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Vector Solutions for Great Circle Navigation

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Traditionally, navigation has been taught with methods employing Napier's rules for spherical triangles while methods derived from vector analysis and calculus appear to have been avoided in the teaching environment. In this document, vector methods are described that allow distance and azimuth at any point on a great circle to be determined. These methods are direct and avoid reliance on the formulae of spherical trigonometry. The vector approach presented here allows waypoints to be established without the need to either ascertain the position of the vertex or select the nearest pole; the method discussed here requires only one spherical triangle having an apex at the North Pole and is also easy to implement on a small computer.

KEY WORDS

1. Geodesy. 2. Vectors. 3. Computing.

1. INTRODUCTION. Although the Earth is a spheroid produced by an ellipsoid of revolution having small eccentricity as currently defined by the World Geodetic Standard of 1984 (WGS84), the spherical earth model is still often used in the teaching of navigation and is sufficiently accurate for practical navigation. The spherical earth model is one for which any great circle has a circumference of $360 \times 60 = 21600$ arc minutes. Each minute spans a distance of one nautical mile. On dividing by 2π, to find the radius R_e, the result is $21600 \div 2\pi$ or 3937·7468 nautical miles. In this case, the nautical mile is the International Nautical mile equal to 1852 metres exactly. It is well understood that the shortest distance between two points on any surface is a geodesic. On the spherical earth a geodesic curve is a great circle, which is defined as the circle formed by the intersection of the spherical surface and a plane that passes through the centre of the sphere. Such a plane divides the sphere into two equal parts.

In dealing with great circle navigation, some texts[1] demonstrate the finding of solutions for positions on a great circle that require the vertex to be established. In fact, in a recent paper[2], the authors comprehensively demonstrate the complexity of the computation procedure based upon the use of conventional methods. The direct vector based approach described here does not require prior knowledge of the vertex, but when required, the vertex location emerges naturally from the great circle equation. In essence, once two positions have been given, the great circle and its

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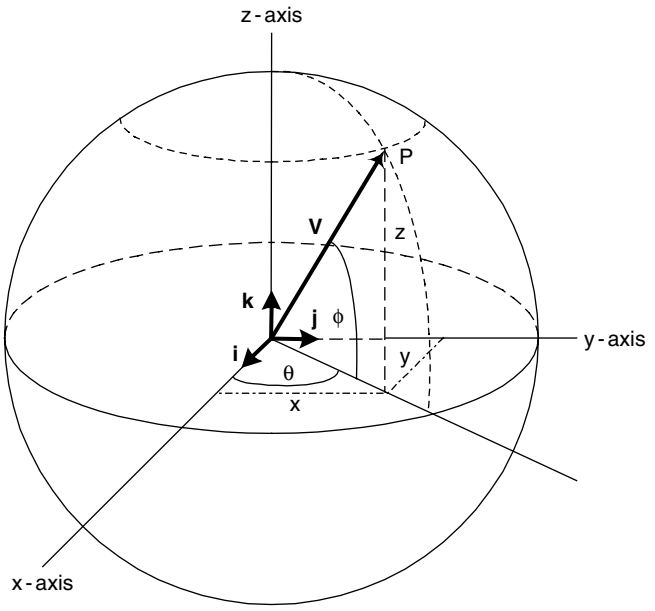


Figure 1. Vector co-ordinates.

properties are fixed. Furthermore, only one spherical triangle is needed from which to determine the azimuth, hence the course to steer for any waypoint. The vector-based approach described here was also employed in an earlier paper[3] aimed at obtaining vector-based results for the great ellipse.

In what follows, a description of the vector method will be provided showing a vector solution for distance between two given locations. Next, the vector solution to finding the azimuth angle, hence the course to steer, will be discussed. This is followed by the extraction of the great circle equation from the scalar triple product of three vectors. With this equation, all other waypoints both within the span of two given locations or beyond can be found through repeated applications of the formula. In addition, its turning points reveal the location of the vertex.

