Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 1–37, *2025*
DOI:10.1017/prm.2025.10075

Normalized multi-bump solutions of nonlinear Hartree equation with steep potential well

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(Received 17 May 2024; revised 24 August 2025; accepted 24 August 2025)

In this paper, we study the existence of solutions to the following Hartree equation

$$\begin{cases} -\Delta u + \lambda V(x)u + \mu u = \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x-y|^{N-\alpha}}\right)|u|^{p-2}u, \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 = \omega, \end{cases}$$

Where $N \geq 3$, $\omega, \lambda > 0$, $p \in \left(\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}\right) \setminus \left\{\frac{N+\alpha+2}{N}\right\}$ and μ will appear as a Lagrange multiplier. We assume that $0 \leq V \in L^{\infty}_{loc}(\mathbb{R}^N)$ has a bottom $intV^{-1}(0)$ composed of ℓ_0 ($\ell_0 \geq 1$) connected components $\{\Omega_i\}_{i=1}^{\ell_0}$, where $intV^{-1}(0)$ is the interior of the zero set $V^{-1}(0) = \{x \in \mathbb{R}^N | V(x) = 0\}$ of V. It is worth pointing out that the penalization technique is no longer applicable to the local sublinear case $p \in \left(\frac{N+\alpha}{N}, 2\right)$. Therefore, we develop a new variational method in which the two deformation flows are established that reflect the properties of the potential. Moreover, we find a critical point without introducing a penalization term and give the existence result for $p \in \left(\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}\right) \setminus \left\{\frac{N+\alpha+2}{N}\right\}$. When ω is fixed and satisfies $\omega^{\frac{-(p-1)}{-Np+N+\alpha+2}}$ sufficiently small, we construct a ℓ-bump ($1 \leq \ell \leq \ell_0$) positive normalization solution, which concentrates at ℓ prescribed components $\{\Omega_i\}_{i=1}^{\ell}$ for large λ. We also consider the asymptotic profile of the solutions as $\lambda \to \infty$ and $\omega^{\frac{-(p-1)}{-Np+N+\alpha+2}} \to 0$.

Keywords: nonlinear Hartree equation; normalized multi-bump solutions; steep potential well; variational method

2020 *Mathematics Subject Classification:* 35B35, 35J20, 35J60

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https://doi.org/10.1017/prm.2025.10075 Published online by Cambridge University Press

1. Introduction

This paper concerns the existence of multibump solutions to the following Hartree equation

$$-\Delta u + \lambda V(x)u + \mu u = \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x-y|^{N-\alpha}}\right)|u|^{p-2}u, \text{ in } \mathbb{R}^N \tag{1.1}$$

under the mass constraint

$$\int_{\mathbb{R}^N} |u|^2 = \omega, \tag{1.2}$$

where $N \geq 3$, $\omega, \lambda > 0$, $p \in \left(\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}\right) \setminus \left\{\frac{N+\alpha+2}{N}\right\}$, $0 \leq V(x) \in L^\infty_{loc}(\mathbb{R}^N)$ and the zero set $V^{-1}(0) = \{x \in \mathbb{R}^N | \ V(x) = 0\}$ of V have a nonempty interior $\Omega := int\ V^{-1}(0)$, $\mu \in \mathbb{R}$ appears as a Lagrange multiplier and is part of the unknowns. It is easy to know that $\frac{N+\alpha+2}{N}$ is the L^2-critical exponent of (1.1) and $\frac{N+\alpha}{N-2}$ is the usual critical Sobolev exponent of (1.1). We call u a normalized solution to (1.1), since (1.2) imposes a normalization on its L^2-mass.

When $V(x) = 0$, for example, the following problem

$$-\Delta u + \mu u = \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x-y|^{N-\alpha}}\right)|u|^{p-2}u, \text{ in } \mathbb{R}^N \tag{1.3}$$

where $N \geq 1$, $\mu \in \mathbb{R}$, $0 < \alpha < N$, $\frac{N+\alpha}{N} \leq p \leq \frac{N+\alpha}{N-2}$, arises in various fields of mathematical physics. The nonlinear term $\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x-y|^{N-\alpha}}\right)|u|^{p-2}u$ describes the interactions between the particles. Physically, $|x|^{-(N-\alpha)}$ is the so-called nonlocal response or Kernel, which represents the nonlocal character of the nonlinearity of the medium. Its width determines the degree of non-locality, see [28]. $|x|^{-(N-\alpha)}$ is a short range potential, if $N - \alpha > 1$ (long range potential, if $N - \alpha \leq 1$, see [18]), and sometimes even considers Dirac. This leads to the so-called Grose–Pitaevsky or cubic Schrödinger equation, which describes the standard Kerr-type local medium. In the physical case $N = 3, p = 2$, $\mu = 1$ and $\alpha = 2$, (1.3) appeared at least as early as in 1954, in a work by S.I. Pekar describing the quantum theory of a polariton at rest [38]. In [28], Choquard applied it as an approximation to the Hartree–Fock theory of one-component plasma. It also arises in multiple particle systems [19] and quantum mechanics [28, 40]. In this context, (1.3) is usually called the nonlinear Schrödinger–Newton equation.

The existence and qualitative properties of solutions to the Hartree equation (1.1) and (1.3) have been studied for a few decades by variational methods. If $N = 3, p = 2$, $\mu = 1$ and $\alpha = 2$, the existence of ground states to equation (1.3) was obtained in [28, 30] by using variational methods. Concerning the qualitative properties of the ground state, uniqueness is proved in [28]. In [36], Moroz and Schaftingen established some results about qualitative properties and decay estimates of ground state solutions to (1.3) with $p > 1$. Ye [51] studied the qualitative properties of the functional related to (1.1), including the existence and nonexistence of minimizers. Cingolani and Tanaka [10] developed the deformation arguments and

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proved the existence of a ground state solution for the nonlinear Hartree equation with prescribed mass, when the nonlinear term satisfies Berestycki–Lions type conditions. In [10, 51], the existence of L^2-normalized solutions to equation (1.3) was investigated when $p \in \left(\frac{N+\alpha}{N}, \frac{N+\alpha+2}{N}\right)$, i.e., in the L^2-subcritical case. Using constrained minimization method, Luo [33] proved the existence of a normalized solution for problem (1.3) with least energy when $\alpha = 2$ and $2 < p < \{4, N\}$. When the nonlinear term satisfies different assumptions, we can see [2, 8, 12, 27, 46, 47] and the references therein.

When $V(x) \neq 0$, there have been some works concerning the study of problem (1.1) under various assumptions on the potential. We refer the readers to [1, 5, 7, 9, 13–17, 20–23, 26, 39, 42, 44, 49, 50, 52] and the references therein. Among them, the authors are mainly interested in the problem of how the properties of the existence, multiplicity, and concentration of solutions. When $V(x) = k|x|^2$, Wang [44] proved the existence of ground state for problem (1.1) by the minimization method on the Nehari manifold. Luo [33] considered the existence and stability/instability of standing waves with a prescribed L^2-norm for problem (1.1) with $\alpha = 2, 2 < p < \min\{4, N\}$. Cao [5] et al. considered the problem with $a|x|^2$ replaced by a more general harmonic potential. They proved the existence and stability of standing waves with a prescribed L^2-norm for problem (1.1). In the semi-linear case, Wei and Winter [45] constructed families of solutions concentrating to the nondegenerate critical points of potential V by using Lyapunov–Schmidt reduction when $N = 3$ and $p = \alpha = 2$. Using the nonlocal penalization method, Moroz et al. [37] proved existence of a single-peak solutions of (1.1) concentrating at a local minima of $V(x)$ for $p \in \left[2, \frac{N+\alpha}{N-2}\right)$. When $p \in \left(\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}\right)$, Cingolani and Tanaka [9] obtained the multiplicity of concentrating solutions by the cup-length of a critical set $Crit_{V_0} = \{x \in \Omega : \ V(x) = V_0\}$, where $\Omega \subset \mathbb{R}^N$ is a bounded set such that $V_0 \equiv \inf\limits_{x\in\Omega} V(x) < \inf\limits_{x\in\partial\Omega} V(x)$. Subsequently, they [11] studied the existence of a family of solutions concentrating to a local maximum or a saddle point of $V(x) \in C^N(\mathbb{R}^N, \mathbb{R})$. The results for the critical problem, we can see [1, 16, 20–22, 32, 41, 48, 54]. For more information on global well-posedness, blow-up solutions, and scattering for the Hartree equation (1.1), we refer the reader to [14, 17, 23, 26, 34] and the references therein.

