac{\langle vH_{-k}, f\rangle}{\langle vH_{-k}, H_{-k}\rangle} H_{-k}(z) + \sum_{k=0}^{\infty} \frac{\langle vH_{+k}, f\rangle}{\langle vH_{+k}, H_{+k}\rangle} H_{+k}(z) \tag{4.7}$$

is a continuous function for $-1 \leqslant z \leqslant 1$.

The terms in (4.7) are the residues of $P(\lambda, z) = \int_{-1}^{1} G(\lambda, z, y)v(y)f(y)\,\mathrm{d}y$ at the eigenvalues $\lambda = \pm\mu_k^{\pm}$, with the Green function

$$G(\lambda, z, y) = \frac{1}{W(\lambda)} \begin{cases} H^L(\lambda, z)H^R(\lambda, y) & \text{for } z \leqslant y, \\ \\ H^L(\lambda, y)H^R(\lambda, z) & \text{for } y \leqslant z. \end{cases} \tag{4.8}$$

Letting C_m be the contour $\lambda = r_m e^{i\theta}$, $-\pi < \theta < \pi$, we have

$$p(0^+, z) - p(0^-, z) = \lim_{m\to\infty} \frac{1}{2\pi i} \int_{C_m} \int_{-1}^{1} G(\lambda, z, y)v(y)f(y)\,\mathrm{d}y\,\mathrm{d}\lambda, \tag{4.9}$$

where r_m is any sequence of radii with $r_m \to \infty$. Since $G(\lambda, z, y)$ satisfies $\mathscr{L}G - \lambda vG = -\delta(z - y)$, $-1 \leqslant y \leqslant 1$, $B^-G = 0$, $B^+G = 0$, we can integrate by parts to find

$$\int_{-1}^{1} G(\lambda, z, y)v(y)f(y)\,\mathrm{d}y = \frac{f(z)}{\lambda} + E(\lambda, z) \tag{4.10}$$

with

$$|E(\lambda, z)| \leqslant \frac{F}{|\lambda|} \int_{-1}^{1} |G(\lambda, z, y)|\,\mathrm{d}y.$$

Substituting (4.10) into (4.9) now yields

$$|p(0^+, z) - p(0^-, z) - f(z)| \leqslant \lim_{m\to\infty} \frac{F}{2\pi} \int_{C_m} \int_{-1}^{1} |G(\lambda, z, y)/\lambda|\,\mathrm{d}y\,\mathrm{d}\lambda.$$

In § 6 we prove that for any $\epsilon > 0$, there is an R_ϵ and a constant b such that $|G(\lambda, z, y)/\lambda| \leqslant b|\lambda|^{-3/2}|v(z)v(y)|^{-1/4}$ when $|\lambda| \geqslant R_\epsilon$ and $\lambda \in S_\epsilon$, where

$$S_\epsilon = \{\text{all } \lambda \text{ with } |\cos(\lambda^{1/2}I_L + \pi/4)| \geqslant \epsilon \text{ and } |\cos((-\lambda)^{1/2}I_R + \pi/4)| \geqslant \epsilon\}. \tag{4.11}$$

Picking a sequence $r_m \to \infty$ so that $\lambda = r_m e^{i\theta}$ is in S_ϵ for all m, now shows that $p(0^+, z) - p(0^-, z) = f(z)$ for all $z \neq 0$. It also holds at $z = 0$ since both of the sides are continuous. So whenever $f(z)$ is C^2 for $-1 \leqslant z \leqslant 1$ and satisfies the boundary conditions at $z = -1$ and $z = 1$, then $p(x, z)$ is a solution of (1.4) and the eigenfunction expansion in (4.7) converges absolutely and uniformly to $f(z)$ over the domain $-1 \leqslant z \leqslant 1$.

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4.1 Extension to $L^2[v]$

To prove Theorem 3.1, we need to extend the expansion in (3.1) to all functions $f(z)$ in $L^2[v]$. Normally such extensions are accomplished by using Bessel's inequality. In our case, however, Parseval's formula would read

$$\langle vf, f\rangle = \sum_{k=0}^{\infty} \frac{|\langle vH_{-k}, f\rangle|^2}{\langle vH_{-k}, H_{-k}\rangle} + \sum_{k=0}^{\infty} \frac{|\langle vH_{+k}, f\rangle|^2}{\langle vH_{+k}, H_{+k}\rangle}.$$

Since $\langle vH_{-k}, H_{-k}\rangle$ and $\langle vH_{+k}, H_{+k}\rangle$ have opposite signs, and $\langle vf, f\rangle$ is of indeterminate sign, there is no obvious means of deriving a Bessel inequality from this formula. To replace Bessel's inequality, in § 8 we prove the following.

Lemma 4.1 *There is a constant b such that*

(a) *if $e(z)$ is any function in $L^2[v]$, then*

$$\sum_{k=0}^{\infty} \frac{|\langle vH_{-k}, e\rangle|^2}{\langle vH_{-k}, H_{-k}\rangle} \leqslant b \parallel \mathrm{e} \parallel_v^2; \quad \sum_{k=0}^{\infty} \frac{|\langle vH_{+k}, e\rangle|^2}{|\langle vH_{+k}, H_{+k}\rangle|} \leqslant b \parallel e \parallel_v^2;$$

(b) *if $\sum_{k=0}^{\infty} |a_k|^2 < \infty$ and $\sum_{k=0}^{\infty} |c_k|^2 < \infty$, then the series*

$$\sum_{k=0}^{\infty} \frac{a_k H_{-k}(z)}{\langle vH_{-k}, H_{-k}\rangle^{1/2}} + \sum_{k=0}^{\infty} \frac{c_k H_{+k}(z)}{|\langle vH_{+k}, H_{+k}\rangle|^{1/2}}$$

converges in $L^2[v]$ to a function $h(z)$ satisfying

$$\parallel h \parallel_v^2 \leqslant b \left\{ \sum_{k=0}^{\infty} |a_k|^2 + \sum_{k=0}^{\infty} |c_k|^2 \right\}.$$

In particular, combining parts (a) and (b) of Lemma 3.2 shows that if $e(z)$ is any function in $L^2[v]$, then the series

$$\sum_{k=0}^{\infty} \frac{\langle vH_{-k}, e\rangle}{\langle vH_{-k}, H_{-k}\rangle} H_{-k}(z), \quad \sum_{k=0}^{\infty} \frac{\langle vH_{+k}, e\rangle}{\langle vH_{+k}, H_{+k}\rangle} H_{+k}(z)$$

both converge in $L^2[v]$ to functions $e_1(z)$ and $e_2(z)$ satisfying

$$\parallel e_1 \parallel_v \leqslant b \parallel e \parallel_v, \quad \parallel e_2 \parallel_v \leqslant b \parallel e \parallel_v . \tag{4.12}$$

Now let $f(z)$ be any function in $L^2[v]$, and consider $p(x, z)$ defined by the series (3.1). The above remark shows that at $x = 0$, these series converge in $L^2[v]$ to functions

$$p_+(z) = \sum_{k=0}^{\infty} \frac{\langle vH_{-k}, f\rangle}{\langle vH_{-k}, H_{-k}\rangle} H_{-k}(z), \quad p_-(z) = \sum_{k=0}^{\infty} \frac{\langle vH_{+k}, f\rangle}{\langle vH_{+k}, H_{+k}\rangle} H_{+k}(z). \tag{4.13}$$

Part (a) of Lemma 4.1 shows that

$$\sum_{k=0}^{\infty} \frac{|\langle vH_{-k}, f\rangle|^2}{\langle vH_{-k}, H_{-k}\rangle} \leqslant b \parallel f \parallel_v^2, \quad \sum_{k=0}^{\infty} \frac{|\langle vH_{+k}, f\rangle|^2}{|\langle vH_{+k}, H_{+k}\rangle|} \leqslant b \parallel f \parallel_v^2 ,$$

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and therefore

$$\sum_{k=0}^{\infty} \frac{|\langle vH_{-k}, f\rangle|^2}{\langle vH_{-k}, H_{-k}\rangle}(e^{-\mu_k^- x} - 1)^2 \to 0 \quad \text{as } x \to 0^+,$$

$$\sum_{k=0}^{\infty} \frac{|\langle vH_{+k}, f\rangle|^2}{|\langle vH_{+k}, H_{+k}\rangle|}(e^{\mu_k^+ x} - 1)^2 \to 0 \quad \text{as } x \to 0^-.$$

Consequently, part (b) of Lemma 4.1 shows that as x goes to zero from above, the limit of $p(x,z)$ exists in $L^2[v]$ and is equal to $p_+(z)$ (*cf.* 4.13). Similarly, as x goes to zero from below, the limit of $p(x,z)$ exists in $L^2[v]$ and is equal to $p_-(z)$ (*cf.* 4.13). We now know that the limits of $p(x,z)$ as $x \to 0^{\pm}$ exist, and that the right side of (4.7) converges in $L^2[v]$ to $p_+(z) - p_-(z)$. We now need to show that it converges to $f(z)$.

Recall that the functions in $\mathscr{B}$ are dense in the metric space $L^2[v]$. Let $\epsilon > 0$ be an arbitrarily small constant, and select a function $\tilde{f}(z)$ in $\mathscr{B}$ such that $\| \tilde{f}(z) - f(z) \|_v < \epsilon$. Define $e(z) = \tilde{f}(z) - f(z)$, and define the partial sum f_m,

$$f_m(z) = \sum_{k=0}^{m} \frac{\langle vH_{-k}, f\rangle}{\langle vH_{-k}, H_{-k}\rangle} H_{-k}(z) + \sum_{k=0}^{m} \frac{\langle vH_{+k}, f\rangle}{\langle vH_{+k}, H_{+k}\rangle} H_{+k}(z), \tag{4.14}$$

and the partial sums $\tilde{f}_m$ and e_m, analogously to (4.14) with f replaced by $\tilde{f}$ and e, respectively. Since $f - f_m = (f - \tilde{f}) + (\tilde{f} - \tilde{f}_m) + e_m$, we have

$$\| f(z) - f_m(z) \|_v \leqslant \| f(z) - \tilde{f}(z) \|_v + \| \tilde{f}(z) - \tilde{f}_m(z) \|_v + \| e_m(z) \|_v . \tag{4.15}$$

For any m, (4.12) implies that $\| e_m \|_v \leqslant 2b \| e \|_v < 2b\epsilon$. Since $\tilde{f}(z)$ is in $\mathscr{B}$, we know that $\tilde{f}(z) - \tilde{f}_m(z) \to 0$ uniformly as $m \to \infty$, so $\| \tilde{f} - \tilde{f}_m \|_v < \epsilon$ for all m large enough. So (4.15) shows that $\| f - f_m \|_v < 2(b+1)\epsilon$ for all large enough m. Since ϵ is arbitrary, this proves that $\| f - f_m \| \to 0$ as $m \to \infty$.

Consequently, if $f(z)$ is any function in $L^2[v]$, the right-hand side of (4.7) converges to $f(z)$ in $L^2[v]$, and (4.7) is a solution to the canonical free-space problem (1.4). The uniqueness of this result follows from a standard argument.