2. SURFACE POSITIONS AND CO-ORDINATES.

Co-ordinate pairs known as Latitude (La) and Longitude (Lo) usually specified in degrees and minutes are used to define a point on the Earth's surface. These two angles are also expressed in radians as ϕ, θ as in Figure 1. Latitude is measured north or south of the equatorial plane, while longitude is measured east or west of a designated prime meridian. At the equator, the latitude is 0 degrees and at the prime meridian the longitude is also 0 degrees. Directions north of the equator are considered as positive numbers for computational purposes and reach 90 degrees at the northern pole. Directions southwards are negative and likewise reach 90 degrees at the southern pole. Longitude is measured east or west from 0 degrees at the prime meridian and does not exceed 180 degrees. For computational purposes east is considered to be positive while west is negative.

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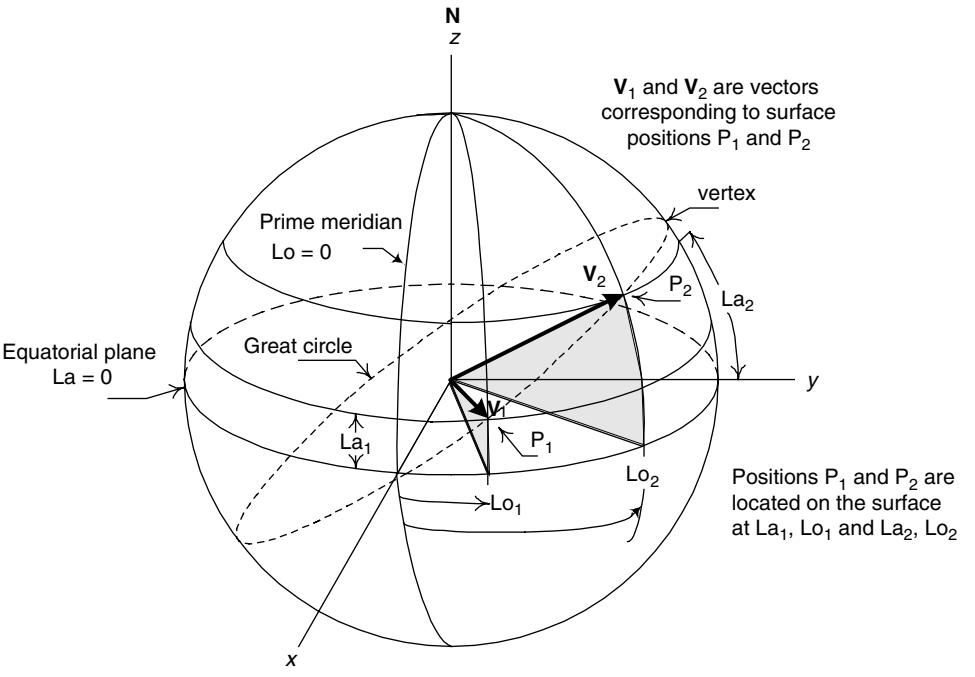


Figure 2. Coordinates and vectors of two positions.

3. VECTOR METHODS. Vectors drawn from the centre of the sphere terminate on the surface and correspond to given positions on that surface. It is convenient to consider a sphere of unit radius and introduce the actual radius R_e into the calculations when it is required. Corresponding to a point P on the surface of the unit sphere with a given latitude and longitude, is a vector of position $\mathbf{V}$ (Figure 1) defined in Cartesian coordinates by:

$$\mathbf{V} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \tag{1}$$

$$\begin{aligned} &\text{Where} \quad x = \cos(\phi)\cos(\theta) \quad y = \cos(\phi)\sin(\theta) \quad z = \sin(\phi) \\ &\text{So that} \quad \mathbf{V} = \mathbf{i}\cos(\phi)\cos(\theta) + \mathbf{j}\cos(\phi)\sin(\theta) + \mathbf{k}\sin(\phi) \end{aligned} \tag{2}$$

In these expressions, x, y and z are the three components of the vector $\mathbf{V}$ that lie on the three co-ordinate axes, while $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors that also lie on each axis at the origin (Figure 1). It is convenient to define a polar vector $\mathbf{V}_p = \mathbf{k}$. With these definitions, the x, y, z co-ordinates form a right-handed set when viewed from the equatorial plane along the polar axis towards the N pole.

