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# On the classification of quaternionic Möbius transformations

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*Abstract*

In this paper we consider quaternionic Möbius transformations preserving the unit ball in the quaternions $\mathbb{H}$. In other words, maps of the form $g(z) = (az + b)(cz + d)^{-1}$ where $a$, $b$, $c$ and $d$ all lie in $\mathbb{H}$ with the property that $|g(z)| < 1$ for all $|z| < 1$. We give an explicit expression for the fixed points of $g$ in terms of $a$, $b$, $c$ and $d$ and we use this to classify quaternionic Möbius transformations into six categories determined by their dynamics.

## 1. *Introduction*

It is well known that the orientation preserving isometries of $n$-dimensional hyperbolic space, $\mathbf{H}^n$, are the Möbius transformations $M(\overline{\mathbb{R}^{n-1}})$. It is important to classify Möbius transformations by their dynamics and their fixed points. It is also generally important that these classes should be conjugacy invariants. When $n = 2$ or $n = 3$ this is very well known and completely standard.

For arbitrary dimensions, Ahlfors [**1**] related Möbius transformations to two-by-two matrices whose entries lie in (a suitably chosen part of) a Clifford algebra. In particular, he established a Clifford algebraic equation for fixed a point $v$ of the Möbius map $g(z) = (az + b)(cz + d)^{-1}$ as follows:

$$t^2 - (cac^{-1} + d)t + cc^{*-1} = 0, \quad t = cv + d.$$

Because of the lack of commutativity, this equation is not trivially solvable. Ahlfors [**1**, **2**] discussed the classification of elements in $M(\overline{\mathbb{R}^{n-1}})$ for $n \geqslant 4$ avoiding directly solving the above equation. Waterman [**11**], Wang et al. [**12**] and others improved

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Ahlfors' results. Cao and Waterman [**3**] gave an alternative method for finding fixed points by analysing the balls preserved by a Möbius transformation.

In the cases of $n = 4$ and $n = 5$ it is possible to relate Möbius transformations to two-by-two quaternionic matrices. This can either be thought of as the restriction of Ahlfors' method to these dimensions [**3**, **7**, **8**], or else as a low dimensional case of the analysis of quaternionic hyperbolic spaces [**4**, **9**]. Kellerhals has recently used quaternionic Möbius transformations to study isometries of hyperbolic 4-space [**7**] and hyperbolic 5-space [**8**] and the geometry of the corresponding hyperbolic manifolds.

In this paper we concentrate on the case $n = 4$. The model of $\mathbf{H}^4$ we choose to work with is the unit ball $\mathbb{B}$ in the quaternions $\mathbb{H}$. The (orientation preserving) isometries of this model are Möbius transformations of the form $g(z) = (az + b)(cz + d)^{-1}$ where $a, b, c, d \in \mathbb{H}$ and $z \in \overline{\mathbb{B}}$. We will show:

LEMMA 1·1. *The quaternionic Möbius transformation* $g(z) = (az + b)(cz + d)^{-1}$, *where* $a, b, c, d \in \mathbb{H}$, *maps the unit ball* $\mathbb{B} \subset \mathbb{H}$ *to itself if and only if*

$$|a| = |d|, \quad |b| = |c|, \quad |a|^2 - |c|^2 = 1, \quad \overline{a}b = \overline{c}d, \quad a\overline{c} = b\overline{d}. \tag{1·1}$$

These conditions may be thought of as an analogy of the familiar result that a Möbius transformation $g(z) = (az + b)/(cz + d)$ in $\mathrm{PSL}(2, \mathbb{C})$ preserves the unit disc in $\mathbb{C}$ (that is $g \in \mathrm{PU}(1, 1; \mathbb{C})$) if and only if $b = \overline{c}$, $d = \overline{a}$ and so $|a|^2 - |c|^2 = 1$. In Section 2 we prove Lemma 1·1 using the connection between Möbius transformations and matrices in $\mathrm{U}(1, 1; \mathbb{H})$; see Chen and Greenberg [**4**]. The group $\mathrm{U}(1, 1; \mathbb{H})$ is sometimes called $\mathrm{Sp}(1, 1)$.

Suppose that $g(z) = (az + b)(cz + d)^{-1}$ is a Möbius transformation preserving $\mathbb{B}$. We will show that the point $v \in \overline{\mathbb{B}} \subset \mathbb{H}$ is fixed by $g$ if and only if $0 = vcv + vd - av - b$. Using this equation we find that $t = cv + d$ is a right-eigenvalue (see [**5**]) for the corresponding matrix

$$\tilde{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{U}(1, 1; \mathbb{H}). \tag{1·2}$$

It then follows that $t$ must satisfy the following equation (which closely resembles Ahlfors' equation):

$$t^2 - (d + \overline{c}^{-1}b\overline{d})t + \overline{c}^{-1}b = 0. \tag{1·3}$$

We solve this equation using methods similar to those used by Huang and So [**6**]. Because $a, b, c, d$ satisfy (1·1), our solutions take a simpler form than those of Huang and So; in particular we do not need to solve a cubic equation. Solving (1·3) for $t$ and hence finding the fixed points $v$ will be the main content of the paper. Our main results give completely explicit formulae for the fixed points of $g$ in terms of $a$, $b$, $c$, $d$ and quantities easily calculated from them.

As we solve the equation (1·3), we are also able to classify $g$ by its fixed points and dynamics. The classification we give has six categories, each preserved under conjugation. We show how to determine which of these classes $g$ belongs to, the calculation being given solely in terms of $a$, $b$, $c$ and $d$.

Given a quaternionic Möbius transformation $g$, consider the associated matrix $\tilde{g}$ in $\mathrm{U}(1, 1; \mathbb{H})$. It may be that $g$ is conjugate to an isometry of the Poincaré disc, that

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is, $\tilde{g}$ is conjugate to an element of $\mathrm{U}(1,1;\mathbb{C})$ (and so is conjugate in $\mathrm{SL}(2,\mathbb{C})$ to an element of $\mathrm{SL}(2,\mathbb{R})$). We say that:

(i) $g$ (or $\tilde{g}$) is *simple* if $\tilde{g}$ is conjugate in $\mathrm{U}(1,1;\mathbb{H})$ to an element of $\mathrm{U}(1,1;\mathbb{C})$;

(ii) $g$ (or $\tilde{g}$) is *compound* otherwise.