Finally, one can show that the Green function $G(\lambda, z, y)$ can be rewritten as

$$G(\lambda, z, y) = \sum_{k=0}^{\infty} \frac{H_{-k}(y)H_{-k}(z)}{(\lambda + \mu_k^-)\langle vH_{-k}, H_{-k}\rangle} + \sum_{k=0}^{\infty} \frac{H_{+k}(y)H_{+k}(z)}{(\lambda - \mu_k^+)\langle vH_{+k}, H_{+k}\rangle}.$$

5 Half-range expansion

We now prove that the function $q(\mu, x, z)$ defined by

$$q(\mu, x, z) = \mathscr{N}(\mu) \sum_{k=0}^{\infty} \frac{H^R(-\mu_k^-, z)e^{-\mu_k^- x}}{(\mu + \mu_k^-)\mathscr{N}'(-\mu_k^-)} = \mathscr{N}(\mu) \sum_{k=0}^{\infty} \frac{H_{-k}^R(z)e^{-\mu_k^- x}}{(\mu + \mu_k^-)\mathscr{N}'(-\mu_k^-)}. \tag{5.1}$$

is the solution to (1.5) with

$$q(\mu, 0, z) = \begin{cases} H^R(\mu, z) & \text{for } 0 \leqslant z \leqslant 1, \\ \text{unknown} & \text{for } -1 \leqslant z < 0. \end{cases} \tag{5.2}$$

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From (6.20) and (7.11) we have that for any $R > 0$ there is a constant b such that

$$\left| \mathcal{N}(\mu) \frac{H^R(-\mu_k^-, z) e^{-\mu_k^- x}}{\mathcal{N}'(-\mu_k^-)(\mu + \mu_k^-)} \right| \leqslant b(\mu_k^-)^{-2/3} e^{-\mu_k^- x} \text{ for } -1 \leqslant z \leqslant 1, x \geqslant 0, \text{ and } |\mu| \leqslant R \quad (5.3)$$

for all k. Since the eigenvalues are distributed according to $\mu_k^- \sim (k + \frac{1}{4})^2 \pi^2 / I_R^2$, (*cf.* 2.7) it shows that for any $R > 0$, the series (5.1) converges absolutely and uniformly for $-1 \leqslant z \leqslant 1, x \geqslant 0$, and $|\mu| \leqslant R$. Consequently, $q(\mu, x, z)$ is a bounded continuous function of x and z for $-1 \leqslant z \leqslant 1, x \geqslant 0$, and any μ. In addition, q is an entire function of μ at each x and z in this domain.

Now let $\delta > 0$ be an arbitrarily small, but fixed, constant. Equations in (6.20) show that if we differentiate the series in (5.1), then $\partial_z^i \partial_x^j q(\mu, x, z)$ converges uniformly over $x \geqslant \delta, -1 \leqslant x \leqslant 1$, for $i = 0, 1, 2$ and any integer $j \geqslant 0$. Thus, $q(\mu, x, z)$ is in $C_z^2 \cap C_x^\infty$ for $x > 0, -1 \leqslant z \leqslant 1$ and the series (5.1) can be differentiated term-by-term. Hence, $v q_x = \mathcal{L} q$, $B^- q = 0$, $B^+ q = 0$ for $x > 0$. From the uniform convergence of (5.1) we already know that $q(\mu, x, z)$ is bounded over $x \geqslant 0, -1 \leqslant z \leqslant 1$, and that (5.1) goes to a continuous function $q(\mu, 0, z)$ as $x \to 0$.

To prove that $q(\mu, 0, z) = H^R(\mu, z)$ for $0 \leqslant z \leqslant 1$, let $\delta > 0$ be an arbitrarily small constant, and let $f(z)$ be any function satisfying $f(z) \in C^2$ for $-1 \leqslant z \leqslant 1$, $f(z) = 0$ for $-1 \leqslant z \leqslant \delta$ and $1 - \delta \leqslant z \leqslant 1$, and we we prove that

$$\Phi \equiv \langle v(z) q(\mu, 0, z), f(z) \rangle \quad (5.4)$$

is equal to $\langle v(z) H^R(\mu, z), f(z) \rangle$.

For any ϵ with $0 < \epsilon < 1/2$, let S_ϵ be as in (4.11). If we substitute (5.1) with $x = 0$ for $q(\mu, 0, z)$ into (5.4), then the terms in the resulting series are the residues of $\langle v(z) Q(\mu, \lambda, z), f(z) \rangle$ with Q defined as

$$Q(\mu, \lambda, z) = \frac{H^R(\mu, z)}{\lambda - \mu} - \frac{E(\lambda)}{(\lambda - \mu)\mathcal{N}(\lambda)} H^R(\lambda, z) \quad \text{for } 0 \leqslant z \leqslant 1. \quad (5.5)$$

So,

$$\Phi = \lim_{m \to \infty} \Phi_m, \text{ where } \Phi_m = \frac{1}{2\pi i} \int_{C_m} \langle v(z) Q(\mu, \lambda, z), f(z) \rangle \, d\lambda. \quad (5.6)$$

Here C_m is the contour $r_m e^{i\theta}, 0 \leqslant \theta < 2\pi$, and we choose the radii so that $r_m e^{i\theta}$ is in S_ϵ for all θ and m, and so that $r_m \to \infty$ as $m \to \infty$. Since Q in (5.5) satisfies $\mathcal{L} Q - \lambda v Q = -v(z) H^R(\mu, z)$ for $z > 0$, integrating by parts yields

$$\Phi_m = \frac{1}{2\pi i} \int_{C_m} \langle (z) H^R(\mu, z), f(z) \rangle \frac{d\lambda}{\lambda} + \frac{1}{2\pi i} \int_{C_m} \langle Q, \mathcal{L} f \rangle \frac{d\lambda}{\lambda}.$$

Consequently,

$$|\Phi_m - \langle v(z) H^R(\mu, z), f(z) \rangle| \leqslant \frac{F}{2\pi} \int_{C_m} Q_{\max}(\mu, \lambda) \frac{d\lambda}{\lambda}, \quad (5.7)$$

where

$$F = \max_{\delta \leqslant z \leqslant 1 - \delta} |\mathcal{L} f|, \quad Q_{\max}(\mu, \lambda) = \max_{\delta \leqslant z \leqslant 1 - \delta} |Q(\mu, \lambda, z)|.$$

Using the estimates (6.16) and (7.5) for $\delta \leqslant z \leqslant 1 - \delta$ and all λ, shows that for some constant b, $|Q_{\max}(\mu, \lambda)| \leqslant b/|\lambda|$ for all λ in S_ϵ with $|\lambda| > R_\epsilon$, so (5.7) becomes

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$|\Phi_m - \langle v(z)H^R(\mu,z), f(z)\rangle| \leqslant Fb/r_m$ for all large enough m. Letting $m \to \infty$ yields

$$\Phi \equiv \langle v(z)q(\mu,0,z), f(z)\rangle = \langle v(z)H^R(\mu,z), f(z)\rangle,$$

which now proves that $q(\mu,0,z) = H^R(\mu,z)$ for $0 \leqslant z \leqslant 1$.

5.1 General half-range expansion

We now prove that the function $p(x,z)$ defined by (3.5) is a solution of the general half-range problem (1.5) whenever $f(z)$ is in $L^2[v]$. By obtaining uniform bounds on the adjoint $\Psi_{-j}(z)$, we first show that $p(x,z)$ satisfies (1.5) for all $x > 0$. We then prove that the limit of $p(x,z)$ as $x \to 0$ exists in $L^2[v]$ with $p(0,z)$ given by (4.6), which converges in $L^2[v]$ to $f(z)$ for $0 \leqslant z \leqslant 1$. As in §4 we initially prove this second step for smooth functions $f(z)$, and then use a Bessel inequality to extend it to all functions in $L^2[v]$. Uniqueness is proved in §5.2.

To bound $\Psi_{-j}(z)$, recall that the eigenvalues are distributed according to (2.7), so there is a constant b such that

$$\sum_{k=0}^{\infty} \frac{(\mu_k^+)^{1/3}(\mu_j^-)^{1/6}}{(\mu_k^+ + \mu_j^-)} \leqslant b \quad \text{for all } j. \tag{5.8}$$

In addition, (6.19, 7.6, 7.12, 5.8) imply that for some constants b and b',

$$|\Psi_{-j}(z)| \leqslant |H^L_{-j}(z)| + b|\mathscr{M}(-\mu_j^-)|(\mu_j^-)^{-1/6} \leqslant b' q_1^{1/6} e^{q_1 I_L} \quad \text{for all } z \text{ and all } j, \tag{5.9}$$

where $q_1 = (\mu_j^-)^{1/2}$. We now suppose that $f(z)$ is in $L^2[v]$ and show that (3.5) defines a smooth function when $x > 0$. From (6.19, 6.26) we know that there is a constant b such that for all j,

$$\left|\partial_z^i \partial_x^k \frac{\langle v\Psi_{-j}, f\rangle}{\langle v\Psi_{-j}, H^R_{-j}\rangle} H^R_{-j}(z)\mathrm{e}^{-\mu_j^- x}\right| \leqslant b \parallel f \parallel_v (\mu_j^-)^{k+i/2+1/6}\mathrm{e}^{-\mu_j^-\delta} \text{ for } x \geqslant \delta, -1 \leqslant z \leqslant 1$$

for $i = 1,2$ and any integer $k \geqslant 0$. If we differentiate the series (3.5), $\partial_z^i \partial_x^k p(x,z)$ converges uniformly for all z and all $x > \delta$, for any $\delta > 0$. Consequently, $p(x,z)$ is in $C_z^2 \cap C_x^\infty$ for all z and all $x > 0$, and (3.5) can be differentiated term–by–term. Thus, $vp_x = \mathscr{L}p$, $B^- p = 0$, $B^+ p = 0$ for $x > 0$, and $p(x,z)$ is bounded for all $x \geqslant \delta$ for any $\delta > 0$. So $p(x,z)$ satisfies (1.5).