Motivated by the above papers, the aim of this paper is to consider the existence of normalized multibump solutions for equation (1.1) in the case of a steep potential well. To illustrate the results of the problem (1.1) in this direction, we assume that:

1. (V_1) $\Omega = \cup_{1\leq j\leq \ell_0}\Omega_j$ ($\ell_0 \in \mathbb{N}\backslash\{0\}$), where $\Omega_1, \cdots, \Omega_{\ell_0}$ are nonempty bounded domains with C^1 smooth boundary such that $\overline{\Omega_j} \cap \overline{\Omega \setminus \Omega_j} = \emptyset$ for each $j = 1, \cdots, \ell_0$.
2. (V_2) $V \in L^\infty_{loc}(\mathbb{R}^N)$ and $V(x) \geq 0$ for a.e. $x \in \mathbb{R}^N$; $\Omega := intV^{-1}(0)$ satisfies (V_1), and $V > 0$ a.e., in $\mathbb{R}^N \setminus \Omega$.
3. (V_3) There exists $M_0 > 0$ such that

$$meas\left(\{x \in \mathbb{R}^N |\ V(x) \leq M_0\}\right) < \infty.$$

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Let $\ell \in [1, \ell_0]$ be an integer. For every fixed subset

$$L := \{j_1, j_2, \cdots, j_\ell\} \subset \{1, \cdots, \ell_0\},$$

we set $\Omega_L := \cup_{i\in L}\Omega_i$. By scaling, we have

$$v(\cdot) = \omega^{\frac{-(2+\alpha)}{2(-Np+N+\alpha+2)}} u(\omega^{\frac{-(p-1)}{-Np+N+\alpha+2}}\cdot)$$

for every $u \in H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} |u|^2 = \omega$. Then (1.1) is equivalent to

$$\begin{cases} -\Delta v + \lambda\omega^{\frac{-2(p-1)}{-Np+N+\alpha+2}} V\left(\omega^{\frac{-(p-1)}{-Np+N+\alpha+2}} x\right) v + \bar{\mu} v = \left(\int_{\mathbb{R}^N} \frac{|v|^p}{|x-y|^{N-\alpha}} dy\right) |v|^{p-2} v, \\ \qquad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |v(x)|^2 = 1. \end{cases}$$

where $\bar{\mu} = \omega^{\frac{-2(p-1)}{-Np+N+\alpha+2}}\mu$. For convenience, let

$$\varepsilon = \omega^{\frac{-(p-1)}{-Np+N+\alpha+2}}.$$

Then (1.1) is equivalent to

$$\begin{cases} -\Delta v + \lambda\varepsilon^2 V(\varepsilon x) v + \bar{\mu} v = \left(\int_{\mathbb{R}^N} \frac{|v|^p}{|x-y|^{N-\alpha}} dy\right) |v|^{p-2} v, \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |v(x)|^2 = 1. \end{cases} \tag{1.4}$$

Clearly, there holds

$$\varepsilon \to 0 \text{ as } \omega \to \begin{cases} +\infty, & \text{if } p \in \left(\frac{N+\alpha}{N}, \frac{N+\alpha+2}{N}\right), \\ 0^+, & \text{if } p \in \left(\frac{N+\alpha+2}{N}, \frac{N+\alpha}{N-2}\right). \end{cases}$$

Set $V_\varepsilon(\cdot) = V(\varepsilon\cdot)$ and consider the functional corresponding to (1.4) by

$$J_{\lambda,\varepsilon}(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + \lambda\varepsilon^2 V_\varepsilon u^2\right) - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{N-\alpha}} dxdy,$$

which is defined on

$$E_\varepsilon := \left\{ u \in H^1(\mathbb{R}^N) \middle| \int_{\mathbb{R}^N} V_\varepsilon u^2 < +\infty \right\}$$

endowed with the norm

$$||u||_\varepsilon := \left(||u||^2 + \int_{\mathbb{R}^N} \varepsilon^2 V_\varepsilon u^2\right)^{\frac{1}{2}},$$

where $||\cdot||$ is the standard $H^1(\mathbb{R}^N)$ norm. Normalized solutions to (1.4) can be obtained by searching critical points of $J_{\lambda,\varepsilon}$ on the constraint

$$M_\varepsilon = \left\{ u \in E_\varepsilon \middle| \int_{\mathbb{R}^N} |u|^2 = 1 \right\}.$$

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From the above analysis, we have the following results.

THEOREM 1.1 *Suppose the conditions* $(V_1)-(V_3)$ *hold. Then for every* ε *sufficiently small, there exists* $\lambda_\varepsilon > 0$ *such that if* $\lambda > \lambda_\varepsilon$, *problem (1.1) has a solution* $(\mu_{\lambda,\omega}, v_{\lambda,\omega})$ *with* $\mu_{\lambda,\omega} > 0$ *and* $v_{\lambda,\omega} > 0$ *in* $\mathbb{R}^N$.

According to the conditions $(V_1)-(V_3)$, we deduce that as $\lambda \to +\infty$, the limit problem related to (1.1) is

$$\begin{cases} -\Delta v + \mu v = \left(\int_{\mathbb{R}^N} \frac{|v|^p}{|x-y|^{N-\alpha}} dy\right) |v|^{p-2} v, \quad \text{in } \Omega, \\ v > 0 \text{ in } \Omega_L,\ v = 0 \text{ in } \Omega \setminus \Omega_L,\ \int_{\Omega_L} |v(x)|^2 = \omega. \end{cases} \tag{1.5}$$

Then it follows from the asymptotic behaviour of the solution $(\mu_{\lambda,\omega}, v_{\lambda,\omega})$ given by Theorem 1.1, we have the following results about problem (1.5).

THEOREM 1.2 *Suppose* (V_1) *holds. Then for sufficiently small* $\varepsilon > 0$, *problem (1.5) admits a normalized solution* $v_\varepsilon \in H_0^1(\Omega)$ *with an associated Lagrange multiplier* $\mu = \mu_\omega$. *Moreover, there exists* $x_\omega^j \in \Omega_j$, *for each* $j \in L$ *such that when* $\varepsilon \to 0$, *there holds*

$$\omega^{\frac{-2(p-1)}{-Np+N+\alpha+2}} \mu_\omega \to \mu_{\ell^{-1}}, \quad \text{in } \mathbb{R},$$

$$\omega^{-\frac{2+\alpha}{2(-Np+N+\alpha+2)}} v_\varepsilon(\omega^{-\frac{p-1}{-Np+N+\alpha+2}} \cdot) - \sum_{j=1}^{\ell} W_j\left(\cdot - \omega^{-\frac{p-1}{-Np+N+\alpha+2}} x_\omega^j\right) \to 0, \quad \text{in } H^1(\mathbb{R}^N),$$

where

$$\mu_{\ell^{-1}} := \frac{2(-Np+N+\alpha+2p)\ell I_{\ell^{-1}}}{Np-N-2-\alpha} > 0,$$

and $\mathrm{W_j}$ *is the normalized ground state solution to*

$$\begin{cases} -\Delta u + \mu u = \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x-y|^{N-\alpha}} dy\right) u, \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 = \ell^{-1}. \end{cases} \tag{1.6}$$

REMARK 1.3. The results in theorems 1.1 and 1.2 hold for $\ell = 1$.

For the nonlinear Schrödinger equation

$$\begin{cases} -\Delta u + \lambda V(x) u + \mu u = |u|^{p-2} u \text{ in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} |u|^2 = \omega, \end{cases} \tag{1.7}$$

the existence and orbital stability of the normalized solutions have been studied in [3, 6, 24, 25, 43, 53]. Among them, Tang et al. [43] studied the existence of normalized multibump solutions to problem (1.7) when V satisfies $(V_1)-(V_3)$ by using a penalized argument [53] and combining with the unique of solutions to problem (1.7) satisfying $V(x) = 0$. However, we have to point out that when

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$V(x) = 0$, problem (1.1) has a unique solution only if $p = \alpha = 2$, and in other cases the solution is not unique due to the existence of a non-local term, that is, a Hartree term. More importantly, the penalization technique depends on the decay property of linear equations outside small balls. To this aim, it is crucial for the exponential decay of solutions. From [36], we see that the solutions decay exponentially only if $p \in \left[2, \frac{N+\alpha}{N-2}\right)$. Hence, the penalization technique in [53] is not valid when $p < 2$.

According to (V_1), we known that $\Omega = \cup_{i=1}^{\ell}\Omega_i (\ell \geq 2)$ is composed of several disjoint bounded domains $\{\Omega_i\}_{i=1}^{\ell}$ and each point in Ω_i is a degenerate minimizer of V. Hence, it is quite nontrivial to prove the existence of normalized multibump solutions to the limit problem (1.5) that do not vanish over multiple components of Ω_i. If we consider the L^2-constrained problem on each component Ω_i separately, then the associated Lagrange multipliers may be different. Moreover, due to the interaction in the L^2-constraint, it is difficult to guarantee the nonvanishing property of gauge resolution for multiple given components.