The reason for the term compound is that, as is well known, if $\tilde{g}$ is compound then it is the product of two simple matrices; compare [**4**, theorem 3.4.1]. In the next section we will prove the following result characterising simple maps in $\mathrm{U}(1,1;\mathbb{H})$; compare [**3**, theorem 5.5].

PROPOSITION 1·2. *Let $\tilde{g} \in \mathrm{U}(1,1;\mathbb{H})$ have the form (1·2). Then $\tilde{g}$ is simple if and only if $b = \overline{c}$ and $\Re(a) = \Re(d)$. Moreover, in this case $\Re(d)$ is invariant under conjugation.*

A second method of classification is the standard trichotomy based on fixed points; see [**4**, page 63], [**11**, page 96]. Let $g$ be a quaternionic Möbius transformation preserving $\mathbb{B}$. We say that:

(i) $g$ is *elliptic* if it has at least one fixed point in $\mathbb{B}$;

(ii) $g$ is *parabolic* if it has exactly one fixed point and this point lies in $\partial\mathbb{B}$;

(iii) $g$ is *loxodromic* if it has exactly two fixed points and these points lie in $\partial\mathbb{B}$.

Simple parabolic maps are also known as pure parabolic or strictly parabolic, compound parabolic maps as screw parabolic, and simple loxodromic maps as hyperbolic; see [**3**, **4**].

Our main theorem is:

THEOREM 1·3. *Let $g(z) = (az+b)(cz+d)^{-1}$ be a Möbius transformation preserving $\mathbb{B} \subset \mathbb{H}$.*

(*a*) *If $b = c = 0$ then $g$ is elliptic and*:

(i) *if $\Re(a) = \Re(d)$ then $g$ is simple elliptic*;

(ii) *if $\Re(a) \neq \Re(d)$ then $g$ is compound elliptic.*

(*b*) *If $b = \overline{c} \neq 0$ then*:

(i) *if $\Re(d)^2 < 1$ then $g$ is simple elliptic*;

(ii) *if $\Re(d)^2 = 1$ then $g$ is simple parabolic*;

(iii) *if $\Re(d)^2 > 1$ then $g$ is simple loxodromic.*

(*c*) *If $b \neq \overline{c} \neq 0$ define: $\Delta = |\Im((\overline{c}^{-1}b - 1)\overline{d})|^2 - |\overline{c}^{-1}b - 1|^2$. Then*:

(i) *if $\Delta < 0$ then $g$ is compound elliptic*;

(ii) *if $\Delta = 0$ then $g$ is compound parabolic*;

(iii) *if $\Delta > 0$ then $g$ is compound loxodromic.*

Observe that the factors $\Re(d)^2 - 1$ and $\Delta$ act as discriminants. For simple maps, $\Re(a) = \Re(d)$ and so the discriminant in this case is the more familiar $(\Re\,\mathrm{tr}(g))^2 - 4$, which is conjugation invariant.

The fixed point set of a quaternionic Möbius transformation $g$ is defined by

$$\mathrm{fix}(g) = \{v \in \overline{\mathbb{B}} : g(v) = v\}.$$

In each of the cases of Theorem 1·3 we will find an explicit expression for $\mathrm{fix}(g)$ as follows.

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THEOREM 1·4. *Suppose that* $g(z) = azd^{-1}$. *Then*:

(i) *if* $\Re(a) = \Re(d)$ *then* $a$ *and* $d$ *are similar, so there is a unit quaternion* $q$ *with* $d = qaq^{-1}$. *Then* $v \in \mathrm{fix}(g)$ *if and only if* $qv \in \mathbb{R}(d)$ *and* $|v| \leqslant 1$;

(ii) *if* $\Re(a) \neq \Re(d)$ *then* $\mathrm{fix}(g) = \{0\}$.

THEOREM 1·5. *Suppose that* $g(z) = (az + \overline{c})(cz + d)^{-1}$ *where* $c \neq 0$. *Then* $v \in \mathrm{fix}(g)$ *if and only if* $v = c^{-1}(t - d)$ *and* $|v| \leqslant 1$ *where the right eigenvalue* $t$ *is given as follows*:

(i)
$$t = \Re(d) + \sqrt{1 - \Re(d)^2}\, u$$
*for any unit modulus imaginary quaternion* $u$ *if* $\Re(d)^2 < 1$;

(ii)
$$t = \Re(d) \pm \sqrt{\Re(d)^2 - 1}$$
*if* $\Re(d)^2 \geqslant 1$.

We remark that the condition in Theorem 1·5 (i) is the same as saying that $N = |t|^2 = 1$ and $T = 2\Re(t) = 2\Re(d)$. These are conjugacy invariants whereas $t$ itself is not; see Section 3.

THEOREM 1·6. *Suppose that* $g(z) = (az+b)(cz+d)^{-1}$ *where* $b \neq \overline{c} \neq 0$. *Define* $\sigma = \overline{c}^{-1}b$ *and* $\tau = d + \sigma\overline{d}$. *Also define* $\Delta = |\Im(\tau)|^2 - |\sigma - 1|^2$. *Then* $v \in \mathrm{fix}(g)$ *if and only if* $|v| \leqslant 1$ *and* $v$ *can be written as* $v = c^{-1}(t-d)$ *for a right eigenvalue* $t$ *given by* $t = (\tau - T)^{-1}(\sigma - N)$ *where* $N = |t|^2$ *and* $T = 2\Re(t)$ *are given as follows*:

(i)
$$T = \Re(\tau) \pm \sqrt{-\Delta}, \quad N = 1$$
*if* $\Delta \leqslant 0$;

(ii)
$$T = \Re(\tau) \pm \sqrt{2X - 2 - \Delta}, \quad N = X \pm \sqrt{X^2 - 1}$$
*where*
$$X = \frac{\Re(\tau)^2 + \Delta + \sqrt{(\Re(\tau)^2 + \Delta - 4)^2 + 16\Delta}}{4}$$
*if* $\Delta > 0$.

The following corollary classifies quaternionic Möbius transformations by their right eigenvalues and their fixed points. It is immediate from the preceding results.