We now suppose that $f(z)$ is in C^2 for $0 \leqslant x \leqslant 1$ and satisfies $B^+ f = 0$, and prove that $p(x,z)$ is continuous for all $x \geqslant 0$ with $p(0,z) = f(z)$ for $0 \leqslant z \leqslant 1$. Let us extend the definition of $f(z)$ to the full domain $-1 \leqslant z \leqslant 1$ in such a way that $f(z)$ is in $\mathscr{B}$. Then we can integrate by parts and use (6.19) and (6.20) to find that for some constant b,

$$|\langle vH^L_{-j}, f\rangle| \leqslant |\langle H^L_{-j}, \mathscr{L}f\rangle/\mu_j^-| \leqslant bFq_1^{-11/6}\mathrm{e}^{q_1 I_L},$$

$$|\langle vH^L_{+k}, f\rangle| \leqslant |\langle H^L_{+k}, \mathscr{L}f\rangle/\mu_k^+| \leqslant bFq^{-11/6},$$

for all j and k, where $F = \max_{-1\leqslant z\leqslant 1}|\mathscr{L}f|$, $q_1 = (\mu_j^-)^{1/2}$, and $q = (\mu_k^+)^{1/2}$. So for some constant b,

$$\left|\langle vH^L_{-j}, f\rangle \frac{H^R_{-j}(z)\mathrm{e}^{-\mu_j^- x}}{\langle vH^L_{-j}, H^R_{-j}\rangle}\right| \leqslant b(\mu_j^-)^{-5/6},$$

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$$\left|\mathscr{M}'(-\mu_j^-)\frac{\langle vH_{+k}^L, f\rangle}{(\mu_k^+ + \mu_j^-)\mathscr{M}'(\mu_k^+)}\right|\left|\frac{H_{-j}^R(z)\mathrm{e}^{-\mu_j^- x}}{\langle vH_{-j}^L, H_{-j}^R\rangle}\right| \leqslant b\frac{(\mu_j^-)^{1/3}(\mu_k^+)^{-2/3}}{(\mu_j^- + \mu_k^+)}$$

for all j and k. Since the eigenvalues are distributed according to (2.7), these estimates imply that the series

$$p(x,z) = \sum_{k=0}^{\infty}\frac{\langle vH_{-k}^R, f\rangle}{\langle vH_{-k}^R, H_{-k}^R\rangle}H_{-k}^R(z)\mathrm{e}^{-\mu_k^- x} + \sum_{k=0}^{\infty}\frac{\langle vH_{+k}^R, f\rangle}{\langle vH_{+k}^R, H_{+k}^R\rangle}q(\mu_k^+, x, z) \tag{5.10}$$

converges absolutely and uniformly for $-1 \leqslant z \leqslant 1$ and $x \geqslant 0$. Therefore $p(x,z)$ is continuous for $x \geqslant 0$ and all z. Interchanging the order of summation in (5.10) gives (3.5). Since $q(\mu_k^+, 0, z) = H_{+k}^R(z)$ for $z \geqslant 0$, at $x = 0$ (5.10) reduces to the full-range expansion (3.2) for $0 \leqslant z \leqslant 1$, and in §4.1 we proved that the full-range expansion converges uniformly to $f(z)$. So when $f(z)$ is in C^2 over $0 \leqslant z \leqslant 1$ and satisfies the boundary condition $B^+f = 0$, then $p(x,z)$ is continuous for all $x \geqslant 0$ with $p(0,z) = f(x)$ for $0 \leqslant z \leqslant 1$, where $p(q,z)$ is given by the series (4.5).

We now need to extend this result to all functions in $L^2[v]$. We do this by using the following lemma, which is proved in §6.

Lemma 5.1 *There is a constant b such that*

$$\sum_{j=0}^{\infty}|d_j|^2 \leqslant b \parallel f \parallel_v^2 \quad \textit{for all functions } f(z) \textit{ in } L^2[v], \tag{5.11}$$

where

$$d_j = \frac{\langle v\Psi_{-j}, f\rangle}{\langle v\Psi_{-j}, H_{-j}^R\rangle}\langle vH_{-j}^R, H_{-j}^R\rangle^{1/2}. \tag{5.12}$$

Combining Lemma 5.1 with part (b) of Lemma 4.1 shows that there is another constant b such that for all functions $f(z)$ in $L^2[v]$,

$$h(z) \equiv \sum_{j=0}^{\infty}\frac{\langle v\Psi_{-j}, f\rangle}{\langle v\Psi_{-j}, H_{-j}^R\rangle}H_{-j}^R(z) = \sum_{j=0}^{\infty}\frac{\langle v\Psi_{-j}, f\rangle H_{-j}(z)}{\langle vH_{-j}, H_{-j}\rangle^{1/2}} \tag{5.13}$$

converges in $L^2[v]$ to a function $h(z)$ satisfying $\parallel h \parallel_v < b \parallel f \parallel_v$.

It follows from (5.11, 5.12, 5.13) that if $f(z)$ is in $L^2[v]$, then the limit of $p(x,z)$ as $x \to 0$ exists in $L^2[v]$ with $p(0,z)$ given by (4.5). In the same manner it can be proved that $p(0,z)$ converges in $L^2[v]$ to $f(z)$ over $0 \leqslant z \leqslant 1$ by using (5.11, 5.12, 5.13) and the fact that functions $f(z)$ in $\mathscr{B}$ are dense in $L^2[v]$.

5.2 Uniqueness

Suppose that there were two half-range expansions that converge in $L^2[v]$ for $-1 \leqslant z \leqslant 1$, and suppose they converged to the same function when $0 \leqslant z \leqslant 1$. Then there would be a set of coefficients d_k, not all zero, such that

$$e_m(z) = \sum_{k=0}^{m} d_k H_{-k}(z)$$

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satisfies

$$I_m \equiv \int_0^1 v(z)|e_m(z)|^2 \mathrm{d}z \to 0 \quad \text{as } m \to \infty. \tag{5.14}$$

The bi-orthogonality of the half-range adjoints shows that for any k,

$$|d_k| = \left| \frac{\langle v\Psi_{-k}, e_m \rangle}{\langle v\Psi_{-k}, H_{-k} \rangle} \right| \leqslant \| \Psi_{-k} \|_v \, I_m^{1/2} / \langle v\Psi_{-k}, H_{-k} \rangle \quad \text{for all } m \geqslant k. \tag{5.15}$$

Equation (5.14) shows that the right-hand side of (5.15) goes to zero as $m \to \infty$, so all coefficients d_k are zero, hence the half-range expansion (4.6) is unique in $L^2[v]$. Similarly, suppose that the solution to the canonical half-range problem (1.5) is not unique, and let $e(x,z)$ in $L^2[v]$ be non-identically zero and satisfy (1.5) with zero half-range boundary conditions, $e(0,z) = 0$ for $0 < z \leqslant 1$. Then $\| e(x,z) \|_v$ is bounded as $x \to \infty$, and

$$\langle v\phi(x,z), e(x,z) \rangle = \int_{-1}^0 v(z)\phi(0,z)e(0,z)\,\mathrm{d}z \quad \text{for any } x > 0 \tag{5.16}$$

for all real functions $\phi(x,z)$ which are in $C_x^1 \cap C_z^2$ for $x > 0$ and all z, are continuous for $x \geqslant 0$ and all z, and satisfy $-v\phi_x = \mathscr{L}\phi$, $0 < x < \infty$, $B^-\phi = 0$, $B^+\phi = 0$, $0 < x < \infty$. Since $e(x,z)$ is in $L^2[v]$, we know that it can be expanded as

$$e(x,z) = \sum_{k=0}^{\infty} a_k(x)H_{-k}(z) + \sum_{k=0}^{\infty} c_k(x)H_{+k}(z) \quad \text{at each } x > 0.$$

Selecting $\phi(x,z) = H_{+k}(z)\mathrm{e}^{-\mu_k^+ x}$ and using (5.16) shows that $c_k(x) \equiv 0$ for $x > 0$ and all k. In particular,

$$\langle v(z)H_{+k}(z), e(x,z) \rangle = 0 \quad \text{for all } x > 0. \tag{5.17}$$

To show that the coefficients $a_k(z)$ are also zero, we select

$$\phi(x,z) = H_{-k}^L(z)\mathrm{e}^{\mu_k^- x} - r(-\mu_k^-, -x, z),$$

with

$$r(\mu, x, z) = \mathscr{M}(\mu) \sum_{j=0}^{\infty} \frac{H_{+j}^L(z)\mathrm{e}^{\mu_j^+ z}}{(\mu - \mu_j^+)\mathscr{M}'(\mu_j^+)} \tag{5.18}$$

and note that $\phi(0,z) = 0$ for $z < 0$ (*cf.* 3.3). Then the right-hand side of (5.16) is zero, so

$$\begin{aligned} \langle vH_{-k}^L, H_{-k} \rangle a_k(x)\mathrm{e}^{\mu_k^- x} &= \langle v(z)H_{-k}^L(z), e(x,z) \rangle \\ &= \langle v(z)r(-\mu_k^-, -x, z), e(x,z) \rangle \quad \text{for all } x > 0. \end{aligned} \tag{5.19}$$

The series (5.18) for $r(-\mu_k^-, -x, z)$ converges absolutely, so we can substitute this series into (5.19) and evaluate the inner product term-by-term. This series contains only the eigenfunctions $H_{+k}(z)$, so (5.17) implies that the right-hand side of (5.19) is zero. Hence $a_k(x) \equiv 0$ for all $x > 0$ and all k, and $e(x,z) \equiv 0$.

5.3 The half-range Green function

If we write the solution to (1.5) in the form

$$p(x,z) = \int_0^1 h_-(x,z,y)v(y)f(y)\mathrm{d}y \tag{5.20}$$

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then the half-space Green function $h_-(x,z,y)$ is the solution to

$$vh_{-x} = \mathscr{L}h_- + \delta(x)\delta(z-y) \quad \text{for all } x>0,\ 0<y<1,\ -1<z<1 \tag{5.21}$$

$$B^-h_- = 0, \quad B^+h_- = 0 \quad \text{for } x>0, \quad h_-(x,z,y) \text{ bounded as } x\to\infty. \tag{5.22}$$

From (3.5) we postulate that

$$h_-(x,z,y) = \sum_{j=0}^{\infty} \frac{\Psi_{-j}(y)\,H^R_{-j}(z)}{\langle v\Psi_{-j}, H^R_{-j}\rangle} \mathrm{e}^{-\mu_j^- x} \text{ for } -1\leqslant z\leqslant 1, -1<y<0, x>0. \tag{5.23}$$

From (5.9, 6.20, 6.26) we know that there is a constant b such that for all j

$$\left|\frac{\Psi_{-j}(y)}{\langle v\Psi_{-j}, H^R_{-j}\rangle} H^R_{-j}(z)\mathrm{e}^{-\mu_j^- x}\right| \leqslant b(\mu_j^-)^{1/6}\mathrm{e}^{-\mu_j^-\delta} \quad \text{when } x\geqslant\delta, -1\leqslant z\leqslant 1\ .$$

Also from (6.19) we know that there is a constant b such that for all j

$$\left|\partial_z^i\partial_x^k \frac{\Psi_{-j}(y)}{\langle v\Psi_{-j}, H^R_{-j}\rangle} H^R_{-j}(z)\mathrm{e}^{-\mu_j^- x}\right| \leqslant b(\mu_j^-)^{k+i/2+1/6}\mathrm{e}^{-\mu_j^-\delta} \quad \text{when } x\geqslant\delta, -1\leqslant z\leqslant 1.$$

Thus the series in (5.20) can be integrated and be differentiated term-by-term, so we conclude that (5.20) satisfies (5.23).

Similarly the half-range reversed Green function $h_+(x,z,y)$, which satisfies

$$vh_{+x} = \mathscr{L}h_+ + \delta(x)\delta(z-y) \quad \text{for all } x<0,\ -1<y<0,\ -1<z<1\ , \tag{5.24}$$

$$B^-h_+ = 0, \quad B^+h_+ = 0 \quad \text{for } x<0, \quad h_+(x,z,y) \text{ bounded as } x\to-\infty, \tag{5.25}$$

is given by

$$h_+(x,z,y) = \sum_{j=0}^{\infty} \frac{\Psi_{+j}(y)\,H^L_{+}(z)}{\langle v\Psi_{+j}, H^R_{+j}\rangle} \mathrm{e}^{\mu_j^- x} \quad \text{for } -1\leqslant z\leqslant 1\ -1<y<0, x<0\ .$$

6 Asymptotic analysis

Here we use the results of Olver [24] to find asymptotic expressions for $H^L(\lambda,z)$ and $H^R(\lambda,z)$ and their first two derivatives. These formulas are used to obtain rigorous bounds on $H^L(\lambda,z)$ and $H^R(\lambda,z)$, on the eigenfunctions $H^L_{\pm k}(z)$ and $H^R_{\pm k}(z)$ and on their derivatives, and to find the asymptotic behaviour of the Wronskian $W(\lambda)$. We also use these expressions to determine the distribution of eigenvalues $\pm\mu_k^{\pm}$ for large k and to find the asymptotic behaviour of, and bounds on, the Green function $G(\lambda,z,y)$.