Motivated by [4], the positive ground state solutions to the following limiting Hartree system

$$\begin{cases} -\Delta v_i + \mu v_i = \left(\int_{\mathbb{R}^N} \frac{|v_i(y)|^p}{|x-y|^{N-\alpha}}\right) |v_i(x)|^{p-2} v_i \quad \text{in } \mathbb{R}^N, \\ v_i(x) > 0, \ \lim\limits_{x\to\infty} v_i(x) = 0, \ i = 1, 2, \ldots, \ell, \\ |v_i|_2^2 = s_i, \ \sum_{i=1}^{\ell} s_i = 1, \end{cases} \tag{1.8}$$

can be used as the building blocks to construct the multi-bump solutions of (1.1). However, the existence of positive ground state solutions is unknown in the literature. Unlike [4], μ is unknown in (1.8) under the L^2-constraint. Therefore, we cannot use their method to prove the existence of positive ground state solutions to problem (1.8). Inspired by [33, 51], combining some analysis tricks, we show the existence of positive ground state solutions to system (1.8), see Lemma 2.3.

In this paper, we adopt a new approach inspired by [9], where we obtain an operator τ_ε defined by Proposition 4.1. It has similar properties to the tail minimization operator via deformation parameters in [9]. Using the operator τ_ε in combination with the standard deformation argument, we can directly deal with the problem of $J_{\lambda,\varepsilon}$ in X_ε^d (see Section 3) without introducing the penalization technique or requiring uniform exponential decay of the solutions, so that we can deal with the case of $p \in \left(\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}\right) \setminus \left\{\frac{N+\alpha+2}{N}\right\}$.

The paper is organized as follows. In Section 2, the existence of the positive ground state solutions to (1.8) will be given. Following, an ε, λ-dependent concentration-compactness (see Proposition 3.3) will be established in Section 3. In Section 4, we develop a deformation parameter for the tail of $u \in X_\varepsilon^d$, that is, the outside of the small balls, to construct a map τ_ε. In Section 5, we use the deformation lemma to search for critical points of $J_{\lambda,\varepsilon}$ restricted to M_ε and give the proof of Theorems 1.1 and 1.2.

2. Preliminary results

In this paper, for convenience, we will use the following notations.

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- $L^q(\mathbb{R}^N)(1 \le q \le \infty)$ is the Lebesgue space with norm $|u|_q = \left(\int_{\mathbb{R}^N} |u|^q dx\right)^{\frac{1}{q}}$.
- $H^1(\mathbb{R}^N)$ is the Sobolev space with norm $\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 dx\right)^{\frac{1}{2}}$.
- $B_R(y) = \{x \in \mathbb{R}^N : \ |x - y| < R\}$.
- $o_n(1), o_\varepsilon(1)$ indicate a generic infinitesimal as $n \to \infty$, $\varepsilon \to 0$.

2.1. Estimates for non-local term

Since we will be dealing with a non-local type problem (1.1), we introduce the two inequalities, which will be frequently used in this article.

Proposition 2.1.

(1) *(Hardy-Littlewood-Sobolev inequality (see [29])) Let $q, r > 1$ and $\alpha \in (0, N)$ with $1/q + 1/r = \frac{N+\alpha}{N}$. Then there exists a constant $C = (N, \alpha, q, r) > 0$ such that*

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^{N-\alpha}} dx\right| \le C|f|_q|g|_r,$$

for all $f \in L^q(\mathbb{R}^N)$ and $g \in L^r(\mathbb{R}^N)$.

(2) *([9]) Assume $q, r > 1$ and $\alpha \in (0, N)$ with $1/q + 1/r < \frac{N+\alpha}{N}$. Then for $\mathrm{R} > 0$, there exists a constant $D_R = D_R(N, \alpha, p, r) > 0$ such that $\lim\limits_{R\to\infty} D_R = 0$ and*

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^{N-\alpha}} dx\right| \le D_R|f|_q|g|_r$$

for all $f \in L^q(\mathbb{R}^N)$ and $g \in L^r(\mathbb{R}^N)$ with $dist(suppf, suppg) \ge R$.

2.2. Limits problems

From [4], we know that the normalized ground state solutions to the following Kirchhoff system

$$\begin{cases} -\Delta v_i + \mu v_i = \left(\int_{\mathbb{R}^N} \frac{|v_i(y)|^p}{|x-y|^{N-\alpha}}\right) |v_i(x)|^{p-2} v_i \quad \text{in } \mathbb{R}^N, \\ v_i(x) > 0, \ \lim\limits_{x\to\infty} v_i(x) = 0, \ i = 1, 2, \ldots, \ell, \\ |v_i|_2^2 = s_i, \ \sum_{i=1}^{\ell} s_i = 1 \end{cases} \tag{2.1}$$

can be used as the building blocks to construct the multi-peak solutions of (1.1). However, the existence of positive normalized ground state solutions for system (2.1) is unknown in the literature. To study system (2.1), we first give the results about the equation

$$\begin{cases} -\Delta v + \mu v = \left(\int_{\mathbb{R}^N} \frac{|v(y)|^p}{|x-y|^{N-\alpha}}\right) |v(x)|^{p-2} v \quad \text{in } \mathbb{R}^N, \\ |v|_2^2 = \omega. \end{cases} \tag{2.2}$$

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Define the energy functional

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2-\frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}dxdy.$$

It follows from [51] that for $\frac{N+\alpha}{N}<p<\frac{N+\alpha+2}{N}$, the minimizing problem

$$I_\omega:=\inf\left\{I(u)|\ |u|_2^2=\omega,\ |\nabla u|_2^2=\frac{Np-N-\alpha}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}dxdy\right\}$$

is achieved and

$$I_\omega=\inf\left\{I(u)|\ |u|_2^2=\omega\right\}<0. \tag{2.3}$$

As for the L^2-supercritical case, when $2<p<\min\{4,N\}$ and $\alpha=2$, Luo [33] proved that I_ω is achieved and

$$I_\omega=\inf\left\{\max_{t>0}I(t^{N/2}u(t\cdot))|\ |u|_2^2=\omega\right\}>0. \tag{2.4}$$

We have to point out that the same conclusions can be obtained when $p\in\left(\frac{N+\alpha+2}{N},\frac{N+\alpha}{N-2}\right)$ by using the minimization method in [33], that is, I_ω is achieved and (2.4) still holds for each $p\in\left(\frac{N+\alpha+2}{N},\frac{N+\alpha}{N-2}\right)$. We will not prove it here. Since the normalized ground state solution of (2.2) is not necessarily unique, for convenience, we define

$$S_\omega=\left\{u\in H^1(\mathbb{R}^N):\ I(u)=I_\omega,|u|_2^2=\omega,\ |\nabla u|_2^2=\frac{Np-N-\alpha}{2p}\right.$$
$$\left.\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}dxdy\right\},$$

where $p\in\left(\frac{N+\alpha}{N},\frac{N+\alpha}{N-2}\right)\setminus\left\{\frac{N+\alpha+2}{N}\right\}$. Let $\omega=\ell^{-1}$, we have

$$\mu_{\ell^{-1}}=\frac{2(-Np+N+\alpha+2p)\ell I_{\ell^{-1}}}{Np-N-2-\alpha}.$$

Clearly, for each $u\in S_{\ell^{-1}}$, $(\mu_{\ell^{-1}},u)$ is the normalized ground state solution of (2.2) when $\omega=\ell^{-1}$.

Set

$$\mathcal{S}_+^\ell:=\left\{s=(s_1,\cdots,s_\ell)\in(0,1)^\ell|\sum_{i=1}^\ell s_i=1\right\}.$$

We define $s^0=\left(\ell^{-1},\cdots,\ell^{-1}\right)$ and $E(s):=\sum_{i=1}^\ell I_{s_i}$ for $s=(s_1,\cdots,s_\ell)\in\mathcal{S}_+^\ell$. Clearly, $(\mu_{\ell^{-1}},(u_{\ell^{-1}},\cdots,u_{\ell^{-1}}))$ is the solution of system (2.1), where $u_{\ell^{-1}}\in S_{\ell^{-1}}$. Next, we claim that $(\mu_{\ell^{-1}},(u_{\ell^{-1}},\cdots,u_{\ell^{-1}}))$ is the normalized ground state solution of (2.1).