COROLLARY 1·7. *A non-trivial Möbius transformation* $g$ *preserving* $\mathbb{B}$ *is*:

(i) *simple elliptic if* $\tilde{g}$ *has right eigenvalues* $t$ *with* $|t| = 1$ *and a two dimensional family of fixed points*;

(ii) *compound elliptic if* $\tilde{g}$ *has right eigenvalues* $t$ *with* $|t| = 1$ *and a unique fixed point in* $\mathbb{B}$;

(iii) *simple parabolic if* $\tilde{g}$ *has the repeated right eigenvalue* $t = \pm 1$;

(iv) *compound parabolic if* $\tilde{g}$ *has a repeated right eigenvalue* $t \neq \pm 1$ *with* $|t| = 1$ *and a fixed point on* $\partial\mathbb{B}$;

(v) *simple loxodromic if* $\tilde{g}$ *has the real right eigenvalues* $t$ *and* $1/t$ *where* $t \neq \pm 1$;

(vi) *compound loxodromic if* $\tilde{g}$ *has non-real right eigenvalues* $t_1$ *and* $t_2$ *with* $|t_2| = |t_1|^{-1}$ *and* $|t_1| \neq 1$.

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## 2. *Preliminaries*

In this section we give conditions that ensure a quaternionic Möbius transformation preserves $\mathbb{B}$. We relate quaternionic Möbius transformations to matrices in $\mathrm{U}(1,1;\mathbb{H})$, that is $\mathrm{Sp}(1,1)$, and we characterise simple quaternionic Möbius transformations.

Let $\mathbb{H}$ denote the division ring of real quaternions. Elements of $\mathbb{H}$ have the form $z = z_1 + z_2\mathbf{i} + z_3\mathbf{j} + z_4\mathbf{k} \in \mathbb{H}$ where $z_i \in \mathbb{R}$ and

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Let $\overline{z} = z_1 - z_2\mathbf{i} - z_3\mathbf{j} - z_4\mathbf{k}$ be the *conjugate* of $z$, and define the *modulus* of $z$ to be:

$$|z| = \sqrt{\overline{z}z} = \sqrt{z_1^2 + z_2^2 + z_3^2 + z_4^2}.$$

We define $\Re(z) = (z + \overline{z})/2$ to be the *real part* of $z$, and $\Im(z) = (z - \overline{z})/2$ to be the *imaginary part* of $z$. Also $z^{-1} = \overline{z}|z|^{-2}$ is the *inverse* of $z$. For $z \in \mathbb{H}$, we let $\mathbb{R}(z)$ be the smallest sub-ring containing $\mathbb{R}$ and $z$. If $z$ is real then $\mathbb{R}(z) = \mathbb{R}$. Otherwise $\mathbb{R}(z)$ is given by $\mathbb{R}(z) = \mathbb{R} \oplus \mathbb{R}z$, which is isomorphic to $\mathbb{C}$. Observe that $\Re(wzw^{-1}) = \Re(z)$ and $|wzw^{-1}| = |z|$ for all $z$ and all non-zero $w$ in $\mathbb{H}$, but when $z \in \mathbb{H} \setminus \mathbb{R}$ we have $\Im(wzw^{-1}) = \Im(z)$ if and only if $w \in \mathbb{R}(z)$. Two quaternions $z$ and $w$ are *similar* if there exists non-zero $q \in \mathbb{H}$ such that $z = qwq^{-1}$. The similarity class of $z$ is the set $\{qzq^{-1} : q \in \mathbb{H} - \{0\}\}$. Quaternions $z$ and $w$ are similar if and only if $|z| = |w|$ and $\Re(z) = \Re(w)$; see Huang and So [**5**].

Let $\mathbb{H}^{1,1}$ be the left vector space of dimension 2 over $\mathbb{H}$ with the unitary structure defined by the Hermitian form

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* J \mathbf{z} = \overline{w_1} z_1 - \overline{w_2} z_2,$$

where $\mathbf{z}$ and $\mathbf{w}$ are the column vectors in $\mathbb{H}^{1,1}$ with entries $(z_1, z_2)$ and $(w_1, w_2)$ respectively, $\mathbf{w}^*$ denotes the conjugate transpose of $\mathbf{w}$ and $J$ is the Hermitian matrix

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We define a *unitary transformation* $g$ to be an automorphism of $\mathbb{H}^{1,1}$, that is, a linear bijection such that $\langle g(\mathbf{z}), g(\mathbf{w}) \rangle = \langle \mathbf{z}, \mathbf{w} \rangle$ for all $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{H}^{1,1}$. We denote the group of all unitary transformations by $\mathrm{U}(1,1;\mathbb{H})$; this is sometimes also denoted $\mathrm{Sp}(1,1)$. For higher dimensions, cf. [**9**].

Following [**4**, section 2], let

$$V_0 = \{\mathbf{z} \in V \setminus \{0\} : \langle \mathbf{z}, \mathbf{z} \rangle = 0\}$$
$$V_- = \{\mathbf{z} \in V : \langle \mathbf{z}, \mathbf{z} \rangle < 0\}.$$

It is obvious that $V_0$ and $V_-$ are invariant under $\mathrm{U}(1,1;\mathbb{H})$. Define a right projection map $P: V_- \cup V_0 \to P(V_- \cup V_0) \subset \mathbb{H}$ by $P(\mathbf{z}) = z_1 z_2^{-1}$ where $\mathbf{z}$ is the column vector $(z_1, z_2)$. Observe that $z_2 \neq 0$ as $\mathbf{z} \in V_- \cup V_0$. We define $\mathbb{B} = P(V_-)$. It is easy to see that $\mathbb{B}$ can be identified with the quaternionic unit ball $\{z \in \mathbb{H} : |z| < 1\}$. Also the unit sphere in $\mathbb{H}$ is $\partial\mathbb{B} = P(V_0)$.