To transfer the equation $\mathscr{L}H = \lambda vH$ to a canonical form we define $\zeta(z)$ by

$$\frac{2}{3}\zeta^{3/2} = \int_0^z \sqrt{v(z')/a(z')}\,\mathrm{d}z' \ \text{ for } z>0, \quad \frac{2}{3}(-\zeta)^{3/2} = \int_z^0 \sqrt{-v(z')/a(z')}\,\mathrm{d}z' \ \text{ for } z<0.$$

Note that $v/(a\zeta)$ is bounded. In fact, it is in C^2 for $-1\leqslant z\leqslant 1$ since $v(z)$ and $a(z)$ are in C^3 with $v'(0)\neq 0$. Define $h(\lambda,\zeta) = (av/\zeta)^{1/4}H(\lambda,z)$, so that (2.4) becomes

$$h_{\zeta\zeta} = (\lambda\zeta - \tilde{c})h, \tag{6.1}$$

where the coefficient $\tilde{c}$

$$\tilde{c}(\zeta) = \frac{a\zeta}{v}\left\{\frac{c}{a} + (v/a\zeta)^{1/4}[(a\zeta/v)^{1/4}]_{zz} - a^{-1/2}(a^{1/2})_{zz}\right\}$$

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is a continuous function. Below we use the notation

$$q \equiv \lambda^{1/2} = q_r + iq_i, \quad q_1 = (-\lambda)^{1/2} \equiv q_r' + iq_i', \tag{6.2}$$

$$I_R = \int_0^1 \sqrt{v/a}\,\mathrm{d}z, \quad I_L = \int_{-1}^0 \sqrt{-v/a}\,\mathrm{d}z. \tag{6.3}$$

$$\theta_R = q_1 I_R + \frac{\pi}{4}, \quad \theta_L = qI_L + \frac{\pi}{4}. \tag{6.4}$$

6.0.1 *The sector* $|\arg\lambda| \leqslant \pi - \delta$.

Let $\delta > 0$ be an arbitrarily small, but fixed, constant. In Olver [24], it has been proved that (6.1) has solutions $h^1(\lambda, \zeta)$ and $h^2(\lambda, \zeta)$ which satisfy

$$h^1 = \begin{cases} \mathrm{Ai}(q^{2/3}\zeta)[1 + C_1/q] & 0 \leqslant z \leqslant 1 \\ \\ \mathrm{Ai}(q^{2/3}\zeta)[1 + C_2/q] + C_3\mathrm{Bi}(q^{2/3}\zeta)/q & -1 \leqslant z \leqslant 0 \end{cases} \tag{6.5}$$

$$h^2 = \begin{cases} \mathrm{Bi}(q^{2/3}\zeta)[1 + C_4/q] & 0 \leqslant z \leqslant 1 \\ \\ \mathrm{Bi}(q^{2/3}\zeta)[1 + C_5/q] + C_6\mathrm{Ai}(q^{2/3}\zeta)/q & -1 \leqslant z \leqslant 0\,. \end{cases} \tag{6.6}$$

Here Ai and Bi denote the Airy functions, and $C_i \equiv C_i(\lambda, z)$ are all functions which are uniformly bounded over the relevant domain. So (6.5) and (6.6) give the asymptotic behaviour of h^1 and h^2 as $\lambda \to \infty$ with explicitly stated error estimates.

From now on we omit the subscripts i, and it is understood that each occurrence of C in any expression refers to a different function $C(\lambda, z)$ which is uniformly bounded for all λ and z in the stated domain of validity of the expression. In a similar spirit, each occurrence of κ in any expression refers to a distinct function $\kappa(\lambda)$ which is bounded for all λ in the relevant sector.

It has also been proven [24] that formally differentiating (6.5) and (6.6) with respect to ζ (without differentiating the error terms C/q) yields the correct asymptotic expressions for the first two derivatives of h^1 and h^2.

We superimpose h^1 and h^2 to form $H^R(\lambda, z)$. Using the asymptotic formulas for the Airy functions in the sector $|\arg\lambda| \leqslant \pi - \delta$ [26] we obtain

$$H^R(\lambda, z) = \sqrt{\pi} q^{1/6} \left| \frac{a(1)v(1)}{a(z)v(z)} \zeta(z) \right|^{1/4} \mathrm{e}^{qI_R} \left\{ \left(1 + \frac{C}{q}\right) \mathrm{Ai}(q^{2/3}\zeta) + \frac{1}{2}\mathrm{e}^{-2qI_R} \left(1 + \frac{C}{q}\right) (q^{2/3}\zeta) \right\}, \quad 0 \leqslant z \leqslant 1 \tag{6.7}$$

$$H^R(\lambda, z) = \sqrt{\pi} q^{1/6} \left| \frac{a(1)v(1)}{a(z)v(z)} \zeta(z) \right|^{1/4} \mathrm{e}^{qI_R} \left\{ \left(1 + \frac{C}{q}\right) \mathrm{Ai}(q^{2/3}\zeta) + \frac{C}{q}(q^{2/3}\zeta) \right\} \quad -1 \leqslant z \leqslant 0. \tag{6.8}$$

The asymptotic formulas for the Airy functions can be used to simplify (6.7) and (6.8)

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when z is bounded away from zero. For any arbitrarily small, but fixed, positive constant δ' we have

$$H^R(\lambda,z)=\left|\frac{a(1)v(1)}{a(z)v(z)}\right|^{1/4}\left\{\left(1+\frac{C}{q}\right)\cosh\left(q\int_z^1\sqrt{v/a}\,\mathrm{d}z'\right)+\frac{C}{q}\sinh\left(q\int_z^1\sqrt{v/a}\,\mathrm{d}z'\right)\right\},\quad \delta'\leqslant z\leqslant 1 \tag{6.9}$$

$$H^R(\lambda,z)=\left|\frac{a(1)v(1)}{a(z)v(z)}\right|^{1/4}\mathrm{e}^{qI_R}\left\{\left(1+\frac{C}{q}\right)\sin\left(q\int_z^0\sqrt{-v/a}\,\mathrm{d}z'+\frac{\pi}{4}\right)+\frac{C}{q}\cos\left(q\int_z^0\sqrt{-v/a}\,\mathrm{d}z'+\frac{\pi}{4}\right)\right\},\quad -1\leqslant z\leqslant-\delta' \tag{6.10}$$

in the sector. In addition, the correct asymptotic behaviour for the first two derivatives of H^R is obtained by formally differentiating (6.7, 6.8, 6.9) and (6.10) with respect to z.

We start from a different pair of basis functions to obtain $H^L(\lambda,z)$. The results in Olver [24] prove that (6.1) also has solutions h^1,h^2 which satisfy

$$h^1=\begin{cases}[\sin\theta_L\mathrm{Ai}(q^{2/3}\zeta)+\cos\theta_L\mathrm{Bi}(q^{2/3}\zeta)](1+C/q)\\ +C[\sin\theta_L\mathrm{Bi}(q^{2/3}\zeta)-\cos\theta_L\mathrm{Ai}(q^{2/3}\zeta)]/q & -1\leqslant z\leqslant 0\\ \left[\cos\theta_L+\kappa/q\sin\theta_L\right]\mathrm{Bi}(q^{2/3}\zeta)](1+C/q)\\ +[\sin\theta_L+\kappa/q\cos\theta_L]\mathrm{Ai}(q^{2/3}\zeta)(1+C/q) & 0\leqslant z\leqslant 1,\end{cases} \tag{6.11}$$

$$h^2=\begin{cases}\left[\sin\theta_L\mathrm{Bi}(q^{2/3}\zeta)-\cos\theta_L\mathrm{Ai}(q^{2/3}\zeta)\right](1+C/q)\\ +C\left[\sin\theta_L\mathrm{Ai}(q^{2/3}\zeta)+\cos\theta_L\mathrm{Bi}(q^{2/3}\zeta)\right]/q & -1\leqslant z\leqslant 0\\ \left[\sin\theta_L+\kappa(\cos\theta_L)/q\right]\mathrm{Bi}(q^{2/3}\zeta)(1+C/q)\\ -\left[\cos\theta_L+\kappa(\sin\theta_L)/q\right]\mathrm{Ai}(q^{2/3}\zeta)(1+C/q) & 0\leqslant z\leqslant 1.\end{cases} \tag{6.12}$$

We now superimpose h^1 and h^2 to form $H^L(\lambda,z)$. This shows that in the sector $|\arg\lambda|\leqslant\pi-\delta$, $H^L(\lambda,z)=h^1(\lambda,z)$ $-1\leqslant z\leqslant 1$. For any constant $\delta'>0$, this reduces to

$$H^L(\lambda,z)=\left|\frac{a(-1)v(-1)}{a(z)v(z)}\right|^{1/4}\left\{(1+C/q)\cos\left(q\int_{-1}^z\sqrt{-v/a}\,\mathrm{d}z'\right)+\frac{C}{q}\sin\left(q\int_{-1}^z\sqrt{-v/a}\,\mathrm{d}z'\right)\right\},\quad -1\leqslant z\leqslant-\delta' \tag{6.13}$$

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$$H^L(\lambda,z)=\left|\frac{a(-1)v(-1)}{a(z)v(z)}\right|^{1/4}\left\{\left(1+\frac{C}{q}\right)\left[\cos\theta_L+\frac{\kappa}{q}\sin\theta_L\right]\exp\left(q\int_0^z\sqrt{v/a}\,\mathrm{d}z'\right)\right.$$
$$\left.+\frac{1}{2}\left(1+\frac{C}{q}\right)\sin\theta_L\exp\left(-q\int_0^z\sqrt{v/a}\,\mathrm{d}z'\right)\right\},\quad \delta'\leqslant z\leqslant 1. \tag{6.14}$$

Moreover, formally differentiating (6.11, 6.13, 6.14) gives the correct asymptotic expressions for H^L_z and H^L_{zz}.