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LEMMA 2.2. *For each $s \in \mathcal{S}_+^\ell \setminus s^0$ ($\ell \geq 2$), there holds*

$$E(s) < E(s^0), \ \text{if } p \in \left(\frac{N+\alpha}{N}, \frac{N+\alpha+2}{N}\right), \tag{2.5}$$

$$E(s) > E(s^0), \ \text{if } p \in \left(\frac{N+\alpha+2}{N}, \frac{N+\alpha}{N-2}\right). \tag{2.6}$$

Proof. Let $s = (s_1, \cdots, s_\ell) \in \mathcal{S}_+^\ell \setminus \{0\}$. For each u with $|u|_2^2 = \ell^{-1}$, we define

$$v_i(x) := (s_i\ell)^{\frac{2+\alpha}{2(-Np+N+\alpha+2)}}\, u\left((s_i\ell)^{\frac{p-1}{-Np+N+\alpha+2}}\right).$$

Then by calculating, we have $|v_i|_2^2 = s_i\ell|u|_2^2$, $|\nabla v_i|_2^2 = (s_i\ell)^{\frac{-Np+N+\alpha+2p}{-Np+N+\alpha+2}}|\nabla u|_2^2$ and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_i(x)|^p|v_i(y)|^p}{|x-y|^{N-\alpha}}dxdy = (s_i\ell)^{\frac{-Np+N+\alpha+2p}{-Np+N+\alpha+2}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}dxdy.$$

Thus, according to the definition of I_{s_i}, we deduce $I_{s_i} = (s_i\ell)^{\frac{-Np+N+\alpha+2p}{-Np+N+\alpha+2}} I_{\ell^{-1}}$. Then combining with (2.3) and (2.4), we calculate that $E(s^0) = \ell I_{\ell^{-1}}$ satisfying $E(s^0) < 0$ if $p \in \left(\frac{N+\alpha}{N}, \frac{N+\alpha+2}{N}\right)$ and $E(s^0) > 0$ if $p \in \left(\frac{N+\alpha+2}{N}, \frac{N+\alpha}{N-2}\right)$. Moreover, it holds

$$\begin{aligned} E(s) =& \sum_{i=1}^{\ell} I_{s_i} = E(s^0)\sum_{i=1}^{\ell} s_i^{\frac{-Np+N+\alpha+2p}{-Np+N+2+\alpha}} (\ell)^{\frac{2p-2}{-Np+N+2+\alpha}} \\ =& E(s^0)\sum_{i=1}^{\ell} s_i^{1+\frac{2p-2}{-Np+N+2+\alpha}} (\ell)^{\frac{2p-2}{-Np+N+2+\alpha}}. \end{aligned} \tag{2.7}$$

Denote

$$d_p := \begin{cases} \frac{2p-2}{-Np+N+2+\alpha}, & \text{if } p \in \left(\frac{N+\alpha}{N}, \frac{N+\alpha+2}{N}\right), \\ 1+\frac{2p-2}{-Np+N+2+\alpha}, & \text{if } p \in \left(\frac{N+\alpha+2}{N}, \frac{N+\alpha}{N-2}\right). \end{cases}$$

It follows from the Höder inequality that for $p \in \left(\frac{N+\alpha}{N}, \frac{N+\alpha+2}{N}\right)$,

$$\begin{aligned} \left[\sum_{i=1}^{\ell} s_i^{1+d_p}(\ell)^{d_p}\right]^{\frac{1}{1+d_p}} =& \left[\sum_{i=1}^{\ell} s_i^{1+d_p}(\ell)^{d_p}\right]^{\frac{1}{1+d_p}} \\ \geq& \sum_{i=1}^{\ell} s_i(\ell)^{d_p/(1+d_p)}(\ell)^{-d_p/(1+d_p)} = 1 \end{aligned} \tag{2.8}$$

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and equality holds if and only if $s_i = \ell^{-1}$, $i = 1, 2, \cdots, \ell$. In a similar way, for $p \in \left(\frac{N+\alpha+2}{N}, \frac{N+\alpha}{N-2}\right)$, we also obtain

$$\left[\sum_{i=1}^{\ell} (\ell)^{1+d_p} (s_i)^{d_p}\right]^{\frac{1}{1+d_p}} \geq 1 \tag{2.9}$$

and equality holds if and only if $s_i = \ell^{-1}$, $i = 1, 2, \cdots, \ell$. According to (2.7)–(2.9), we complete the proof. □

LEMMA 2.3. *If $(\mu, (u_1, \cdots, u_\ell))$ is the solution of system (2.1), then we have $\sum_{i=1}^{\ell} I(u_i) \geq E(s^0)$. Moreover, equalities hold if and only if $u_i \in S_{\ell^{-1}}$ for each $i = 1, \cdots, \ell$.*

Proof. Let $|u_i|_2^2 = s_i$, $\sum\limits_{i=1}^{\ell} s_i = 1$ for each $i = 1, \cdots, \ell$. For convince, we denote

$$A = \frac{2(-Np + N + \alpha + 2p)}{-Np + N + 2 + \alpha}.$$

If $p \in \left(\frac{N+\alpha}{N}, \frac{N+\alpha+2}{N}\right)$, then $A > 0$ and a direct calculation that

$$\mu = -A\frac{I(u_i)}{s_i} \leq -A\frac{I_{s_i}}{s_i} = -A\,(s_i\ell)^{\frac{-Np+N+\alpha+2p}{-Np+N+\alpha+2}}\,\frac{I_{\ell^{-1}}}{s_i} \tag{2.10}$$

and

$$\mu = -A\sum_{i=1}^{\ell} I(u_i) \leq -A\sum_{i=1}^{\ell} I_{s_i} = -AE(s). \tag{2.11}$$

where $i = 1, \cdots, \ell$. If $s_i = \ell^{-1}$ for each $i = 1, \cdots, \ell$, by $I(u_i) \geq I_{\ell^{-1}}$, we easily obtain $E(s) \geq E(s^0)$ and equalities holds if and only if $I(u_i) = I_{\ell^{-1}}$, i.e., $u_i \in S_{\ell^{-1}}$. Otherwise, there exists i_0 such that $s_{i_0} < \ell^{-1}$. Combining with (2.10) and (2.11), we obtain that $\mu = -A\sum_{i=1}^{\ell} I(u_i) \leq -As_{i_0}^{\frac{2p-2}{-Np+N+\alpha+2}}\ell^{\frac{-Np+N+\alpha+2p}{-Np+N+\alpha+2}} I_{\ell^{-1}} < -A\ell I_{\ell^{-1}}$. This implies $\sum\limits_{i=1}^{\ell} I(u_i) > \ell I_{\ell^{-1}} = E(s^0)$. If $p \in \left(\frac{N+\alpha+2}{N}, \frac{N+\alpha}{N-2}\right)$, then by Lemma 2.2, we have $\sum\limits_{i=1}^{\ell} I(u_i) \geq \sum\limits_{i=1}^{\ell} I_{s_i} = E(s) \geq E(s^0)$. □

3. Concentration-compactness type results

Without loss of generality, we may suppose that $L = \{1, 2, \cdots, \ell\}$ and give the following notations: For $i = 1, \cdots, \ell$, $\varepsilon \in (0, 1)$, and $t > 0$,

$$\Omega_{i,\varepsilon} := \{x \in \mathbb{R}^N | \varepsilon x \in \Omega_i\}, \quad \Omega_{L,\varepsilon} := \{x \in \mathbb{R}^N | \varepsilon x \in \Omega_L\}, \quad \Omega_\varepsilon := \{x \in \mathbb{R}^N | \varepsilon x \in \Omega\},$$
$$\Omega_{i,\varepsilon}^t := \{x \in \mathbb{R}^N | dist(\varepsilon x, \Omega_i) < t\}, \quad \Omega_{L,\varepsilon}^t := \{x \in \mathbb{R}^N | dist(\varepsilon x, \Omega_L) < t\}.$$

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If follows from (V_1) that we may assume that

$$r := \frac{1}{20}\min_{1\le\ell\le\ell_0} dist(\overline{\Omega}_i, \overline{\Omega\setminus\Omega_i}) > 0.$$

Define

$$X_\varepsilon = \left\{ \sum_{i=1}^{\ell} (\xi_\varepsilon w_i)(\cdot - x_i/\varepsilon) |\ x_i \in \Omega_i,\ w_i \in S_{\ell-1} \right\},$$

where $\xi_\varepsilon(\cdot) = \xi(\varepsilon\cdot)$ with $\xi \in C_0^\infty(B_{2r}(0), [0,1])$ satisfying $\xi = 1$ in $B_r(0)$ and $|\nabla\xi| \le 2/r$. Note that $supp\xi_\varepsilon(\cdot - x_i/\varepsilon) \cap \cup_{j\neq i}\Omega_{j,\varepsilon} = \emptyset$ for $x_i \in \overline{\Omega}_i$. Set

$$d_\ell := \min\left\{1, \left\{ \left(\frac{1}{2A_p} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{N-\alpha}} \right)^{\frac{1}{2p}} :\ u \in S_{\ell-1} \right\}\right\},$$

where A_p is the Sobolev embedding constant from $H^1(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$. For sufficiently small $\varepsilon > 0$, we will find a solution to (1.1) in the following set

$$X_\varepsilon^d = \left\{ u \in M_\varepsilon |\ \inf_{v\in X_\varepsilon} \|u - v\|_\varepsilon \le d \right\}.$$

where $0 < d < d_\ell$.