An element

$$\tilde{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{U}(1,1;\mathbb{H})$$

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acts on $\mathbb{H}^{1,1}$ by left matrix multiplication. Therefore, projecting (on the right), we see that

$$\begin{aligned} P(\tilde{g}(\mathbf{z})) &= P\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ &= P\begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix} \\ &= (az_1 + bz_2)(cz_1 + dz_2)^{-1} \\ &= \big(az_1 z_2^{-1} + b\big) z_2 z_2^{-1} \big(cz_1 z_2^{-1} + d\big)^{-1}. \end{aligned}$$

Hence we see that $\tilde{g}$ acts on $\overline{\mathbb{B}}$ by the *quaternionic Möbius transformation* $g$ given by

$$g(z) = (az + b)(cz + d)^{-1}.$$

So we may identify the Möbius group $M(\overline{\mathbb{R}^3})$ with $\mathrm{PU}(1,1;\mathbb{H}) = \mathrm{U}(1,1;\mathbb{H})/\{\pm I\}$.

If the matrix $\tilde{g}$ defined above is in $\mathrm{U}(1,1;\mathbb{H})$ then, by definition, $\tilde{g}$ preserves the Hermitian form. Hence

$$\mathbf{w}^* J\mathbf{z} = \langle \mathbf{z}, \mathbf{w}\rangle = \langle \tilde{g}\mathbf{z}, \tilde{g}\mathbf{w}\rangle = \mathbf{w}^* \tilde{g}^* J\tilde{g}\mathbf{z}$$

for all $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{H}^{1,1}$. Letting $\mathbf{z}$ and $\mathbf{w}$ vary over a basis for $\mathbb{H}^{1,1}$ we see that $J = \tilde{g}^* J\tilde{g}$. From this we find $\tilde{g}^{-1} = J^{-1}\tilde{g}^* J$. That is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \overline{a} & -\overline{c} \\ -\overline{b} & \overline{d} \end{pmatrix}.$$

Use of the identities $\tilde{g}\tilde{g}^{-1} = \tilde{g}^{-1}\tilde{g} = I$ proves Lemma 1·1. That is, $a$, $b$, $c$ and $d$ must satisfy the equations (1·1):

$$|a| = |d|, \quad |b| = |c|, \quad |a|^2 - |c|^2 = 1, \quad \overline{a}b = \overline{c}d, \quad a\overline{c} = b\overline{d}.$$

We remark that $\mathbb{B}$ is a model of the quaternionic hyperbolic line $\mathbf{H}^1_{\mathbb{H}}$ which is equivalent to (real) hyperbolic 4-space $\mathbf{H}^4$. The group $M(\overline{\mathbb{R}^3})$, that is $\mathrm{PU}(1,1;\mathbb{H})$, is the group of orientation preserving isometries of $\mathbf{H}^4$.

We conclude this section with the proof of our characterisation of simple matrices.

*Proof of Proposition* 1·2. First, we show that the conditions $b = \overline{c}$ and $\Re(a) = \Re(d)$ are preserved by conjugation within $\mathrm{U}(1,1;\mathbb{H})$. Suppose that $b = \overline{c}$ and $\Re(a) = \Re(d)$. Then:

$$\begin{aligned} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} &= \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a & \overline{c} \\ c & d \end{pmatrix}\begin{pmatrix} \overline{a}' & -\overline{c}' \\ -\overline{b}' & \overline{d}' \end{pmatrix} \\ &= \begin{pmatrix} a'a\overline{a}' + b'c\overline{a}' - a'\overline{c}\overline{b}' - b'd\overline{b}' & -a'a\overline{c}' - b'c\overline{c}' + a'\overline{c}\overline{d}' + b'd\overline{d}' \\ c'a\overline{a}' + d'c\overline{a}' - c'\overline{c}\overline{b}' - d'd\overline{b}' & -c'a\overline{c}' - d'c\overline{c}' + c'\overline{c}\overline{d}' + d'd\overline{d}' \end{pmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} b_1 - \overline{c}_1 &= -a'a\overline{c}' - b'c\overline{c}' + a'\overline{c}\overline{d}' + b'd\overline{d}' - a'\overline{a}\overline{c}' - a'\overline{c}\overline{d}' + b'c\overline{c}' + b'\overline{d}\overline{d}' \\ &= -a'(a + \overline{a})\overline{c}' + b'(d + \overline{d})\overline{d}' \\ &= -a'\overline{c}'(a + \overline{a}) + b'\overline{d}'(d + \overline{d}). \end{aligned}$$

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Because $\Re(a) = \Re(d)$ and $a'\overline{c}' = b'\overline{d}'$, from (1·1), we see that $b_1 = \overline{c}_1$. Moreover

$$\Re(a_1) = \Re(a'a\overline{a}') - \Re(b'd\overline{b}') = |a'|^2\Re(a) - |b'|^2\Re(d),$$
$$\Re(d_1) = \Re(d'd\overline{d}') - \Re(c'a\overline{c}') = |d'|^2\Re(d) - |c'|^2\Re(a).$$

Since $|a'| = |d'|$, $|b'| = |c'|$, from (1·1), and $\Re(a) = \Re(d)$ we have $\Re(a_1) = \Re(d_1)$ as claimed. Moreover, as $|d'|^2 - |c'|^2 = 1$, from (1·1), we see that $\Re(d_1) = \Re(d)$.

If $\tilde{g}$ lies in $\mathrm{U}(1,1;\mathbb{C})$ then $\tilde{g}$ satisfies $b = \overline{c}$ and $d = \overline{a}$, so $\Re(d) = \Re(\overline{a}) = \Re(a)$. Therefore any conjugate of $\tilde{g}$ satisfies $b = \overline{c}$ and $\Re(a) = \Re(d)$.

Now we show that if $\tilde{g} \in \mathrm{U}(1,1;\mathbb{H})$ satisfies $b = \overline{c}$ and $\Re(a) = \Re(d)$ then $\tilde{g}$ is conjugate to an element of $\mathrm{U}(1,1;\mathbb{C})$. First, suppose that $b = \overline{c} \neq 0$. Then

$$\begin{pmatrix} c/|c| & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & \overline{c} \\ c & d \end{pmatrix}\begin{pmatrix} \overline{c}/|c| & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \overline{d} & |c| \\ |c| & d \end{pmatrix}.$$

This matrix preserves the subspace $\mathbb{R}(d)$. There is a unit quaternion $u$ so that $ud\,\overline{u} \in \mathbb{C}$. Then conjugation of the above matrix by $uI$ will send it to an element of $\mathrm{U}(1,1;\mathbb{C})$.