To prove convergence in §4 and §5, we need to bound $H^L(\lambda,z)$ and $H^R(\lambda,z)$, and to obtain uniform bounds on the eigenfunctions $H^L(\mu_k^+,z)$ and $H^L(\mu_k^-,z)$. To bound $H^L(\lambda,z)$ for $z\leqslant 0$, we break q into its real and imaginary parts, as in (6.2), and use (6.11) to obtain for $-1\leqslant z\leqslant 0$

$$|\sin\theta_L\mathrm{Ai}(q^{2/3}\zeta)+\cos\theta_L\mathrm{Bi}(q^{2/3}\zeta)|\leqslant b|q|^{-1/6}|\zeta|^{-1/4}\exp\left(|q_i|\int_{-1}^z\sqrt{-v/a}\,\mathrm{d}z'\right),$$
$$|\sin\theta_L\mathrm{Bi}(q^{2/3}\zeta)-\cos\theta_L\mathrm{Ai}(q^{2/3}\zeta)|\leqslant b|q|^{-1/6}|\zeta|^{-1/4}\exp\left(|q_i|\int_{-1}^z\sqrt{-v/a}\,\mathrm{d}z'\right), \tag{6.15}$$

over the sector $|\arg\lambda|\leqslant\pi-\delta$ for some constant b. Similarly, for all $t\geqslant 0$,

$$|\mathrm{Ai}(-q^{2/3}t)|\leqslant b|q|^{-1/6}t^{-1/4}\exp\{\tfrac{2}{3}|q_i|t^{3/2}\},\ |\mathrm{Bi}(-q^{2/3}t)|\leqslant b|q|^{-1/6}t^{-1/4}\exp\{\tfrac{2}{3}|q_i|t^{3/2}\},$$

$$|\mathrm{Ai}(q^{2/3}t)|\leqslant b|q|^{-1/6}t^{-1/4}\exp\{-\tfrac{2}{3}q_rt^{3/2}\},\quad |\mathrm{Bi}(q^{2/3}t)|\leqslant b|q|^{-1/6}t^{-1/4}\exp\{\tfrac{2}{3}q_rt^{3/2}\}$$

over the sector $|\arg\lambda|\leqslant\pi-\delta$ for some other constant b. Consequently, for all λ in this sector,

$$|H^L(\lambda,z)|\leqslant\frac{b}{|a(z)v(z)|^{1/4}}\begin{cases}\exp(|q_i|\int_{-1}^z\sqrt{-v/a}\,\mathrm{d}z') & -1\leqslant z\leqslant 0\\ \exp(|q_i|I_L)\exp(q_r\int_0^z\sqrt{v/a}\,\mathrm{d}z') & 0\leqslant z\leqslant 1\,,\end{cases}$$

$$|H^R(\lambda,z)|\leqslant\frac{b}{|a(z)v(z)|^{1/4}}\begin{cases}\exp(q_rI_R)\exp(|q_i|\int_z^0\sqrt{-v/a}\,\mathrm{d}z') & -1\leqslant z\leqslant 0\\ \exp(q_r\int_z^1\sqrt{v/a}\,\mathrm{d}z') & 0\leqslant z\leqslant 1\,.\end{cases} \tag{6.16}$$

for some constant b.

To bound the eigenfunctions $H^L_{+k}(z)\equiv H^L(\mu_k^+,z)$ and $H^R_{+k}(z)\equiv H^R(\mu_k^+,z)$ we recall that $H^L(\lambda,z)$ and $H^R(\lambda,z)$ are linearly dependent at $\lambda=\mu_k^+$. Hence,

$$H^L_{+k}(z)\equiv\frac{H^L_{+k}(0)}{H^R_{+k}(0)}H^R_{+k}(z),\quad \frac{\mathrm{d}}{\mathrm{d}z}H^L_{+k}(z)\equiv\frac{H^L_{+k}(0)}{H^R_{+k}(0)}\frac{\mathrm{d}}{\mathrm{d}z}H^R_{+k}(z), \tag{6.17}$$

and so on. Since the eigenvalues are real and positive, (6.7, 6.8, 6.11) show that for some constant b, $|H^L_{+k}(0)/H^R_{+k}(0)|\leqslant b\exp(-qI_R)$ for all k. When λ is real and positive, $\mathrm{Ai}(q^{2/3}\zeta)$ is bounded for all ζ, $\mathrm{Bi}(q^{2/3}\zeta)$ is bounded for all $\zeta\leqslant 0$, and $(\zeta/av)^{1/4}$ is bounded for $-1\leqslant z\leqslant 1$. Thus (6.7) and (6.8) imply that for any fixed $\delta''>0$,

$$|H^R(\lambda,z)|\leqslant bq^{1/6}\mathrm{e}^{qI_R}\quad\text{for } -1\leqslant z\leqslant 1\text{ and all real }\lambda\geqslant\delta'' \tag{6.18}$$

for some constant b. Choosing $\delta''\leqslant\mu_0^+$ now shows that (6.18) holds at all the eigenvalues

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$\lambda = \mu_k^+$, and by using (6.17) we obtain

$$|H_{+k}^R(z)| \leqslant bq^{1/6}e^{qI_R}, \quad |H_{+k}^L(z)| \leqslant bq^{1/6} \quad \text{for all k}$$

for some b. Similarly, differentiating (6.7, 6.8, 6.11) we derive the bounds

$$\left.\begin{aligned} &|H_{+k}^R(z)| \leqslant bq^{1/6}e^{qI_R}, \ \left|\frac{d}{dz}H_{+k}^R(z)\right| \leqslant bqe^{qI_R}, \ \left|\frac{d^2}{dz^2}H_{+k}^R(z)\right| \leqslant bq^2e^{qI_R}, \text{ for } -1 \leqslant z \leqslant 1 \\ &|H_{+k}^L(z)| \leqslant bq^{1/6}, \ \left|\frac{d}{dz}H_{+k}^L(z)\right| \leqslant bq, \ \left|\frac{d^2}{dz^2}H_{+k}^L(z)\right| \leqslant bq^2, \text{ for } -1 \leqslant z \leqslant 1 \end{aligned}\right\} \tag{6.19}$$

for all k, some constant b, with $q = (\mu_k^+)^{1/2}$.

6.0.2 *The sector* $|\arg(-\lambda)| \leqslant \pi - \delta$

We conduct analogous calculations to find that for some constant b the eigenfunctions $H_{-k}^L(z)$ and $H_{-k}^R(z)$ and their derivatives are uniformly bounded by

$$\left.\begin{aligned} &|H_{-k}^R(z)| \leqslant bq_1^{1/6}, \qquad \left|\frac{d}{dz}H_{-k}^R(z)\right| \leqslant bq_1, \qquad \left|\frac{d^2}{dz^2}H_{-k}^R(z)\right| \leqslant bq_1^2 \\ &|H_{-k}^L(z)| \leqslant bq^{1/6}e^{q_1I_L}, \quad \left|\frac{d}{dz}H_{-k}^L(z)\right| \leqslant bq_1e^{q_1I_L}, \quad \left|\frac{d^2}{dz^2}H_{-k}^L(z)\right| \leqslant bq_1^2e^{q_1I_L} \end{aligned}\right\} \tag{6.20}$$

for all k, where $q = (\mu_k^+)^{1/2}$, $q_1 = (\mu_k^-)^{1/2}$ and $-1 \leqslant z \leqslant 1$.

6.1 Asymptotic behaviour of the Wronskian

Recalling that $W(\lambda) = B^-H^R(\lambda, -1)$ and using the asymptotic expression for $H^R(\lambda, z)$ in (6.10) we then prove that

$$W(\lambda) = |a(-1)v(-1)a(1)v(1)|^{1/4}e^{qI_R}\{(q+\kappa)\cos(qI_L + \pi/4) + \kappa\sin(qI_L + \pi/4)\} \tag{6.21}$$

for $|\arg\lambda| \leqslant \pi - \delta$, where $q = \lambda^{1/2}$, Similarly, for $|\arg(-\lambda)| \leqslant \pi - \delta$, where $q_1 = (-\lambda)^{1/2}$:

$$W(\lambda) = |a(-1)v(-1)a(1)v(1)|^{1/4}e^{q_1I_L}\{(q_1+\kappa)\cos(q_1I_R + \pi/4) + \kappa\sin(q_1I_R + \pi/4)\}. \tag{6.22}$$

The asymptotic behaviour of $1/W(\lambda)$ is the reciprocal of (6.21, 6.22). To bound the error in this expression for $1/W(\lambda)$, select a constant ϵ with $0 < \epsilon < 1/2$ and the set S_ϵ as defined in (4.11) and note that

$$\left|\frac{\sin(qI_L + \pi/4)}{\cos(qI_L + \pi/4)}\right| \leqslant 2/\epsilon, \quad \left|\frac{\sin(q_1I_R + \pi/4)}{\cos(q_1I_R + \pi/4)}\right| \leqslant 2/\epsilon \quad \text{for } \lambda \text{ in } S_\epsilon.$$

Therefore, for any ϵ with $0 < \epsilon < 1/2$ there is an R_ϵ such that

$$\frac{1}{W(\lambda)} = |a(-1)v(-1)a(1)v(1)|^{-1/4}\begin{cases} \dfrac{e^{-qI_R}}{q\cos(qI_L + \pi/4)}\left(1 + \dfrac{\kappa}{q}\right) & |\arg\lambda| \leqslant \pi - \delta \\ \dfrac{e^{-q_1I_L}}{q_1\cos(q_1I_R + \pi/4)}\left(1 + \dfrac{\kappa}{q_1}\right) & |\arg(-\lambda)| \leqslant \pi - \delta \end{cases} \tag{6.23}$$

for all λ in S_ϵ with $|\lambda| > R_\epsilon$.

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To determine the asymptotic behaviour of $W'(\lambda)$, we start with the fact that $W'(\lambda) = \langle vH^L(\lambda, z), H^R(\lambda, z)\rangle$. Substituting the asymptotic expressions (6.9, 6.10, 6.13, 6.14) for $H^R(\lambda, z)$ and $H^L(\lambda, z)$ into the inner product and carrying out the integration, we find that for all real λ,

$$W'(\lambda) = \frac{1}{2}|a(-1)v(-1)a(1)v(1)|^{1/4}e^{qI_R}\left\{ -\left(1+\frac{\kappa}{q}\right) I_L \sin(qI_L + \pi/4) \right.$$
$$\left. + \left(1+\frac{\kappa}{q}\right) I_R \cos(qI_L + \pi/4)\right\} \quad \text{as} \quad \lambda = q^2 \to \infty, \tag{6.24}$$

$$W'(\lambda) = \frac{1}{2}|a(-1)v(-1)a(1)v(1)|^{1/4}e^{q_1 I_L}\left\{ \left(1+\frac{\kappa}{q_1}\right) I_R \sin(q_1 I_R + \pi/4) \right.$$
$$\left. - \left(1+\frac{\kappa}{q_1}\right) I_L \cos(q_1 I_R + \pi/4)\right\} \quad \text{as} \quad \lambda = -q_1^2 \to -\infty\,. \tag{6.25}$$

Since $W'(\lambda)$ is non-zero at all eigenvalues, this proves that for some constant b,

$$|W'(-\mu_k^-)| = |\langle vH^L_{-k}, H^R_{-k}\rangle| \geqslant be^{q_1 I_L}, \quad |W'(\mu_k^+)| = |\langle vH^L_{+k}, H^R_{+k}\rangle| \geqslant be^{qI_R}, \tag{6.26}$$

for all k, where $q = (\mu_k^+)^{1/2}$ and $q_1 = (\mu_k^-)^{1/2}$.