Remark 3.1. For $0 < d < d_\ell$, we have $\|u\| \le \limsup_{\varepsilon\to 0^+} \|u\|_\varepsilon \le \ell \max_{i=1,\cdots,\ell} \|w_i\| + 1$ for $u \in X_\varepsilon^d$. Here, $w_i \in S_{\ell-1}$ is defined as in X_ε. Moreover,

$$\sup\{\|u\|_\varepsilon | \varepsilon \in (0,1), u \in X_\varepsilon^d\} < +\infty.$$

In fact, since $d_\ell \le 1$ and $\|V_\varepsilon\|_{L^\infty(\Omega_{L,\varepsilon}^{2r}\setminus\Omega_{L,\varepsilon})} = \|V\|_{L^\infty(\Omega_L^{2r}\setminus\Omega_L)} < +\infty$, we have

$$\|\xi_\varepsilon w_i\| \to \|w_i\| \text{ and } \sup_{x_\varepsilon\in\overline{\Omega}_{L,\varepsilon}} \int_{\mathbb{R}^N} \varepsilon^2 V_\varepsilon(x) |(\xi_\varepsilon u(x - x_\varepsilon))|^2 \to 0 \text{ as } \varepsilon \to 0.$$

So the conclusion holds.

Define

$$G(u) := \frac{1}{2}\int_{\mathbb{R}^N} |u|^2, \text{ for } u \in E_\varepsilon.$$

Then we give an ε, λ-dependent concentration-compactness type results, which will be useful to develop deformation theory in Section 5.

Proposition 3.2. *For any $\varepsilon \in (0,1)$ fixed, the Palais–Smale condition holds for $J_{\lambda,\varepsilon}$ in the set X_ε^d. That is, $\lambda_\varepsilon^* > 0$ exists such that for any $\lambda > \lambda_\varepsilon^*$, if $\{u_n\} \subset X_\varepsilon^d$ satisfies*

$$\limsup_{n\to\infty} J_{\lambda,\varepsilon}(u_n) \le E(s^0), \tag{3.1}$$

$$\lim_{n\to\infty} \inf_{\mu_n\in\mathbb{R}} \|J'_{\lambda,\varepsilon}(u_n) + \mu_n G'(u_n)\|_{E_\varepsilon^*} = 0, \tag{3.2}$$

then $\mathrm{u_n}$ *has a strong convergent subsequence in* $\mathrm{E_\varepsilon}$.

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Proof. From Remark 3.1, we easily obtain that $\{u_n\}$ is bounded in E_ε. Since

$$J_{\lambda,\varepsilon}(u_n)+\frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n|^p|u_n|^p}{|x-y|^{N-\alpha}}=\frac{1}{2}|\nabla u_n|_2^2+\frac{1}{2}\int_{\mathbb{R}^N}\lambda\varepsilon^2V_\varepsilon(x)u_n^2+o_n(1),$$

combining with the Hardy-Little-wood-Sobolev inequality and the assumption (3.1), we obtain that for large n,

$$\frac{1}{2}\int_{\mathbb{R}^N}\lambda\varepsilon^2V_\varepsilon(x)u_n^2\le J_{\lambda,\varepsilon}(u_n)+C\|u_n\|^{2p}\le C_1$$

and

$$\begin{aligned}|\mu_n|=&|\mu_nG'(u_n)u_n|=|J'_{\lambda,\varepsilon}(u_n)u_n+o_n(1)|\\ \le& C\left(\|u_n\|_\varepsilon^2+\|u_n\|_\varepsilon^{2p}+\int_{\mathbb{R}^N}\lambda\varepsilon^2V_\varepsilon(x)u^2\right)\le C_1,\end{aligned}$$

where the constant $C_1>0$ independent of ε and λ. Since $\{u_n\}$ is bounded in E_ε, we assume, up to a subsequence, $u_n\rightharpoonup u_0$ in E_ε.

Next, we claim $u_n\to u_0$ in E_ε. We first claim that $\lambda_\varepsilon^*>0$ exists such that for any $\lambda>\lambda_\varepsilon^*$ and $\delta>0$, $R>1$ exists such that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B_{2R}(0)}\left(|\nabla u_n|^2+\left(\frac{\lambda}{2}\varepsilon^2V_\varepsilon(x)+1\right)u_n^2\right)<\delta. \tag{3.3}$$

we fix a cut-off function $\overline{\psi}\in C^\infty(\mathbb{R}^N,[0,1])$ satisfying $\overline{\psi}(x)=1$ in $|x|\ge 2$, $\overline{\psi}(x)=0$ in $|x|\le 1$ and $|\nabla\overline{\psi}(x)|\le 2$ for all $x\in\mathbb{R}^N$. For convenience, set $\overline{\psi}_R(\cdot)=\overline{\psi}(\cdot/R)$. Then we text (3.2) by $\overline{\psi}_R^2u_n$ to get

$$\begin{aligned}&\int_{\mathbb{R}^N}\left[|\nabla(\overline{\psi}_Ru_n)|^2+(\lambda\varepsilon^2V_\varepsilon(x)+\mu_n)(\overline{\psi}_Ru_n)^2\right]\\ =&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\overline{\psi}_R^2|u_n(x)|^p|u_n(y)|^p}{|x-y|^{N-\alpha}}+\int_{\mathbb{R}^N}|\nabla\overline{\psi}_R|^2u_n^2+o_n(1).\end{aligned} \tag{3.4}$$

Using the two inequalities of Proposition 2.1, we derive that

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\overline{\psi}_R^2|u_n(x)|^p|u_n(y)|^p}{|x-y|^{N-\alpha}}\le&\int_{\mathbb{R}^N\setminus B_{2R}(0)}\int_{\mathbb{R}^N\setminus B_{2R}(0)}\frac{|u_n(x)|^p|u_n(y)|^p}{|x-y|^{N-\alpha}}\\ &+2\int_{R<|x|<2R}\int_{\mathbb{R}^N}\frac{\overline{\psi}_R^2(x)|u_n(x)|^p|u_n(y)|^p}{|x-y|^{N-\alpha}}\\ &+\int_{B_R(0)}\int_{\mathbb{R}^N\setminus B_{2R}(0)}\frac{|u_n(x)|^p|u_n(y)|^p}{|x-y|^{N-\alpha}}\\ \le&C|u_n|^{2p}_{L^{2Np/(N+\alpha)}(\mathbb{R}^N\setminus B_{2R}(0))}+D_R\\ \le&C\|\overline{\psi}_Ru_n\|^2|u_n|^{2(p-1)}_{L^{2Np/(N+\alpha)}(\mathbb{R}^N\setminus B_{2R}(0))}+D_R\end{aligned}$$

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and

$$\int_{\mathbb{R}^N} |\nabla \overline{\psi}_R|^2 u_n^2 = \int_{R<|x|<2R} |\nabla \overline{\psi}_R|^2 u_n^2 \leq \frac{C}{R}.$$

where C is a positive constant independent of ε, R, and large n, $\lim\limits_{R\to\infty} D_R = 0$. Then combining with (3.4), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} \left[|\nabla(\overline{\psi}_R u_n)|^2 + (\lambda\varepsilon^2 V_\varepsilon(x) + \mu_n)(\overline{\psi}_R u_n)^2\right]\\ \leq& C_2 \|\overline{\psi}_R u_n\|^2 |u_n|^{2(p-1)}_{L^{2Np/(N+\alpha)}(\mathbb{R}^N\setminus B_{2R}(0))} + D_R + \frac{C_2}{R} + o_n(1),\end{aligned} \tag{3.5}$$

where C_2 is a constant independent of ε, λ, R and large n. According to the condition (V_3), we denote

$$K_0(R) := \left(\mathbb{R}^N \setminus B_R(0)\right) \cap \{x \in \mathbb{R}^N | V_\varepsilon(x) \leq M_0\}$$

and

$$L_0(R) := \left(\mathbb{R}^N \setminus B_R(0)\right) \cap \{x \in \mathbb{R}^N | V_\varepsilon(x) > M_0\}.$$

Then we deduce that for every $\varepsilon, \lambda > 0$,

$$\lim_{R\to\infty} meas\ K_0(R) = 0, \quad \lim_{R\to\infty} \limsup_{n\to\infty} \int_{K_0(R)} |u_n|^{\frac{2Np}{N+\alpha}} = 0 \tag{3.6}$$

and

$$\begin{aligned}&\lim_{R\to\infty} \limsup_{n\to\infty} \int_{K_0(R)} \left(\lambda\varepsilon^2 V_\varepsilon(x) + |\mu_n|\right) (\overline{\psi}_R u_n)^2\\ \leq& \lim_{R\to\infty} \limsup_{n\to\infty} \int_{K_0(R)} \left(\lambda\varepsilon^2 M_0 + C_1\right) (\overline{\psi}_R u_n)^2 = 0.\end{aligned} \tag{3.7}$$