When $c = 0$ we know that $a$ and $d$ are similar, so there is a unit quaternion $q$ with $qa\overline{q} = \overline{d}$. Then

$$\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} \overline{q} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \overline{d} & 0 \\ 0 & d \end{pmatrix}.$$

Again, conjugation of by $uI$ for some unit quaternion $u$ sends this matrix to an element of $\mathrm{U}(1,1;\mathbb{C})$.

## 3. *Fixed points and eigenvalues*

In this section we prove Theorems 1·3, 1·4, 1·5 and 1·6. First we consider the relationship between fixed points of the Möbius transformation $g$ and right eigenvalues of the matrix $\tilde{g}$.

The point $v \in \overline{\mathbb{B}}$ is in $\mathrm{fix}(g)$ if and only if $v = g(v) = (av + b)(cv + d)^{-1}$. Multiplying both sides on the right by $cv + d$ we see that $v$ satisfies the equation

$$vcv + vd - av - b = 0. \tag{3·1}$$

Lemma 3·1. *If $v$ is a fixed point of $g(z) = (az + b)(cz + d)^{-1}$ then $t = cv + d$ is a right eigenvalue of $\tilde{g}$.*

*Proof.* Applying $\tilde{g}$ to the column vector $(v, 1)$ and using equation (3·1) we obtain

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} v \\ 1 \end{pmatrix} = \begin{pmatrix} av + b \\ cv + d \end{pmatrix} = \begin{pmatrix} vcv + vd \\ cv + d \end{pmatrix} = \begin{pmatrix} v \\ 1 \end{pmatrix}(cv + d).$$

Right eigenvalues are conjugacy invariants for matrices $\tilde{g}$ in $\mathrm{U}(1,1;\mathbb{H})$; see Zhang [**13**]. In order to see this observe that if $\tilde{g}\mathbf{z} = \mathbf{z}t$ then

$$(\tilde{h}\tilde{g}\tilde{h}^{-1})(\tilde{h}\mathbf{z}) = \tilde{h}\tilde{g}(\mathbf{z}) = (\tilde{h}\mathbf{z})t.$$

If $t$ is a right eigenvalue of $\tilde{g}$ then so is any quaternion in the similarity class of $t$. In order to see this, observe that if $\tilde{g}\mathbf{z} = \mathbf{z}t$ and $q$ is any non-zero quaternion then

$$\tilde{g}(\mathbf{z}q) = \tilde{g}\mathbf{z}q = \mathbf{z}tq = (\mathbf{z}q)(q^{-1}tq).$$

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When we speak of a right eigenvalue $t$ associated to the fixed point $v$ of a quaternionic Möbius transformation $g$ we want the right eigenvalue in its similarity class whose eigenvector is $\mathbf{v}$, the column vector with entries $(v, 1)$. Therefore, in this sense, right eigenvalues are *not* conjugacy invariants of quaternionic Möbius transformations. Thus the only conjugacy invariants, from the point of view of Möbius transformations, are the real part and absolute value of the right eigenvalue $t$ (later we will write $N = |t|^2$ and $T = 2\Re(t)$). Thus it is possible to show that Möbius transformations of the form $g(z) = aza^{-1}$, which have the unique right eigenvalue $a$, are conjugate to more general simple elliptic Möbius transformations, which have a continuous family of eigenvalues in the same similarity class.

We now convert equation (3·1) into a statement about eigenvalues.

PROPOSITION 3·2. *Suppose that* $g(z) = (az + b)(cz + d)^{-1}$ *where* $c \neq 0$. *The set* $\mathrm{fix}(g)$ *of fixed points of* $g(z)$ *is the set of all points* $v = c^{-1}(t - d)$ *in* $\overline{\mathbb{B}}$ *where* $t$ *is a solution of the equation* (1·3):

$$t^2 - (d + \overline{c}^{-1}b\overline{d})t + \overline{c}^{-1}b = 0.$$

*Proof.* Substituting $v = c^{-1}(t - d)$ in equation (3·1) we have

$$\begin{aligned} 0 &= vcv + vd - av - b \\ &= c^{-1}(t - d)(t - d) + c^{-1}(t - d)d - ac^{-1}(t - d) - b \\ &= c^{-1}t^2 - c^{-1}dt - ac^{-1}t + ac^{-1}d - b. \end{aligned}$$

Multiplication on the left by $c$ gives

$$t^2 - (d + cac^{-1})t + cac^{-1}d - cb = 0.$$

Using the identities (1·1), we have $cac^{-1} = \overline{c}^{-1}a\overline{c} = \overline{c}^{-1}b\overline{d}$. Hence, the constant term becomes

$$cac^{-1}d - cb = \overline{c}^{-1}b|d|^2 - \overline{c}^{-1}b|c|^2 = \overline{c}^{-1}b.$$

Therefore we see that $t$ satisfies equation (1·3).

We now solve equations (1·3) and (3·1). This will take the rest of this section and will be divided into three cases.

3·1. *The case* $c = 0$: *proofs of Theorems* 1·4 *and* 1·3(*a*)

We first consider the case where $c = 0$, which yields $b = 0$. Thus $g(z) = azd^{-1}$ and equation (3·1) becomes

$$vd = av. \tag{3·2}$$

Obviously, $v = 0 \in \mathbb{B}$ is a solution to equation (3·2) and thus $g$ is elliptic. If $\Re(a) \neq \Re(d)$ then $a$ is not similar to $d$, and equation (3·2) has the unique solution $v = 0$. Otherwise, there exists non-zero $q \in \mathbb{H}$ with $|q| = 1$ so that $a = qdq^{-1}$. This means that $v = q$ is also a solution to equation (3·2). Letting $h(z) = qz$ we obtain $h^{-1}gh(z) = q^{-1}aqzd^{-1} = dzd^{-1}$. Thus $v' \in \mathrm{fix}(h^{-1}gh)$ if and only if $|v'| \leqslant 1$ and $v' \in \mathbb{R}(d)$. From this we obtain $\mathrm{fix}(g) = h\mathrm{fix}(h^{-1}gh)$ and so $v = qv'$. This proves Theorem 1·4.

It is clear that, since $0 \in \mathbb{B}$ is a fixed point of $g$, then $g$ must be elliptic. The distinction between simple and compound elliptic maps follows from Proposition 1·2. This proves Theorem 1·3(a).