6.2 Zeros of the eigenfunctions

We now need to determine the distribution of eigenvalues $\pm\mu_k^{\pm}$ for large k by using (6.21, 6.22) and to locate the zeros of $W(\lambda)$ for large λ. Consider first the positive eigenvalues $\lambda = \mu_k^+$. Equation (6.21) implies that there is a constant b_1 such that all positive zeros of $W(\lambda)$ must occur when $|\cos(qI_L + \pi/4)| \leqslant b_1/q$ where $q = \lambda^{1/2}$ is real and positive. Therefore, there is a constant Q such that any positive zero of $W(\lambda)$ with $q > Q$ must lie in one of the intervals

$$\frac{(k+1/4)\pi}{I_L} - \frac{3}{2}\frac{b_1}{(k+1/4)\pi} \leqslant q \leqslant \frac{(k+1/4)\pi}{I_L} + \frac{3}{2}\frac{b_1}{(k+1/4)\pi}, \quad k = 0, 1, 2, \ldots\,. \tag{6.27}$$

For some constant k, each such interval with $k > K$ contains at least one zero, since (6.21) implies that $W(q^2)$ changes sign at least once in each of these intervals. For some constant b_2,

$$\left| -\left(1+\frac{\kappa}{q}\right) I_L \sin(qI_L + \pi/4) + \left(1+\frac{\kappa}{q}\right) I_R \cos(qI_L + \pi/4)\right| \geqslant b_2\left(I_L + \frac{\kappa}{q}\right)$$

for all q in the intervals (6.27). Therefore (6.21) implies that $W'(\lambda) \neq 0$ in all intervals (6.27) with k large enough. So for some K, each interval in (6.27) with $k > K$ contains exactly one eigenvalue μ_j^+. Recall that the eigenfunction $H(\mu_k^+, z)$ has exactly k zeros. From (6.7) and (6.8) we note that for q sufficiently large, zeros of $H^R(q^2, z)$ occur for $-1 \leqslant z < 0$ and only if $|\sqrt{\pi}(3t/2)^{1/6}\mathrm{Ai}(-[3t/2]^{3/2})| \leqslant \epsilon$ where $t \equiv 2q|\zeta|^{3/2}/3$. We now count the zeros of the eigenfunction $H^R(q^2, t)$ whose eigenvalue lies in the interval (6.27). Denoting by

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$-z_j$ the j-th zero of the Airy function, and noting that $2z_j^{3/2}/3 = \pi(j - 1/4) + O(j)$ [26], we see that for k large enough, this eigenfunction has exactly k zeros, and $(\mu_k)^{1/2}$ lies in the interval defined in (6.27). By choosing a new constant b we write for all k

$$\left|\mu_k^+ - \frac{(k+\tfrac{1}{4})^2\pi^2}{I_L^2}\right| \leqslant b \quad \text{for all } k = 0, 1, 2, \ldots \tag{6.28}$$

for some constant b. A similar argument establishes

$$\left|\mu_k^- - \frac{(k+\tfrac{1}{4})^2\pi^2}{I_R^2}\right| \leqslant b \quad \text{for all } k = 0, 1, 2, \ldots \tag{6.29}$$

for some other constant b.

6.3 The function $E(\lambda)$

Now we show that the entire function $E(\lambda) \equiv \mathcal{N}(\lambda)\mathcal{M}(\lambda)/W(\lambda)$ introduced in §3 is constant. Equations (6.23, 7.4, 7.6) imply that for any small enough $\epsilon > 0$, there is a R_ϵ and a constant b such that

$$|E(\lambda)| < b \quad \text{for all } \lambda \text{ in } S_\epsilon \text{ with } |\lambda| > R_\epsilon, \tag{6.30}$$

where S_ϵ is defined in (4.11). The growth restriction (6.30) is enough to conclude that $E(\lambda)$ is constant. To see this, note that the Cauchy integral formula yields

$$E'(\lambda) = \frac{1}{2\pi i}\int_{C_R} \frac{E(\zeta)}{(\zeta - \lambda)^2}\,\mathrm{d}\zeta \quad \text{for all } |\lambda| < R,$$

since $E(\lambda)$ is entire, where C_R is the contour $\zeta = Re^{i\theta}$, $0 < \theta < 2\pi$. From (6.30) then, $|E'(\lambda)| \leqslant 4b/R$ for all $|\lambda| < R/2$, for any $R > R_\epsilon$ with $Re^{i\theta}$ in S_ϵ for $0 \leqslant \theta \leqslant 2\pi$. Letting $R \to \infty$ now shows that $E'(\lambda) = 0$ at all λ.

6.4 Asymptotic behaviour of the free space Green function

To evaluate the asymptotic behaviour of $G(\lambda, z, y)$ in (4.8) as $\lambda \to \infty$ in the sector $|\arg\lambda| < \pi - \delta$, we substitute the asymptotic expressions for H^L, H^R, and W into (4.8) (*cf.* 6.7, 6.8, 6.23) and use (6.15) and (6.16), to bound the error. Writing

$$G(\lambda, z, y) = G^0(\lambda, z, y) + E(\lambda, z, y),$$

we find that when $|\arg\lambda| < \pi - \delta$,

$$G^0(\lambda, z, y) = \frac{\pi}{q^{2/3}}\left|\frac{\zeta(z)\zeta(y)}{a(z)v(z)a(y)v(y)}\right|^{1/4} \cdot [\mathrm{Bi}(q^{2/3}\zeta(z)) + \tan\theta_L \mathrm{Ai}(q^{2/3}\zeta(z))] \times \left[\mathrm{Ai}(q^{2/3}\zeta(y)) + \frac{1}{2}\mathrm{e}^{-2qI_R}\mathrm{Bi}(q^{2/3}\zeta(y))\right]. \tag{6.31}$$

Moreover, for any ϵ with $0 < \epsilon < 1/2$, there is an R_ϵ and a constant b such that the error

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is bounded by

$$|E(\lambda,z,y)| \leqslant \frac{b}{|q|^2}|a(z)v(z)a(y)v(y)|^{-1/4} \times \begin{cases} \exp(-|q_i|\int_z^y \sqrt{-v/a}\,\mathrm{d}z') & \text{if } z \leqslant y \leqslant 0, \\ \exp(-|q_j|\int_z^0 \sqrt{-v/a}\,\mathrm{d}z')\exp(-q_r\int_0^y \sqrt{v/a}\,\mathrm{d}z') & \text{if } z \leqslant 0 \leqslant y, \\ \exp(-q_r\int_z^y \sqrt{v/a}\,\mathrm{d}z') & \text{if } 0 \leqslant z \leqslant y\ , \end{cases} \tag{6.32}$$

for all λ in S_ϵ with $|\lambda| > R_\epsilon$ and $|\arg\lambda| < \pi - \delta$. Similarly, as $\lambda \to \infty$ in the sector $|\arg(-\lambda)| < \pi - \delta$ we have

$$G^0(\lambda,z,y) = \frac{\pi}{q_1^{2/3}}\left|\frac{\zeta(z)\zeta(y)}{a(z)v(z)a(y)v(y)}\right|^{1/4} \cdot \left[\mathrm{Ai}(-q_1^{2/3}\zeta(z)) + \frac{1}{2}\mathrm{e}^{-2q_1 I_L}\mathrm{Bi}(-q_1^{2/3}\zeta(z))\right] \times [\mathrm{Bi}(-q_1^{2/3}\zeta(y)) + \tan\theta_R \mathrm{Ai}(-q_1^{2/3}\zeta(y))]. \tag{6.33}$$

For any ϵ with $0 < \epsilon < 1/2$, there is an R_ϵ and a constant b such that the error is bounded by

$$|E(\lambda,z,y)| \leqslant \frac{b}{|q_1|^2}|a(z)v(z)a(y)v(y)|^{-1/4} \times \begin{cases} \exp(-q_r'\int_z^y \sqrt{-v/a}\,\mathrm{d}z') & \text{if } z \leqslant y \leqslant 0, \\ \exp(-q_r'\int_z^0 \sqrt{-v/a}\,\mathrm{d}z')\mathrm{e}^{-|q|\int_0^y \sqrt{v/a}\,\mathrm{d}z'} & \text{if } z \leqslant 0 \leqslant y, \\ \exp(-|q_i'|\int_y^z \sqrt{v/a}\,\mathrm{d}z') & \text{if } 0 \leqslant z \leqslant y, \end{cases} \tag{6.34}$$

for all λ in S_ϵ with $|\lambda| > R_\epsilon$ and $|\arg(-\lambda)| < \pi - \delta$.

Equations (6.31)–(6.34) imply that for any ϵ with $0 < \epsilon < 1/2$, there is an R_ϵ and a constant b such that the Green function is bounded by

$$|G(\lambda,z,y)| \leqslant b|\lambda|^{-1/2}|v(z)v(y)|^{-1/4} \quad \text{for } -1 \leqslant z,y \leqslant 1.$$

7 Properties of $\mathscr{N}(\lambda)$ and $\mathscr{M}(\lambda)$

Recalling (6.29) we define $s_k = (k+1/4)^2\pi^2/I_k^2$ and we decompose $\mathscr{N}(\lambda)$ in (2.9) as $\mathscr{N}(\lambda) = C\mathscr{A}(\lambda)\widetilde{\mathscr{N}}(\lambda)$, where

$$C = \prod_{k=0}^{\infty} s_k/\mu_k^-, \quad \mathscr{A}(\lambda) = \prod_{k=0}^{\infty}\left\{1 + \frac{\mu_k^- - s_k}{\lambda + s_k}\right\}, \quad \widetilde{\mathscr{N}}(\lambda) = \prod_{k=0}^{\infty}(1 + \lambda/s_k).$$

Equation (6.29) shows that C converges, and $\mathscr{A}(\lambda)$ converges uniformly in every compact region of the λ plane which does not contain any of the poles $\lambda = -s_k$. Hence $\mathscr{A}(\lambda)$ is a meromorphic function whose only singularities are simple poles at $\lambda = -s_k$. Finally, $\widetilde{\mathscr{N}}(\lambda)$ is an entire function of λ.

The infinite product representation of the gamma function [26] enables us to recognize

$$\widetilde{\mathscr{N}}(\lambda) = -\frac{1}{\pi}\Gamma^2\left(\frac{1}{4}\right)\sin(q_1 I_R - \pi/4)\frac{\Gamma(q_1 I_R/\pi + 3/4)}{\Gamma(q_1 I_R/\pi + 1/4)}. \tag{7.1}$$

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Since $\widetilde{\mathscr{N}}(\lambda)$ is entire, which branch we choose in (7.1) is irrelevant. To be definite, we choose $|\arg(-\lambda)| \leqslant \pi, |\arg q_1| \leqslant \pi$. The asymptotic behaviour of the gamma function implies that

$$\widetilde{\mathscr{N}}(\lambda) = -\frac{1}{\pi}\Gamma^2\left(\frac{1}{4}\right)(I_R/\pi)^{1/2}q_1^{1/2}\sin\left(q_1 I_R - \frac{\pi}{4}\right)\left(1 + \frac{\kappa(\lambda)}{\lambda}\right), \tag{7.2}$$

where $\kappa(\lambda)$ is bounded for all λ. We prove that $\mathscr{A}(\lambda) \to 1$ uniformly in $|\lambda|^{-1}$ as $\lambda \to \infty$ in S'_ϵ where $S'_\epsilon = \{\text{all } \lambda \text{ with } |\cos((-\lambda)^{1/2}I_R + \pi/4)| \geqslant \epsilon\}$ for some $0 < \epsilon < 1/2$. Then for all K, and $\lambda \in S'_\epsilon$ there is an R_ϵ such that $b/|\lambda + s_k| < 1/2$ for $|\lambda| > R_\epsilon$, with the constant b as defined in (6.29). Thus,

$$|\log \mathscr{A}(\lambda)| \leqslant -\sum_{k=0}^{\infty}\log\left(1 - \frac{b}{|\lambda + s_k|}\right) \leqslant 2b\sum_{k=0}^{\infty}\frac{1}{|\lambda + s_k|}.$$

Evaluating the sum gives

$$|\log \mathscr{A}(\lambda)| \leqslant \frac{8bI_R}{\epsilon|\lambda|^{1/2}}\{1 + \log(6I_R|\lambda|^{1/2}/\epsilon)\} \quad \text{for all } \lambda \text{ in } S'_\epsilon \text{ with } |\lambda| > R_\epsilon. \tag{7.3}$$

Since $\mathscr{A}(\lambda)$ has no poles in S'_ϵ, (7.3) implies that there is a constant b_1 such that

$$|\mathscr{A}(\lambda) - 1| \leqslant b_1|\lambda|^{-1/4} \quad \text{for all } \lambda \text{ in } S'_\epsilon.$$

Together with (7.2) this proves that for any ϵ in $0 < \epsilon < \frac{1}{2}$, there is an R_ϵ such that

$$\mathscr{N}(\lambda) = K_1\sin(q_1 I_R - \pi/4)\{q_1^{1/2} + \kappa_1(\lambda)\} \text{ for all } \lambda \text{ in } S'_\epsilon \text{ with } |\lambda| > R_\epsilon, \tag{7.4}$$

where

$$q_1 = (-\lambda)^{1/2}, \quad K_1 = -\frac{1}{\pi}\Gamma^2\left(\frac{1}{4}\right)(I_R/\pi)^{1/2}\prod_{k=0}^{\infty}s_k/\mu_k^-,$$

and where $\kappa_1(\lambda)$ is bounded for all λ. In particular, (7.4) implies that there is an R'_ϵ and a constant b_2 such that

$$\frac{1}{|\mathscr{N}(\lambda)|} \leqslant b_2|q_1|^{-1/2}\mathrm{e}^{-|q'_i|I_R}, \quad |\mathscr{N}(\lambda)| \leqslant b_2|q_1|^{1/2}\mathrm{e}^{|q'_i|I_R} \tag{7.5}$$

for all λ in S'_ϵ, where q'_i is the imaginary part of q_1.