It follows from the Gagliardo–Nirenberg inequality that

$$\begin{aligned}\int_{L_0(R)} |u_n|^{\frac{2Np}{N+\alpha}} \leq& C_3 |\nabla u_n|^{\frac{(Np-N-\alpha)N}{N+\alpha}}_{L^2(L_0(R))} |u_n|^{\frac{(-Np+N+\alpha+2p)N}{N+\alpha}}_{L^2(L_0(R))}\\ \leq& C_3 \left(M_0^{-1}\lambda^{-1}\varepsilon^{-2}\right)^{\frac{(-Np+N+\alpha+2p)N}{N+\alpha}},\end{aligned} \tag{3.8}$$

where C_3 is a positive constant independent of ε, λ and large n. Using (3.6) and (3.8), and fixing λ^*_ε large enough such that for $\lambda > \lambda^*_\varepsilon$,

$$\lim_{R\to\infty} \limsup_{n\to\infty} |u_n|^{2(p-1)}_{L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N\setminus B_{2R}(0))} < (2C_2)^{-1}, \ \frac{\lambda}{2}\varepsilon^2 M_0 - C_1 > 1. \tag{3.9}$$

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Then by (3.7) and (3.9),

$$\begin{aligned}
&\limsup_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}(\lambda\varepsilon^2V_\varepsilon(x)+\mu_n)(\overline{\psi}_Ru_n)^2\\
&=\limsup_{R\to\infty}\limsup_{n\to\infty}\int_{L_0(R)}(\lambda\varepsilon^2V_\varepsilon(x)+\mu_n)(\overline{\psi}_Ru_n)^2\\
&\geq\limsup_{R\to\infty}\limsup_{n\to\infty}\int_{L_0(R)}\left(\frac{1}{2}\lambda\varepsilon^2V_\varepsilon(x)+1\right)(\overline{\psi}_Ru_n)^2\\
&=\limsup_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(\frac{1}{2}\lambda\varepsilon^2V_\varepsilon(x)+1\right)(\overline{\psi}_Ru_n)^2. \qquad (3.10)
\end{aligned}$$

Thus, it follows from (3.5), (3.9), and (3.10) that

$$\limsup_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left[|\nabla(\overline{\psi}_Ru_n)|^2+\left(\frac{\lambda}{2}\varepsilon^2V_\varepsilon(x)+1\right)(\overline{\psi}_Ru_n)^2\right]=0,$$

which implies (3.3). Then by the Rellich theorem and (3.3), we deduce that

$$\int_{\mathbb{R}^N}V_\varepsilon(x)|u_n-u_0|^2\to 0 \text{ and } u_n\to u_0 \text{ in } L^q(\mathbb{R}^N), q\in\left[2,\frac{N+\alpha}{N-2}\right). \qquad (3.11)$$

According to this, we text (3.2) by u_n-u_0 and get

$$\begin{aligned}
\int_{\mathbb{R}^N}\nabla u_n\nabla(u_n-u_0)=&-\int_{\mathbb{R}^N}\lambda\varepsilon^2V_\varepsilon(x)u_n(u_n-u_0)-\mu_n(u_n-u_0)u_n\\
&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n|^pu_n^{p-1}(u_n-u_0)}{|x-y|^{N-\alpha}}\\
=&o_n(1),
\end{aligned}$$

which implies

$$\int_{\mathbb{R}^N}|\nabla u_n-\nabla u_0|^2\to 0. \qquad (3.12)$$

By (3.11)–(3.12), we easily get $u_n\to u_0$ in E_ε. Thus, we complete the proof. □

Since $meas(\Omega_{L,\varepsilon}^{8r}\backslash\Omega_{L,\varepsilon})<+\infty$ and $V_\varepsilon>0$ a.e. in $\Omega_{L,\varepsilon}^{8r}\backslash\Omega_{L,\varepsilon}$, we choose $\alpha_\varepsilon\in(0,1)$ such that

$$meas(A^\varepsilon:=\{x\in\Omega_{L,\varepsilon}^{8r}\setminus\Omega_{L,\varepsilon}:\ V_\varepsilon(x)<\alpha_\varepsilon\})\to 0 \text{ as } \varepsilon\to 0. \qquad (3.13)$$

Then we have the following result.

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PROPOSITION 3.3. *For sufficiently small* $\mathrm{d}>0$, $J_{\lambda,\varepsilon}$ *has the following property: let* $\{\varepsilon_n\}$, $\{\lambda_n\}$, $\{\mu_n\}\subset\mathbb{R}$, *and* $\{u_n\}\subset X_\varepsilon^d$ *satisfying that*

$$\begin{aligned}&\varepsilon_n\to 0^+,\quad \lambda_n\varepsilon_n^2\alpha_{\varepsilon_n}\to+\infty,\\ &\lim_{n\to\infty} J_{\lambda_n,\varepsilon_n}(u_n)\le E(s^0),\end{aligned}\tag{3.14}$$

$$\|J'_{\lambda_n,\varepsilon_n}(u_n)+\mu_n G'(u_n)\|_{E^*_{\varepsilon_n}}\to 0,\tag{3.15}$$

as $n\to\infty$. *Then* $\mu_n\to\mu_{\ell^{-1}}$ *and for each* $i=1,\cdots,\ell$, *there exist* $q_{n,i}\in\Omega_{i,\varepsilon_n}$, $u_i\in S_{\ell^{-1}}$ *such that*

$$dist(q_{n,i},\partial\Omega_{i,\varepsilon_n})\to\infty\quad and\quad \|u_n-\sum_{i=1}^{\ell}(\xi_{\varepsilon_n}u_i(\cdot-q_{n,i}))\|_{\varepsilon_n}\to 0.$$

Proof. Since $u_n\in X_\varepsilon^d$, there exist $q_{n,i}\in\overline{\Omega}_{i,\varepsilon_n}$, $u_i\in S_{\ell^{-1}}$ and $h_n\in E_{\varepsilon_n}$ satisfying $\|h_n\|_{\varepsilon_n}\le d$ such that $u_n=\sum_{i=1}^{\ell}(\xi_{\varepsilon_n}u_i)(\cdot-q_{n,i})+h_n$. By Remark 3.1, we obtain that $\|u_n\|$ and $\|u_n\|_{\varepsilon_n}$ are bounded for large n. Similar to Proposition 3.2, by (3.14), we also deduce that for large n,

$$|\mu_n|,\ \int_{\mathbb{R}^N}\lambda_n\varepsilon_n^2V_{\varepsilon_n}(x)u_n^2\le C$$

where the constant $C>0$ is independent of n. Then by (3.13),

$$\begin{aligned}\int_{\mathbb{R}^N\setminus\Omega_{L,\varepsilon}}|u_n|^2&\le\int_{\mathbb{R}^N\setminus\Omega_{L,\varepsilon}^{8r}}|u_n|^2+\int_{\Omega_{L,\varepsilon}^{8r}\setminus(A^{\varepsilon_n}\cup\Omega_{L,\varepsilon})}u_n^2+\int_{A^{\varepsilon_n}}|u_n|^2\\&\le\int_{\mathbb{R}^N\setminus\Omega_{L,\varepsilon_n}^{8r}}|u_n|^2+(\lambda\varepsilon^2\alpha_{\varepsilon_n})^{-1}+|measA^{\varepsilon_n}|^{\frac{\alpha+2}{N+\alpha}}|u_n|^2_{\frac{N+\alpha}{N-2}},\end{aligned}$$

where $measA^{\varepsilon_n}\to 0$ as $n\to\infty$. Now we claim that $\lim_{n\to\infty}\int_{\mathbb{R}^N\setminus\Omega_{L,\varepsilon_n}^{8r}}u_n^2=0$. Otherwise, by Lions' Lemma, there is $x_{\varepsilon_n}\in\mathbb{R}^N\setminus\Omega_{L,\varepsilon_n}^{8r}$ such that

$$\limsup_{n\to\infty}\int_{B_1(x_{\varepsilon_n})}|u_n|^2>0.$$

Since $|q_{n,i}-x_{\varepsilon_n}|\to\infty$, we have

$$|\sum_{i=1}^{\ell}(\xi_{\varepsilon_n}u_i)(\cdot-q_{n,i})|^2_{L^2(B_1(x_{\varepsilon_n}))}=o_n(1).$$

It follows from $\|u_n-\sum_{i=1}^{\ell}(\xi_{\varepsilon_n}u_i)(\cdot-q_{n,i})\|_{B_1(x_{\varepsilon_n})}\le 2d$ that we obtain a contradiction for sufficiently small d. Hence, we conclude that $\int_{\mathbb{R}^N\setminus\Omega_{L,\varepsilon_n}^{8r}}|u_n|^2=o_n(1)$.