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3·2. *The case* $b = \overline{c} \neq 0$: *proofs of Theorems* 1·5 *and* 1·3(*b*)

Secondly, we consider the case $b = \overline{c}$ and $c \neq 0$. Substituting $\overline{c}^{-1}b = 1$ into equation (1·3) we see that $v = c^{-1}(t - d)$ is a fixed point of $g(z) = (az + b)(cz + d)^{-1}$ provided $|v| \leqslant 1$ and

$$t^2 - (d + \overline{d})t + 1 = t^2 - 2\Re(d)t + 1 = 0.$$

Therefore

$$(t - \Re(d))^2 = \Re(d)^2 - 1.$$

Taking square roots, we obtain the solutions given in Theorem 1·5.

Observe that $b = \overline{c} \neq 0$ implies that $\Re(a) = \Re(d)$: From (1·1) we have

$$a = b\overline{d}\overline{c}^{-1} = \overline{c}\overline{d}\overline{c}^{-1}.$$

Thus $\Re(a) = \Re(\overline{d}) = \Re(d)$ as claimed. Therefore $\tilde{g}$ is conjugate to an element of $\mathrm{U}(1, 1; \mathbb{C})$. Using the fact that $\Re(d)$ is invariant under conjugation, we see that Theorem 1·3(b) follows from the standard classification of elements in $\mathrm{PU}(1, 1; \mathbb{C})$.

3·3. *The case* $b \neq \overline{c} \neq 0$: *proofs of Theorems* 1·6 *and* 1·3(*c*)

For the rest of this section we suppose that $b \neq \overline{c}$ and $c \neq 0$. We define

$$\tau = d + \overline{c}^{-1}b\overline{d} \quad \text{and} \quad \sigma = \overline{c}^{-1}b \neq 1.$$

For later use, we observe that

$$\sigma\overline{\tau} + \tau\overline{\sigma} = \overline{c}^{-1}b\overline{d} + d\overline{b}c^{-1} + |c|^{-4}cb(d + \overline{d})\overline{b}\overline{c} = \tau + \overline{\tau}. \tag{3·3}$$

Then equation (1·3) becomes $t^2 - \tau t + \sigma = 0$. Following the idea of Niven [**10**], we define $T = t + \overline{t}$ and $N = \overline{t}t = t\overline{t}$. Hence we have $N \geqslant T^2/4$ with equality if and only if $t$ is real. Since $|\sigma| = 1$ we see $t \neq 0$ and so $N > 0$. Then

$$t^2 - Tt + N = t^2 - (t + \overline{t})t + \overline{t}t = 0.$$

Hence

$$0 = (t^2 - \tau t + \sigma) - (t^2 - Tt + N) = -(\tau - T)t + (\sigma - N).$$

Because we are assuming that $b \neq \overline{c}$ then $\sigma - N$ is not real. In particular, it is not zero and thus $\tau - T \neq 0$. Therefore, following [**6**], we see that $t$ is given by

$$t = (\tau - T)^{-1}(\sigma - N). \tag{3·4}$$

Substituting for $t$ in the defining equations for $T$ and $N$, we obtain

$$T = (\tau - T)^{-1}(\sigma - N) + (\overline{\sigma} - N)(\overline{\tau} - T)^{-1},$$
$$N = (\tau - T)^{-1}(\sigma - N)(\overline{\sigma} - N)(\overline{\tau} - T)^{-1}.$$

Multiplying by $(\tau - T)$ on the left and $(\overline{\tau} - T)$ on the right we have

$$(\tau - T)T(\overline{\tau} - T) = (\sigma - N)(\overline{\tau} - T) + (\tau - T)(\overline{\sigma} - N)$$
$$(\tau - T)N(\overline{\tau} - T) = (\sigma - N)(\overline{\sigma} - N).$$

Expanding and using $|\sigma| = 1$ and equation (3·3) we have

$$|\tau|^2T - 2\Re(\tau)T^2 + T^3 = 2\Re(\tau) - 2\Re(\sigma)T - 2\Re(\tau)N + 2NT, \tag{3·5}$$
$$|\tau|^2N - 2\Re(\tau)NT + NT^2 = 1 - 2\Re(\sigma)N + N^2. \tag{3·6}$$

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Let $\Delta = |\tau|^2 - \Re(\tau)^2 - 2 + 2\Re(\sigma) = |\Im(\tau)|^2 - |\sigma - 1|^2$. Then equations (3·5) and (3·6) become

$$(N-1)^2 = N(T - \Re(\tau))^2 + N\Delta, \tag{3·7}$$

$$2(N-1)(T - \Re(\tau)) = T(T - \Re(\tau))^2 + T\Delta. \tag{3·8}$$

Since $N > 0$, equation (3·7) implies that $(T - \Re(\tau))^2 \geqslant -\Delta$ with equality if and only if $N = 1$.

LEMMA 3·3. *If $b \neq \overline{c}$ and $T - 2\Re(\tau)N + NT = 0$ then $t$ is not real.*

*Proof.* Suppose, for a contradiction, that $t$ is real. Then $N = t^2$ and $T = 2t$. Therefore

$$0 = T - 2\Re(\tau)N + NT = 2t - 2\Re(\tau)t^2 + 2t^3.$$

Since we know that $t \neq 0$ we must have $0 = t^2 - \Re(\tau)t + 1$. Taking this equation, together with equation (1·3), we have

$$\begin{aligned} 0 &= 2(t^2 - \Re(\tau)t + 1) - (t^2 - \tau t + \sigma) - (t^2 - \overline{\tau}t + \overline{\sigma}) \\ &= |\sigma - 1|^2. \end{aligned}$$

Therefore $\sigma = 1$ and $b = \overline{c}$, a contradiction.

Theorem 1·6 will follow from the next lemma.