The asymptotic behaviour of $\mathscr{M}(\lambda)$ can be evaluated in a similar fashion. For $0 < \epsilon < \frac{1}{2}$, define $S''_\epsilon = \{\text{all } \lambda \text{ with } |\cos(\lambda^{1/2}I_L + \pi/4)| \geqslant \epsilon\}$. Then there is a constant R_ϵ and a constant b_3 such that

$$\mathscr{M}(\lambda) = K_2\sin(qI_L - \pi/4)\{q^{1/2} + \kappa_2(\lambda)\}, \tag{7.6}$$

$$\frac{1}{|\mathscr{M}(\lambda)|} \leqslant b_3|q|^{-1/2}\mathrm{e}^{-|q_i|I_L}, \quad |\mathscr{M}(\lambda)| \leqslant b_3|q|^{1/2}\mathrm{e}^{|q_i|I_L}$$

for all λ in S''_ϵ with $|\lambda| > R_\epsilon$. In the above, $\kappa_2(\lambda)$ is bounded for all λ and

$$q = \lambda^{1/2}, \quad K_2 = -\frac{1}{\pi}\Gamma^2\left(\frac{1}{4}\right)(I_L/\pi)^{1/2}\prod_{k=0}^{\infty}\frac{(k + 1/4)^{1/2}\pi^2}{I_L^2\mu_k^+}.$$

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7.0.1 *Derivatives*

At the eigenvalues, the derivatives of $\mathcal{N}(\lambda)$ and $\mathcal{M}(\lambda)$ can be expressed as

$$\mathcal{N}'(-\mu_m^-) = \frac{1}{\mu_m^-}\prod_{\substack{k=0\\k\neq m}}^{\infty}(1+\lambda/\mu_k^-) \quad \text{at } \lambda = -\mu_m^-, \tag{7.7}$$

$$\mathcal{M}'(\mu_m^+) = -\frac{1}{\mu_m^+}\prod_{\substack{k=0\\k\neq m}}^{\infty}(1-\lambda/\mu_k^+) \quad \text{at } \lambda = \mu_m^+. \tag{7.8}$$

The asymptotic behaviour of $\mathcal{N}'(-\mu_m^-)$ and $\mathcal{M}'(\mu_m^+)$ for large m can be found by evaluating the infinite products in (7.7, 7.8) the same manner as above. This yields

$$\mathcal{N}'(-\mu_m^-) = \frac{1}{2}(-1)^{m-1}K_1 I_R(\mu_m^-)^{-1/4}\{1+\kappa_m(\mu_m^-)^{-1/4}\}, \tag{7.9}$$

$$\mathcal{M}'(\mu_m^+) = \frac{1}{2}(-1)^{m}K_2 I_L(\mu_m^+)^{-1/4}\{1+\kappa'_m(\mu_m^+)^{-1/4}\}, \tag{7.10}$$

where the sequences κ_m and κ'_m are bounded . In particular, (7.9, 7.10) imply that there are constants b_1 and b_2 such that

$$b_1(\mu_m^-)^{-1/4} \leqslant |\mathcal{N}'(-\mu_m^-)| \leqslant b_2(\mu_m^-)^{-1/4}, \tag{7.11}$$

$$b_1(\mu_m^+)^{-1/4} \leqslant |\mathcal{M}'(\mu_m^+)| \leqslant b_2(\mu_m^+)^{-1/4}, \tag{7.12}$$

for all m.

8 Proofs of Lemmas 4.1 and 5.1

8.1 Proof of Lemma 4.1

For this section only, we normalize the eigenfunctions so that $\langle vH_{-k}, H_{-k}\rangle = 1$, $\langle vH_{+k}H_{+k}\rangle = -1$ for $k = 0, 1, 2, \ldots$. We consider the partial sums

$$h_-^m(z) = h_-(z) = \sum_{k=0}^{m} a_k H_{-k}(z), \quad h_+^m(z) = h_+(z) = \sum_{k=0}^{m} c_k H_{+k}(z), \tag{8.1}$$

where a_k and c_k are arbitrary sequences of complex numbers. The eigenfunctions are orthogonal with the weighting factor $v(z)$, but they are not orthogonal with $|v(z)|$, so

$$\langle |v|h_-, h_-\rangle = \sum_{j,k=0}^{m} a_j^* a_k \langle |v|H_{-j}, H_{-k}\rangle, \quad \langle |v|h_+, h_+\rangle = \sum_{j,k=0}^{m} c_j^* c_k \langle |v|H_{+j}, H_{+k}\rangle. \tag{8.2}$$

First, we show that for some constant b, and for all sequences a_j, c_j and all m

$$\langle |v|h_-, h_-\rangle \leqslant b\sum_{k=0}^{m}|a_k|^2, \quad \langle |v|h_+, h_+\rangle \leqslant b\sum_{k=0}^{m}|c_k|^2.$$

Since $\langle vH_{-j}, H_{-k}\rangle = \delta_{jk}$, we have

$$\langle |v|H_{-j}, H_{-k}\rangle = \delta_{jk} + 2\int_{-1}^{0}|v(z)|H_{-j}(z)H_{-k}(z)\mathrm{d}z, \tag{8.3}$$

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$$\langle |v|H_{+j}, H_{+k}\rangle = \delta_{jk} + 2\int_0^1 v(z)H_{+j}(z)H_{+k}(z)\mathrm{d}z.$$

We define the real symmetric $(m+1)\times(m+1)$ matrices $\mathbf{D}^-$ and $\mathbf{D}^+$ by

$$D^-_{jk} = \int_{-1}^0 |v(z)|H_{-j}(z)H_{-k}(z)\mathrm{d}z,\ D^+_{jk} = \int_0^1 v(z)H_{+j}(z)H_{+k}(z)\mathrm{d}z,\ j,k = 0,1,\ldots,m. \quad (8.4)$$

We will estimate the off-diagonal terms in (8.4) by establishing bounds on the eigenvalues of $\mathbf{D}^-$ and $\mathbf{D}^+$ which are independent of m. Recall that the Airy functions are bounded by $0 \leqslant \mathrm{Ai}(t) \leqslant bt^{-1/4}\exp\{-2t^{3/2}/3\}$, $\quad 0 \leqslant \mathrm{Bi}(t) \leqslant bt^{-1/4}\exp\{2t^{3/2}/3\}$ for all $t \geqslant 0$ for some constant b [26]. Substituting these bounds into (8.4) and using

$$\int_0^\infty t\mathrm{Ai}(\alpha t)\mathrm{Ai}(\beta t)\mathrm{d}t = \frac{1}{\alpha^2+\alpha\beta+\beta^2} \quad \text{if } \alpha > 0 \text{ or } \beta > 0,$$

we prove that for some constant b,

$$|D^-_{jk}| \leqslant \frac{b(q_jq_k)^{1/6}}{q_j^{4/3}+q_j^{2/3}q_k^{2/3}+q_k^{4/3}}, \quad \text{with } q_j = (\mu_j^-)^{1/2} \text{ and } q_k = (\mu_k^-)^{1/2},$$

$$|D^+_{jk}| \leqslant \frac{b(q_jq_k)^{1/6}}{q_j^{4/3}+q_j^{2/3}q_k^{2/3}+q_k^{4/3}}, \quad \text{with } q_j = (\mu_j^+)^{1/2} \text{ and } q_k = (\mu_k^+)^{1/2},$$

for all j and k. Since the eigenvalues $\mu_k^\pm$ are distributed according to (6.28, 6.29), we conclude that there is a constant b such that

$$|D^-_{jk}| \leqslant \frac{b(j+1/4)^{1/6}(k+1/4)^{1/6}}{(j+1/4)^{4/3}+(j+1/4)^{2/3}(k+1/4)^{2/3}+(k+1/4)^{4/3}},$$

$$|D^+_{jk}| \leqslant \frac{b(j+1/4)^{1/6}(k+1/4)^{1/6}}{(j+1/4)^{4/3}+(j+1/4)^{2/3}(k+1/4)^{4/3}+(k+1/4)^{4/3}},$$

for all j and k.

Since $\mathbf{D}^-$ and $\mathbf{D}^+$ are real symmetric matrices, their eigenvalues are real. Moreover, provided that at least one of the first m coefficients a_k is non-zero, we have

$$\sum_{j,k=0}^m a_j^*a_kD^-_{jk} = \int_{-1}^0 |v(z)||h_-(z)|^2\mathrm{d}z > 0, \quad \sum_{j,k=0}^m a_j^*a_kD^+_{jk} = \int_0^1 v(z)|h_+(z)|^2\mathrm{d}z > 0. \quad (8.5)$$

Since (8.5) holds for arbitrary sequences, clearly all eigenvalues of $\mathbf{D}^-$ and $\mathbf{D}^+$ are strictly positive for any m.