Given two points P_1 and P_2 on the surface, where P_1 is the departure point and P_2 the destination, their corresponding position vectors are (Figure 2):

$$\mathbf{V}_1 = \mathbf{i}\cos(\phi_1)\cos(\theta_1) + \mathbf{j}\cos(\phi_1)\sin(\theta_1) + \mathbf{k}\sin(\phi_1) \tag{3a}$$

$$\text{i.e.} \quad \mathbf{V}_1 = \mathbf{i}x_1 + \mathbf{j}y_1 + \mathbf{k}z_1 \tag{3b}$$

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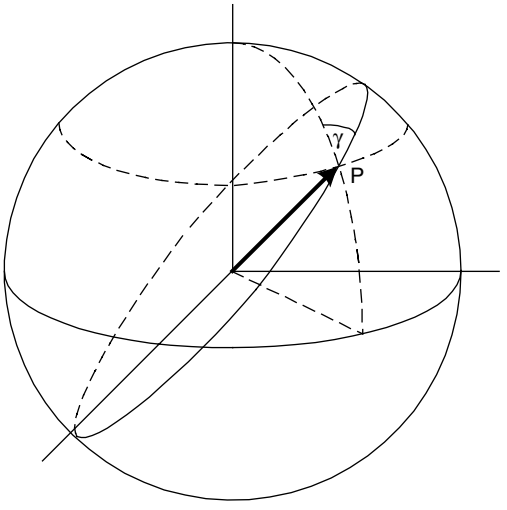


Figure 3. Azimuth angle γ at point P.

$$\text{and} \qquad \mathbf{V_2} = \mathbf{i}\cos(\phi_2)\cos(\theta_2) + \mathbf{j}\cos(\phi_2)\sin(\theta_2) + \mathbf{k}\sin(\phi_2) \tag{3c}$$

$$\text{i.e.} \qquad \mathbf{V_2} = \mathbf{i}x_2 + \mathbf{j}y_2 + \mathbf{k}z_2 \tag{3d}$$

From these equations, the great circle can now be defined.

4. DISTANCE BETWEEN TWO POINTS. Distance along the great circle passing through the two given points P_1 and P_2 is found from the following vector dot product expression[4]:

$$D = R_e \arccos[\mathbf{V_1} \cdot \mathbf{V_2}]. \tag{4}$$

Where D is the distance along the arc P_1, P_2. On expanding, the dot product of Equation 4 gives:

$$D = R_e \arccos(x_1x_2 + y_1y_2 + z_1z_2). \tag{5}$$

Further expansion of Equation 5 into its trigonometric terms is unnecessary if computer evaluation proceeds from this point.

5. AZIMUTH ANGLE AND COURSE TO STEER. Azimuth angle γ is the angle between the meridian through P and any track that crosses that meridian (Figure 3.). This angle is the same as the angle between the normal to the meridian plane containing P and the normal to the plane of the great circle. It is found from the following vector expression[4,3]:

$$\text{Cos}\ \gamma = \left(\frac{(\mathbf{V_1} \mathbf{x} \mathbf{V_p})}{|\mathbf{V_1} \mathbf{x} \mathbf{V_p}|} \cdot \frac{(\mathbf{V_1} \mathbf{x} \mathbf{V_2})}{|\mathbf{V_1} \mathbf{x} \mathbf{V_2}|} \right) \tag{6}$$

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Upon expanding the two key terms of Equation 6 we first get for the normal vector to the meridian plane:

$$\mathbf{V_1 x V_p} = \mathbf{i} y_1 - \mathbf{j} x_1 \tag{7}$$

Next, since the vector cross product $\mathbf{V_1 x V_2}$ is determined from two fixed vectors, it is therefore a unique vector constant normal to the plane of the great circle that passes through P_1 and P_2. It has a value of

$$\mathbf{V_1 x V_2} = \mathbf{i} m + \mathbf{j} n + \mathbf{k} p \tag{8}$$

for which the constants m, n and p are

$$m = (y_1 z_2 - y_2 z_1) \qquad n = (x_2 z_1 - x_1 z_2) \qquad p = (x_1 y_2 - x_2 y_1) \tag{9}$$

On substituting Equations 7 and 8 into Equation 6, the azimuth angle emerges as:

$$\gamma = \arccos\left(\frac{m y_1 - n x_1}{\sqrt{(m^2 + n^2 + p^2)(x_1^2 + y_1^2)}}\right) \tag{10}$$

Here again it is remarked that for computer evaluation no further expansion into trigonometric terms is required.