LEMMA 3·4. *Suppose that $b \neq \overline{c}$. Then the equations* (3·7) *and* (3·8) *have the following solutions.*

(i) *If $\Delta \leqslant 0$ then*

$$T = \Re(\tau) \pm \sqrt{-\Delta}, \quad N = 1.$$

(ii) *If $\Delta > 0$ then*

$$T = \Re(\tau) \pm \sqrt{2X - 2 - \Delta}, \quad N = X \pm \sqrt{X^2 - 1}$$

*where*

$$X = \frac{\Re(\tau)^2 + \Delta + \sqrt{(\Re(\tau)^2 + \Delta - 4)^2 + 16\Delta}}{4}.$$

*Proof.* Eliminating $\Delta$ from (3·7) and (3·8) we find

$$(N-1)(T - 2\Re(\tau)N + NT) = 0 \tag{3·9}$$

and so either $N = 1$ or else $T - 2\Re(\tau)N + NT = 0$.

First, if $N = 1$ then from equation (3·7) we have $(T - \Re(\tau))^2 = -\Delta$. So we must have $\Delta \leqslant 0$ and also $T = \Re(\tau) \pm \sqrt{-\Delta}$.

Secondly, suppose that $N \neq 1$. Therefore $T - 2\Re(\tau)N + NT = 0$. Thus

$$T = \frac{2\Re(\tau)N}{N+1}. \tag{3·10}$$

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Substituting for $T$ in (3·7) we obtain

$$(N-1)^2 = N\frac{\Re(\tau)^2(N-1)^2}{(N+1)^2} + N\Delta.$$

A rearrangement of this equation gives

$$(N+1/N)^2 - (\Re(\tau)^2+\Delta)(N+1/N) + 2\Re(\tau)^2 - 2\Delta - 4 = 0.$$

Writing $N+1/N = 2X$ we obtain $N = X \pm \sqrt{X^2-1}$ where

$$X = \frac{\Re(\tau)^2 + \Delta + \sqrt{(\Re(\tau)^2+\Delta-4)^2+16\Delta}}{4}.$$

Because $N \neq 1$ we have $X > 1$.

Substituting for $N$ in equation (3·7), we obtain

$$(T-\Re(\tau))^2 = \frac{(N-1)^2}{N} - \Delta = 2X - 2 - \Delta,$$

which yields the formula for $T$.

It remains to show that $\Delta > 0$ when $N \neq 1$. Suppose, for a contradiction, that $\Delta \leqslant 0$.

Using $(N+1)^2 = 2N(X+1)$ and (3·10) we see that the condition $N > T^2/4$ (which follows from Lemma 3·3) implies that $X + 1 - \Re(\tau)^2/2 > 0$. This gives

$$\begin{aligned} 0 &< 4X + 4 - 2\Re(\tau)^2 \\ &= -\Re(\tau)^2 + \Delta + 4 + \sqrt{(\Re(\tau)^2+\Delta-4)^2+16\Delta} \\ &= -\Re(\tau)^2 + \Delta + 4 + \sqrt{(\Re(\tau)^2-\Delta-4)^2+4\Delta\Re(\tau)^2} \\ &\leqslant -\Re(\tau)^2 + \Delta + 4 + |\Re(\tau)^2 - \Delta - 4|. \end{aligned}$$

The last inequality follows from $\Delta \leqslant 0$. Since $|\Re(\tau)^2 - \Delta - 4| > \Re(\tau)^2 - \Delta - 4$ we must have $\Re(\tau)^2 - 4 < \Delta$. Similarly, using $X > 1$ we have:

$$\begin{aligned} 0 &< 4(X-1) \\ &= \Re(\tau)^2 + \Delta - 4 + \sqrt{(\Re(\tau)^2+\Delta-4)^2+16\Delta} \\ &\leqslant \Re(\tau)^2 + \Delta - 4 + |\Re(\tau)^2 + \Delta - 4|. \end{aligned}$$

Therefore $\Re(\tau)^2 - 4 > -\Delta$. Thus $\Delta \leqslant 0$ implies $0 \leqslant -\Delta < \Re(\tau)^2 - 4 < \Delta \leqslant 0$ which is a contradiction.

Therefore (when $b \neq \overline{c}$), if $N = 1$ then $\Delta \leqslant 0$ and if $N \neq 1$ then $\Delta > 0$. In both cases the solutions for $N$ and $T$ are as we claimed.

We now prove Theorem 1·3(c). From Proposition 1·2 we see that $g$ is compound. We will need the following lemma.

LEMMA 3·5. *Suppose that $g(z) = (az+b)(cz+d)^{-1}$ with $b \neq \overline{c}$. Let $v$ be a fixed point of $g$. Then*

$$|v|^2 - 1 = \frac{(T - 2\Re(d))(T - 2\Re(\tau)N + NT)}{|c|^2|\tau - T|^2}.$$

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*Proof.* We know that $v = c^{-1}(t-d) = c^{-1}(\tau - T)^{-1}((\sigma - N) - (\tau - T)d)$ where $N = t\overline{t}$ and $T = t + \overline{t}$. This means that

$$\begin{aligned}
&(|v|^2 - 1)|c|^2|\tau - T|^2 \\
&\quad = |(\sigma - N) - (\tau - T)d|^2 - |c|^2|\tau - T|^2 \\
&\quad = |\sigma - N|^2 - (\sigma - N)\overline{d}(\overline{\tau} - T) - (\tau - T)d(\overline{\sigma} - N) + (|d|^2 - |c|^2)|\tau - T|^2 \\
&\quad = N|\tau - T|^2 - (\tau - d - N\overline{d})(\overline{\tau} - T) - (\tau - T)(\overline{\tau} - \overline{d} - Nd) + |\tau - T|^2 \\
&\quad = N|\tau|^2 - NT2\Re(\tau) + NT^2 - 2|\tau|^2 + T2\Re(\tau) + (d\overline{\tau} + \tau\overline{d}) - T2\Re(d) \\
&\qquad + N2\Re(\tau)2\Re(d) - N(d\overline{\tau} + \tau\overline{d}) - NT2\Re(d) + |\tau|^2 - T2\Re(\tau) + T^2 \\
&\quad = T^2 - NT2\Re(d) + NT^2 - T2\Re(d) + N2\Re(d)2\Re(\tau) - NT2\Re(d) \\
&\quad = (T - 2\Re(d))(T - 2\Re(\tau)N + NT).
\end{aligned}$$

In the third line we have used $N = |\tau - T|^{-2}|\sigma - N|^2$ and in the penultimate line we have used $(d\overline{\tau} + \tau\overline{d}) = 2|d|^2 + \sigma\overline{d}^2 + d\overline{\sigma}^2 = |\tau|^2$.