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Since $|\mu_n|$ and $\|u_n\|$ are bounded, up to a subsequence, we may assume that $\mu_n \to \mu_0$ in $\mathbb{R}$ and

$$\begin{aligned} &u_n(\cdot + q_{n,i}) \rightharpoonup v_i \quad \text{in } H^1(\mathbb{R}^N) \text{ and } L^q(\mathbb{R}^N), q \in \left[2, \frac{N+\alpha}{N-2}\right) \\ &u_n(\cdot + q_{n,i}) \to v_i \quad \text{in } L^q_{loc}(\mathbb{R}^N), q \in \left[2, \frac{N+\alpha}{N-2}\right) \\ &u_n(\cdot + q_{n,i}) \to v_i \quad a.e. \text{ in } \mathbb{R}^N. \end{aligned}$$

Obviously, there holds

$$\begin{aligned} &\lim_{R\to\infty}\lim_{n\to\infty}\int_{B_R(0)}\int_{B_R(0)}\frac{|u_n(\cdot+q_{n,i})|^p|u_n(\cdot+q_{n,i})|^p}{|x-y|^{N-\alpha}} \\ &=\lim_{R\to\infty}\int_{B_R(0)}\int_{B_R(0)}\frac{|v_i|^p|v_i|^p}{|x-y|^{N-\alpha}}=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_i|^p|v_i|^p}{|x-y|^{N-\alpha}}. \end{aligned} \tag{3.16}$$

On the other hand, since $|q_{n,i}-q_{n,j}|\to\infty$ if $i\neq j$, we get that for each fixed $R>0$,

$$\int_{B_R(0)}\int_{B_R(0)}\frac{|u_n(\cdot+q_{n,i})|^p|u_n(\cdot+q_{n,i})|^p}{|x-y|^{N-\alpha}} \geq \int_{B_R(0)}\int_{B_R(0)}\frac{|u_i|^p|u_i|^p}{|x-y|^{N-\alpha}} - A_p d^{2p} + o_n(1).$$

For d sufficiently small, according to the above inequality and combining with (3.16), we deduce that

$$\begin{aligned} \int_{R^N}\int_{R^N}\frac{|v_i(x)|^p\,|v_i(y)|^p}{|x-y|^{N-\alpha}} &\geq \int_{R^N}\int_{R^N}\frac{|u_i|^p\,|u_i|^p}{|x-y|^{N-\alpha}} - A_p d^{2p} \\ &\geq \frac{1}{2}\int_{R^N}\int_{R^N}\frac{|u_i|^p\,|u_i|^p}{|x-y|^{N-\alpha}} \geq 0. \end{aligned} \tag{3.17}$$

By Fatou's lemma, we have

$$|v_i|_2 \leq \lim_{n\to\infty}\inf|u_n|^2 = 1 < 2. \tag{3.18}$$

The following proof proceeds in four claims.

Claim 1 $\lim_{n\to\infty} dist(q_{n,i}, \partial\Omega_{i,\varepsilon_n}) \to \infty$.

Suppose, on the contrary, that for some i, up to a subsequence, $\lim_{n\to\infty} dist(q_{n,i}, \partial\Omega_{i,\varepsilon_n}) = M \in [0,\infty)$, and for some $q_i \in \partial\Omega_i$, $\varepsilon_n q_{n,i} \to q_i$ as $n\to\infty$. By a transformation, we may assume that $e_N := (0,\cdots,0,1)$ is inward normal to $\partial\Omega_i$ at q_i. Denote $A = \{x\in\mathbb{R}^N | x_N > -M\}$ and let $B + q_{n,i} = \{x + q_{n,i} | x \in B\}$ where $B \subset \text{int}(\mathbb{R}^N \setminus A)$ is a closed ball. Then we claim

$$\lim_{n\to\infty}\inf dist(B+q_{n,i}, \Omega_{i,\varepsilon_n}) > 0. \tag{3.19}$$

Through the compactness and indirect argument, we assume $|x+q_{n,i}-x_n|\to 0$ for some $x\in B$ and $x_n \in \Omega_{i,\varepsilon_n}$. Let $q_n \in \partial\Omega_{i,\varepsilon_n}$ be such that $|q_{n,i}-q_n| = dist(q_{n,i}, \partial\Omega_{i,\varepsilon_n}) \to M$. Clearly, it holds $\{q_{n,i}, x+q_{n,i}, x_n\} \subset B_{|x|+M+1}(q_n)$ for large

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n. Since $\varepsilon_n q_n \to q_i$, we derive that $(q_{n,i} - q_n) \cdot e_N \to M$ and $\sup\{(q_n - y) \cdot e_N | y \in \Omega_{i,\varepsilon_n} \cap B_{|x|+M+1}(q_n)\} \to 0$. By calculating, we conclude that

$$0 = \lim_{n\to\infty} \sup(x + q_{n,i} - x_n) \cdot e_N = x \cdot e_N + M + \lim_{n\to\infty} \sup(q_n - x_n) \cdot e_N < 0,$$

which contradicts with (3.19). Since

$$\begin{aligned} 0 &= \lim_{n\to\infty} |u_n|_{L^2(\mathbb{R}^N \setminus \Omega_{\varepsilon_n})} \le \lim_{n\to\infty} |u_n|_{L^2(B+q_{n,i})} \\ &= \lim_{n\to\infty} |u_n(\cdot + q_{n,i})|_{L^2(B)} = |v_i|_{L^2(B)}, \end{aligned}$$

we obtain that $v_i \in H_0^1(A)$ by the arbitrary choice of B. Next, we text (3.15) by $\psi(\cdot - q_{n,i})$ for any $\psi \in C_0^\infty(A)$. Similar to (3.19), we show that $supp\psi(\cdot - q_{n,i}) \subset \Omega_{L,\varepsilon_n}$ for large n,

$$\begin{aligned} o_n(1) = &\int_{\mathbb{R}^N} \nabla u_n \nabla \psi(x - q_{n,i}) + \int_{\mathbb{R}^N} \lambda_n \varepsilon_n^2 V_{\varepsilon_n}(x) u_n \psi(x - q_{n,i}) + \mu_n \\ &\times \int_{\mathbb{R}^N} u_n \psi(x - q_{n,i}) - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(y)|^p |u_n(x)|^{p-2} u_n(x) \psi(x - q_{n,i})}{|x - y|^{N-\alpha}} \\ = &\int_{\mathbb{R}^N} \nabla u_n(x + q_{n,i}) \nabla \psi(x) + \mu_n \int_{\mathbb{R}^N} u_n(x + q_{n,i}) \psi(x) \\ &- \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(y + q_{n,i})|^p |u_n(x + q_{n,i})|^{p-2} u_n(x + q_{n,i}) \psi(x)}{|x - y|^{N-\alpha}} + o_n(1) \\ = &\int_A \nabla v_i \nabla \psi(x) + \mu_0 \int_A v_i \psi(x) - \int_A \int_A \frac{|v_i|^p |v_i|^{p-2} \psi(x) v_i(x)}{|x - y|^{N-\alpha}} + o_n(1). \end{aligned}$$

According to this, we deduce that v_i is a nontrivial solution to $-\Delta v_i + \mu_0 v_i = \left(\int_A \frac{|v_i|^p}{|x-y|^{N-\alpha}}\right) |v_i|^{p-2} v_i$, which is impossible since this equation has no nontrivial solution on a half space by [35].

Claim 2. $\mu_0 > 0$ and $v_i > 0$ for $i = 1, \cdots, \ell$.

Since $\lim_{n\to\infty} dist(q_{n,i}, \partial\Omega_{i,\varepsilon_n}) = \infty$, it holds

$$-\Delta v_i + \mu_0 v_i = \left(\int_{\mathbb{R}^N} \frac{|v_i|^p}{|x - y|^{N-\alpha}}\right) |v_i|^{p-2} v_i, \quad \text{in } H^1(\mathbb{R}^N). \tag{3.20}$$

By the Pohozaev identity, we deduce $\mu_0 > 0$. Now we claim $v_i > 0$. Since $u > 0$, we deduce that for any $d < 1$,

$$\int_{\mathbb{R}^N} |u_n^-|^2 \le \int_{\mathbb{R}^N} \left| u_n - \sum_{j=1}^{\ell} (\xi_{\varepsilon_n} u_i)(\cdot - q_{n,i}) \right|^2 = \int_{\mathbb{R}^N} |h_n|^2 \le d^2$$

From Fatou's Lemma, one has

$$|v_i^-|_2^2 \le \liminf_{n\to\infty} |u_n^-(\cdot + q_{n,j})|_2^2 < d. \tag{3.21}$$

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By Remark 3.1, $\|v_i^-\| \le \ell \max_{i=1,\cdots,\ell}\|u_i\| + d$. According to equation (3.20), we get

$$\int_{\mathbb{R}^N} |\nabla v_i^-|^2 + \mu (v_i^-)^2 = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_i^-|^p |v_i^-|^p}{|x-y|^{N-\alpha}} dxdy \le C_\beta |(v_i^-)^2|_2^{2p} + D\beta |\nabla v_i^-|_2^2,$$

where β is chosen so that $D\beta = \beta(\ell\|v_i^-\|+1)^{2(p-1)} = 1$. This implies

$$\mu \le C_\beta \left(\int_{\mathbb{R}^N} (v_i^-)^2 \right)^{p-1}. \tag{3.22}$$

Combining with (3.21)–(3.22), we deduce $v_i^- = 0$ by controlling d small. Using the maximum principle, we have $v_i > 0$.