The initial course to steer C_s, depends upon whether the course is from W to E in which case $C_s = \gamma$, otherwise $C_s = 2\pi - \gamma$. Since these angles are in radians they are converted to degrees using $C_s{}^{o} = 180\ C_s \div \pi$

6. EQUATION OF THE GREAT CIRCLE. All points on the great circle passing through the given points P_1 and P_2 can be found once the equation for the great circle has been determined. The equation of the great circle passing through the two points P_1 and P_2 is found from the scalar triple product of three vectors[4,3], two of which are the known vectors for the two given points of the track and the third one $\mathbf{V}$, is unknown and is to be determined. This unknown vector lies in the plane of the great circle defined by the vectors $\mathbf{V_1}$, $\mathbf{V_2}$ when their scalar triple product[4,3] is zero,

$$\text{i.e. when} \quad \mathbf{V} \bullet (\mathbf{V_1 x V_2}) = 0 \tag{11}$$

Since the vector cross product $\mathbf{V_1 x V_2}$ contains the constants m, n, p for the great circle as described previously (Equation 8), then the scalar triple product expression becomes:

$$\mathbf{V} \bullet (\mathbf{V_1 x V_2}) = m \cos(\phi) \cos(\theta) + n \cos(\phi) \sin(\theta) + p \sin(\phi) = 0 \tag{12}$$

The solution for which is the equation for the great circle namely:

$$\phi = -\arctan[\lambda \cos(\theta) + \mu \sin(\theta)] \tag{13}$$

In which $\lambda = m/p$ and $\mu = n/p$. From this equation, the latitude of any point on the great circle can be established once the longitude is specified.

7. WAYPOINTS. Having determined the equation for the great circle, then for a selected waypoint longitude Lo_x, (i.e. θ_x), the corresponding waypoint

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latitudes ϕ_x can be found using Equation 13. The waypoint vector V_x can then be constructed as:

$$V_x = \mathbf{i}\cos(\phi_x)\cos(\theta_x) + \mathbf{j}\cos(\phi_x)\sin(\theta_x) + \mathbf{k}\sin(\phi_x) \tag{14}$$

Substituting Equation 14 into Equations 4 and 6 will determine distance and azimuth angle between the waypoint at P_x, and the destination point P_2. This procedure can be repeated for other waypoints.

8. THE VERTEX. For sailings over large distances, the great circle track has the least length. It also reaches a maximum latitude N and S called a vertex. Knowledge of where the vertex will occur may be important to the navigator in that it may attain latitudes that are unsafe, for example, due to ice. The navigator may then choose to sail part of his track at constant latitude, thereby following a compound track with specified limiting latitude. The N and S vertexes ϕ_v exist at the turning points of the great circle which occur when $d\phi/d\theta = 0$. On differentiating Equation 13 we find that this condition occurs at a value of longitude θ_v satisfying the condition:

$$\frac{d\phi}{d\theta} = \frac{\lambda\sin\theta - \mu\cos\theta}{\sec^2\phi} = 0 \tag{15}$$

i.e. when

$$\theta_v = \arctan\left(\frac{\mu}{\lambda}\right) \tag{16}$$

Upon substituting this value of longitude into Equation 13, the latitude of the vertex is found to occur at

$$\phi_v = \pm\arctan\sqrt{\lambda^2 + \mu^2} \tag{17}$$

Each great circle crosses the equator at two points called nodes, which occur either side of a vertex at $\theta_o = \theta_v \pm \pi/2$.

9. CONCLUSION. A vector-based method of computing the properties of great circle paths and of establishing waypoints has been described and which offers computational simplicity without recourse to Napier's rules or any prior establishment of the vertex. Only one spherical triangle is required from which vector expression have been derived. The method described will provide satisfactory results differing slightly from those found from the spheroidal model and which will be well within what practical course keeping provides. No attention has been given to presenting actual computational algorithms since these are now conveniently found among the toolsets of commercial mathematical packages. For those proficient in computer programming languages, the expressions presented lend themselves readily to that particular discipline.

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