If $\Delta > 0$ we see that equations (3·7) and (3·8) have two solutions for $N$ and $T$ and hence for $t = (\tau - T)^{-1}(\sigma - N)$ and for $v = c^{-1}(t - d)$. Furthermore, both these solutions satisfy $T - 2\Re(\tau)N + NT = 0$. Thus for both solutions $|v| = 1$ and so $g$ is loxodromic.

If $\Delta = 0$ then $N = 1$ and $T = \Re(\tau)$. This gives a unique solution for $t$ and hence $v$. Moreover, $T - 2\Re(\tau)N + NT = 0$ and so $|v| = 1$. Therefore $g$ is parabolic.

If $\Delta < 0$ then $N = 1$ and $T = \Re(\tau) \pm \sqrt{-\Delta}$. This gives exactly two solutions for $t$ and hence for $v$; call these $v_+$ and $v_-$. We now show that exactly one of these solutions is in $\mathbb{B}$ and the other is in $\mathbb{H} \setminus \overline{\mathbb{B}}$. It suffices to show that $(|v_+|^2 - 1)(|v_-|^2 - 1) < 0$. First, we have

$$T - 2\Re(\tau)N + NT = \pm 2\sqrt{-\Delta} \neq 0.$$

Also,

$$T - 2\Re(d) = \Re(\sigma\overline{d} + d - d - \overline{d}) \pm \sqrt{-\Delta} = \Re((\sigma - 1)\overline{d}) \pm \sqrt{-\Delta}.$$

Therefore

$$|v_\pm|^2 - 1 = \frac{(\pm\Re((\sigma - 1)\overline{d}) + \sqrt{-\Delta})2\sqrt{-\Delta}}{|c|^2|\Im(\tau) \mp \sqrt{-\Delta}|^2}.$$

Hence $(|v_+|^2 - 1)(|v_-|^2 - 1)$ has the same sign as

$$\begin{aligned}
&(\Re((\sigma - 1)\overline{d}) + \sqrt{-\Delta})(-\Re((\sigma - 1)\overline{d}) + \sqrt{-\Delta}) \\
&\quad = -\Re((\sigma - 1)\overline{d})^2 - |\Im((\sigma - 1)\overline{d})|^2 + |\sigma - 1|^2 \\
&\quad = -|\sigma - 1|^2|d|^2 + |\sigma - 1|^2 \\
&\quad = -|\sigma - 1|^2|c|^2 < 0.
\end{aligned}$$

This means that one of $v_\pm$ lies in $\mathbb{B}$ and the other in $\mathbb{H} \setminus \overline{\mathbb{B}}$. Hence $g$ is elliptic. This completes the proof of Theorem 1·3(c).

## 4. *Examples*

In this section we give examples to show that all the classes we indicated are non-empty and to illustrate our method of finding fixed points.

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*Example* 4·1. Let $g(z) = (\sqrt{3}\mathbf{j}z + \sqrt{2})(\sqrt{2}z - \sqrt{3}\mathbf{j})^{-1}$. We can check that $b = \overline{c} = \sqrt{2}$ and $\mathfrak{R}(d)^2 = 0 < 1$. Thus $g$ is simple elliptic. Any purely imaginary $u$ with modulus 1 is a right eigenvalue. The fixed point corresponding to such a $u$ is $v = \sqrt{2}^{-1}(u + \sqrt{3}\mathbf{j})$. The fixed points in $\overline{\mathbb{B}}$ must also satisfy $|v| \leqslant 1$.

*Example* 4·2. Let $g(z) = ((1 + \mathbf{k})z - \mathbf{k})(\mathbf{k}z + 1 - \mathbf{k})^{-1}$. We have $b = \overline{c}$ and $\mathfrak{R}(d) = 1$. Therefore $g$ is simple parabolic and clearly $t = 1$ and $g$ fixes the point 1.

*Example* 4·3. Let $g(z) = (2z + \mathbf{i} + \mathbf{j} + \mathbf{k})((-\mathbf{i} - \mathbf{j} - \mathbf{k})z + 2)^{-1}$. Clearly $b = \overline{c}$ and $\mathfrak{R}(d)^2 = 4 > 1$. Thus $g$ is simple loxodromic. Its right eigenvalues are $t = 2 \pm \sqrt{3}$ and fixed points $v = \pm(\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$.

*Example* 4·4. Let $g(z) = ((1 + \mathbf{j})z - \mathbf{j})(z - 1 - \mathbf{j})^{-1}$. We see that $b \neq \overline{c}$, $\sigma = -\mathbf{j}$ and $\tau = 0$. Therefore $\Delta = -2 < 0$ and $g$ is compound elliptic. We find that $T = \pm\sqrt{2}$ and $N = 1$. Therefore $t = \mp\sqrt{2}^{-1}(-1 - \mathbf{j})$ and $v = (1 \mp 1/\sqrt{2})(1 + \mathbf{j})$. Taking the minus sign gives the unique fixed point in $\mathbb{B}$.

*Example* 4·5. Let $g(z) = ((\mathbf{k} - 1)z + 1)(-z + \mathbf{k} + 1)^{-1}$. We have $b \neq \overline{c}$, $\sigma = -1$ and $\tau = 2\mathbf{k}$. Therefore $\Delta = 4 - 4 = 0$ and $g$ is compound parabolic. Clearly $g$ fixes 1. We have $t = (2\mathbf{k})^{-1}(-2) = \mathbf{k}$.

*Example* 4·6. Let $g(z) = ((1-\mathbf{i})z+\mathbf{k})(\mathbf{j}z+1+\mathbf{i})^{-1}$. We have $b \neq \overline{c}$, $\sigma = \mathbf{i}$ and $\tau = 2+2i$. Then $\Delta = 4 - 2 = 2 > 0$ so $g$ is compound loxodromic. We find that $T = 2 \pm \sqrt{2}$ and $N = 3 \pm 2\sqrt{2}$. This means that $t = (1 + \mathbf{i})(2 \pm \sqrt{2})/2$ and $v = \pm(\mathbf{k} - \mathbf{j})/\sqrt{2}$.

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