We now bound the eigenvalues from above. Pick any $m > 0$, and let γ^- be any eigenvalue of $\mathbf{D}^-$. Then

$$\begin{aligned} 0 < \gamma^- &\leqslant \max_{0\leqslant j\leqslant m}\sum_{k=0}^m |D^-_{jk}| \\ &\leqslant \max_{0\leqslant j\leqslant m}\sum_{k=0}^m \frac{b(j+1/4)^{1/6}(k+1/4)^{1/6}}{(j+1/4)^{4/3}+(j+1/4)^{2/3}(k+1/4)^{2/3}+(k+1/4)^{4/3}}. \end{aligned} \quad (8.6)$$

Since

$$\int_0^\infty \frac{(j+1/4)^{1/6}(k+1/4)^{1/6}}{(j+1/4)^{4/3}+(j+1/4)^{2/3}(k+1/4)^{2/3}+(k+1/4)^{4/3}}\mathrm{d}k \leqslant \sqrt{3}\pi \text{ for all } j \geqslant 0, \quad (8.7)$$

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the eigenvalues γ^- of $\mathbf{D}^-$ are bounded independent of m. A similar argument shows that the eigenvalues of $\mathbf{D}^+$ are bounded independent of m. Thus there is a constant γ_{max} such that all eigenvalues γ^- of $\mathbf{D}^-$ and all eigenvalues of γ^+ of $\mathbf{D}^+$ are in the interval

$$0 < \gamma^- < \gamma_{max}, \quad 0 < \gamma^+ < \gamma_{max} \quad \text{for all } m. \tag{8.8}$$

To prove part (a) of Lemma 4.1, let $\mathrm{e}(z)$ be any function in $L^2[v]$, and define $a_k = \langle vH_{-k}, \mathrm{e}\rangle$, $\quad c_k = \langle vH_{+k}, e\rangle$, and

$$\tilde{e}(z) = \begin{cases} e(z) & \text{for } z > 0 \\ -e(z) & \text{for } z < 0. \end{cases}$$

Then

$$a_k = \langle |v| H_{-k}, \tilde{e}\rangle, \quad c_k = \langle |v| H_{+k}, \tilde{e}\rangle, \quad \langle |v| e, e\rangle = \langle |v| \tilde{e}, \tilde{e}\rangle. \tag{8.9}$$

Pick any integer $m \geqslant 0$, and define $e_m(z) = \sum_{k=0}^{m} t_k H_{-k}(z)$ where the coefficients $\mathbf{t} = (t_0, t_l, \ldots, t_m)^T$ will be picked shortly. We have,

$$0 \leqslant \langle |v|(\tilde{e} - e_m), (\tilde{e} - e_m)\rangle =$$
$$\langle |v| \tilde{e}, \tilde{e}\rangle - \sum_{j=0}^{m} \{t_j^* \langle |v| H_{-j}, \tilde{e}\rangle + t_j \langle |v| H_{-j}, \tilde{e}\rangle^*\} + \sum_{j,k=0}^{m} t_j^* t_k \langle |v| H_{-j}, H_{-k}\rangle.$$

Using (8.3) and (8.9) (and switching to matrix notation), we obtain

$$\mathbf{t}^* \cdot \mathbf{a} + \mathbf{a}^* \cdot \mathbf{t} - \mathbf{t}^* \cdot (\mathbf{I} + 2\mathbf{D}^-)\mathbf{t} \leqslant \langle |v| e, e\rangle, \tag{8.10}$$

where $\mathbf{a} = (a_0, a_1, \ldots, a_m)^T$. Since the eigenvalues of $\mathbf{D}^-$ are all positive, the matrix $\mathbf{I} + 2\mathbf{D}^-$ is invertible. We select $\mathbf{t} = (\mathbf{I} + 2\mathbf{D}^-)^{-1}\mathbf{a}$ to make the inequality (8.10) as sharp as possible; it then becomes $\mathbf{a}^*(\mathbf{I} + 2\mathbf{D}^-)^{-1}\mathbf{a} \leqslant \langle |v| e, e\rangle$. Equation (8.8) implies that all eigenvalues of $\mathbf{I} + 2\mathbf{D}^-$ are between 1 and $1 + 2\gamma_{max}$, and so $\mathbf{a}^* \cdot (\mathbf{I} + 2\mathbf{D}^-)^{-1} \mathbf{a} > \mathbf{a}^* \cdot \mathbf{a}/(1 + 2\gamma_{max})$ for any integer $m \geqslant 0$. Hence,

$$\sum_{k=0}^{m} |a_k|^2 < (1 + 2\gamma_{max}) \langle |v| e, e\rangle$$

for all integers $m \geqslant 0$. In a similar manner we can prove that

$$\sum_{k=0}^{m} |c_k|^2 < (1 + 2\gamma_{max}) \langle |v| e, e\rangle$$

for all integers $m \geqslant 0$. This establishes part (a) of Lemma 4.1.

To prove part (b), let a_k and c_k be any sequence of complex numbers. Note that for any $n \geqslant m \geqslant 0$, we have

$$\left\langle |v(z)| \sum_{k=m}^{n} a_k H_{-k}(z), \sum_{k=m}^{n} a_k H_{-k}(z) \right\rangle = \sum_{j,k=m}^{n} a_j^* a_k (\delta_{jk} + 2D_{jk}^-).$$

Since the eigenvalues of $\mathbf{I} + 2\mathbf{D}^-$ are all between 1 and $1 + 2\gamma_{max}$, this yields

$$\left\langle |v(z)| \sum_{k=m}^{n} a_k H_{-k}(z), \sum_{k=m}^{n} a_k H_{-k}(z) \right\rangle < (1 + 2\gamma_{max}) \sum_{k=m}^{n} |a_k|^2.$$

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Similarly,

$$\left\langle |v(z)| \sum_{k=m}^{n} c_k H_{+k}(z), \sum_{k=m}^{n} c_k H_{+k}(z) \right\rangle < (1 + 2\gamma_{max}) \sum_{k=m}^{n} |c_k|^2$$

for any $n \geqslant m \geqslant 0$. If we now suppose that

$$\sum_{k=0}^{\infty} |a_k|^2 < \infty, \quad \sum_{k=0}^{\infty} |c_k|^2 < \infty,$$

then $h_-^m(z)$ and $h_+^m(z)$ in (8.1) are Cauchy sequences in $L^2[v]$. Since $L^2[v]$ is complete, both

$$h_-(z) = \sum_{k=0}^{\infty} a_k H_{-k}(z), \quad h_+(z) = \sum_{k=0}^{\infty} c_k H_{+K}(z)$$

must converge in $L^2[v]$ to functions $h_-(z)$ and $h_+(z)$ which satisfy

$$\langle |v| h_-, h_- \rangle < (1 + 2\gamma_{max}) \sum_{k=0}^{\infty} |a_k|^2, \quad \langle |v| h_+, h_+ \rangle < (1 + 2\gamma_{max}) \sum_{k=0}^{\infty} |c_k|^2.$$

In particular, $h_-(z) + h_+(z)$ must also be in $L^2[v]$ and satisfy

$$\langle |v|(h_- + h_+), (h_- + h_+) \rangle < (2 + 4\gamma_{max}) \sum_{k=0}^{\infty} \{|a_k|^2 + |c_k|^2\},$$

which establishes part (b) of Lemma 4.1.

8.2 Proof of Lemma 5.1

We use the original eigenfunctions $H_{\pm k}^L(z)$ and $H_{\pm k}^R(z)$, which are normalized to unity at the left and right boundaries, respectively. We know that the series (3.3) for $\Psi_{-j}(z)$ converges uniformly, so we can integrate term-by-term to obtain

$$d_j = \frac{\langle vH_{-j}^R, H_{-j}^R \rangle^{1/2}}{\langle vH_{-j}^L, H_{-j}^R \rangle} \langle vH_{-j}^L, f \rangle + \frac{\langle vH_{-j}^R, H_{-j}^R \rangle^{1/2}}{\langle vH_{-j}^L, H_{-j}^R \rangle} \mathscr{M}(-\mu_j^-) \sum_{k=0}^{\infty} \frac{\langle vH_{+k}^L, f \rangle}{(\mu_k^+ + \mu_j^-)\mathscr{M}'(\mu_k^+)} \tag{8.11}$$

and then rewrite (8.11) as

$$d_j = a_j + \sum_{k=0}^{\infty} D_{jk} c_k,$$

where

$$D_{jk} = \frac{\langle vH_{-j}^R, H_{-j}^R \rangle^{1/2}}{\langle vH_{-j}^L, H_{-j}^R \rangle} \mathscr{M}(-\mu_j^-) \frac{|\langle vH_{+k}^L, H_{+k}^L \rangle|^{1/2}}{(\mu_k^+ + \mu_j^-)\mathscr{M}'(\mu_k^+)}$$

and

$$a_j = \frac{\langle vH_{-j}^R, H_{-j}^R \rangle^{1/2}}{\langle vH_{-j}^L, H_{-j}^R \rangle} \langle vH_{-j}^L, f \rangle = \frac{\langle vH_{-j}^L, f \rangle}{\langle vH_{-j}^L, H_{-j}^L \rangle^{1/2}}, \quad c_j = \frac{\langle vH_{+j}^L, f \rangle}{|\langle vH_{+j}^L, H_{+j}^L \rangle|^{1/2}}.$$

Consider the $(m+1) \times (m+1)$ matrix given by $\mathbf{D} = [D_{jk}]$ with j and k running from 0 to m. Part (a) of Lemma 4.1 shows that for some constant b,

$$\sum_{j=0}^{\infty} |a_j|^2 < b \parallel f \parallel_v^2, \sum_{j=0}^{\infty} |c_j|^2 < b \parallel f \parallel_v^2 .$$

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So we can prove Lemma 5.1, by obtaining a bound on the eigenvalues of **D**.

We need to bound the entries in the matrix **D**. Equations (6.9, 6.10, 6.20, 7.11, 7.12) show that for some constants b and b' we then have

$$|D_{jk}| < b\frac{(\mu_j^- \mu_k^+)^{1/4}}{\mu_j^- + \mu_k^+} < b'\frac{[(j+1/4)(k+1/4)]^{1/4}}{(j+1/4)^2+(k+1/4)^2}, \quad \text{for all } j \text{ and } k.$$

For each m, the eigenvalues γ of **D** are bounded by

$$|\gamma| \leqslant \max_{0\leqslant j\leqslant m} \sum_{k=0}^{m} |D_{jk}| \leqslant \max_{0\leqslant j\leqslant m} \sum_{k=0}^{m} b'\frac{[(j+1/4)(k+1/4)]^{1/4}}{(j+1/4)^2+(k+1/4)^2}.$$

The latter sum is bounded independent of m, so there is a constant b such that the eigenvalues γ of **D** are bounded by $|\gamma| < b$ for all m. Consequently, there is a constant b such that

$$\left|\sum_{k=0}^{\infty} D_{jk}c_k\right|^2 < b\sum_{k=0}^{\infty}|c_k|^2,$$

which establishes Lemma 5.1.

9 Conclusion

In this paper we considered the transport problem 1.1 with half-range boundary conditions 1.2. We assumed that the transport operator $\mathscr{L}$ is a negative-definite Sturm–Liouville operator with simple eigenvalues. We explicitly constructed the solutions to the half-range boundary value problems, and showed that the expansions are unique and converge to the conditions prescribed on the boundary. Thus, we validated and extended earlier formal constructions.

One type of analysis of the half-range expansion problem is concerned with projecting eigenfunctions of the full range eigenvalue problem onto the space spanned by half of the eigenfunctions, which correspond (say) to the negative eigenvalues. Earlier theoretical results established the existence and uniqueness of such a projection [10]. In contrast, we have explicitly constructed this projection, and expressed it in the form of infinite products and sums. We have provided the rate of convergence of these products and sums. This would allow one to estimate the number of terms needed to solve a particular problem to within a desired precision.

In our analysis we relied on the asymptotic form of the eigenvalues of the operator $\mathscr{L}$. However, the techniques presented here can be extended to treat a more general class of transport operators with discrete spectrum, as motivated by formal constructions [17, 18, 19, 20].

There are many open problems associated with half-range expansion problems. The treatment of operators with continuous, or mixed continuous-discrete spectra has yet to be tackled. Other types of boundary conditions are also important, and require non-trivial extensions to the methods used here.

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