Claim 3. $u_n - \sum_{i=1}^{\ell} v_i(\cdot - q_{n,i}) \to 0$ in $L^q(\mathbb{R}^N)$ for any $q \in \left[2, \frac{N+\alpha}{N-2}\right)$.

Suppose, on the contrary, that according to the Lions' vanishing lemma [31], there is a point sequence $\{x_n\} \subset \mathbb{R}^N$ that satisfies

$$\limsup_{n\to\infty} \int_{B_1(x_n)} \left| u_n - \sum_{i=1}^{\ell} v_i(\cdot - q_{n,i}) \right|^2 > 0. \tag{3.23}$$

For any fixed $R>0$, there holds

$$\int_{B_R(q_{n,i})} |u_n - \sum_{j=1}^{\ell} v_j(\cdot - q_{n,j})|^2 = \int_{B_R(0)} |u_n(\cdot + q_{n,i}) - v_i|^2 + o_n(1) \to 0.$$

Combining with (3.23), we have $|x_n - q_{n,i}| \to \infty$ for $i = 1, 2, \cdots, \ell$. Hence, we deduce $\|v_i(\cdot + x_n - q_{n,i})\|_{L^2(B_1(0))} \to 0$ and

$$\limsup_{n\to\infty} |u_n(\cdot + x_n)|_{L^2(B_1(0))} > 0. \tag{3.24}$$

We assume, up to a subsequence, $u_n(\cdot + x_n) \rightharpoonup v_0$ in $H^1(\mathbb{R}^N)$. It follows from (3.24) that $v_0 \neq 0$.

For $R>0$, we set $\psi_R = 1 - \overline{\psi}(\cdot/R) \in C_0^\infty(B_{2R}(0))$ where $\overline{\psi}$ is defined as Proposition 3.2. Then we test (3.15) by $\psi_{R,n}u_n := \psi_R(\cdot - x_n)u_n$ and obtain that

$$\begin{aligned} o_n(1) &= \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + \mu_n u_n^2 + \lambda_n \varepsilon_n^2 V_{\varepsilon_n} u_n^2 \right) \psi_{R,n} + \int_{\mathbb{R}^N} u_n \nabla u_n \nabla \psi_{R,n} \\ &\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(y)|^p |u_n(x)|^p \psi_{R,n}}{|x-y|^{N-\alpha}} \\ &\ge \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + \mu_n u_n^2 \right) \psi_{R,n} - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(y)|^p |u_n(x)|^p \psi_{R,n}}{|x-y|^{N-\alpha}} - \frac{C}{R}. \end{aligned}$$

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By the Rellich theorem and $\mu_n \to \mu_0$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \mu_n u_n^2 \psi_{R,n} - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(y)|^p |u_n(x)|^p \psi_{R,n}}{|x-y|^{N-\alpha}} \\
=&\int_{\mathbb{R}^N} \mu_n u_n(\cdot + x_n)^2 \psi_R - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(y+x_n)|^p |u_n(x+x_n)|^p \psi_R}{|x-y|^{N-\alpha}} \\
\to&\int_{\mathbb{R}^N} \mu_0 v_0^2 \psi_R - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_0|^p |v_0|^p \psi_R}{|x-y|^{N-\alpha}}.
\end{aligned}$$

Then we get that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(|\nabla v_0|^2 + \mu_0 v_0^2\right) \psi_R - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_0(y)|^p |v_0(x)|^p \psi_R(x)}{|x-y|^{N-\alpha}} \\
\le& \limsup_{n\to\infty} \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + \mu_n u_n^2\right) \psi_R - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(y)|^p |u_n(x)|^p \psi_{R,n}}{|x-y|^{N-\alpha}} \\
\le& \frac{C}{R}.
\end{aligned}$$

Passing to the limit $R \to \infty$ in the above equation yields that it holds

$$\int_{\mathbb{R}^N} |\nabla v_0|^2 + \mu_0 v_0^2 \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_0(x)|^p |v_0(y)|^p}{|x-y|^{N-\alpha}}.$$

Arguing in a similar way to (3.17) and (3.22) yields

$$|v_i|_2 \ge |u_i|_2 - d = \ell^{\frac{-1}{2}} - d$$

and

$$\mu_0 \le C_\beta \left(\int_{\mathbb{R}^N} |v_0|^2\right)^{p-1}.$$

Then when d sufficiently small,

$$\sum_{i=1}^{\ell} |v_i|_2^2 + |v_0|_2^2 \ge 1 - 2d\ell^{1/2} + \ell d^2 + \ell |v_0|_2^2 > 1.$$

This is impossible since

$$\begin{aligned}
1 = \lim_{n\to\infty} |u_n|_2^2 \ge& \lim_{R\to\infty} \limsup_{n\to\infty} \left(\sum_{i=1}^{\ell} \|u_n(\cdot + q_{n,i})\|^2_{L^2(B_R(0))} + \|u_n(\cdot + x_n)\|^2_{L^2(B_R(0))}\right) \\
=& \sum_{i=1}^{\ell} |v_i|_2^2 + |v_0|_2^2.
\end{aligned}$$

Thus, claim 3 is proved.

Claim 4. $\|u_n - \sum_{i=1}^{\ell} (\xi_{\varepsilon_n} v_i)(\cdot - q_{n,i})\|_{\varepsilon_n} \to 0.$

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Let $R_n := \frac{1}{2}\lim\limits_{n\to\infty}\min\limits_{1\le i\le \ell} dist\,(q_{n,i}, \partial\Omega_{i,\varepsilon_n})$. According to Claim 1, we have $R_n \to \infty$. Set $\widetilde{\psi}_n = 1 - \overline{\psi}(\cdot/R_n) \in C_0^\infty(B_{2R_n}(0))$. It is easy to check that $\|\,(\xi_{\varepsilon_n} v_i)\,(\cdot - q_{n,i}) - (\widetilde{\psi}_n v_i)(\cdot - q_{n,i})\|_{\varepsilon_n} \to 0$. Thus, we only need to claim

$$\|u_n - \sum_{i=1}^{\ell}(\widetilde{\psi}_n v_i)(\cdot - q_{n,i})\|_{\varepsilon_n} \to 0. \tag{3.25}$$

Noting that $supp\ \widetilde{\psi}_n(\cdot - q_{n,i}) \subset B_{2R_n}(q_{n,i}) \subset\subset \Omega_{i,\varepsilon_n}$ and (3.15) is tested by $u_n - \sum\limits_{i=1}^{\ell}(\widetilde{\psi}_n v_i)(\cdot - q_{n,i})$, as well as claim 3, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla u_n \nabla \left(u_n - \sum_{i=1}^{\ell}(\widetilde{\psi}_n v_i)(\cdot - q_{n,i})\right) \\
&+ \int_{\mathbb{R}^N} (\lambda_n \varepsilon_n^2 V_{\varepsilon_n} + \mu_n) u_n \left(u_n - \sum_{i=1}^{\ell}(\widetilde{\psi}_n v_i)(\cdot - q_{n,i})\right) \le o_n(1).
\end{aligned}$$

Due to $\widetilde{\psi}_n v_j \to v_j$ and $u_n(\cdot + q_{n,j}) \rightharpoonup v_j$ in $H^1(\mathbb{R}^N)$ for each $j = 1, \cdots, \ell$, and $|q_{n,j} - q_{n,i}| \to \infty$ for $i \neq j$, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla(\widetilde{\psi}_n v_j)(\cdot - q_{n,j}) \nabla \left(u_n - \sum_{i=1}^{\ell}(\widetilde{\psi}_n v_i)(\cdot - q_{n,i})\right) \\
&= \int_{\mathbb{R}^N} \nabla v_j \nabla \left(u_n(\cdot + q_{n,i}) - v_j - \sum_{i=1, i\neq j}^{\ell} v_i(\cdot + q_{n,j} - q_{n,i})\right) + o_n(1) \\
&= \int_{\mathbb{R}^N} \nabla v_j \nabla \left(u_n(\cdot + q_{n,j}) - v_j\right) + o_n(1) = o_n(1).
\end{aligned}$$

Moreover, it follows from $V_{\varepsilon_n} = 0$ in $supp\ \varphi_n(\cdot - q_{n,j}) \subset \Omega_{\varepsilon_n}$ that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (\lambda_n \varepsilon_n^2 V_{\varepsilon_n} + \mu_n) \left(u_n - \sum_{i=1}^{\ell}(\widetilde{\psi}_n v_i)(\cdot - q_{n,i})\right) (\widetilde{\psi}_n v_j)(\cdot - q_{n,j}) \\
&= \int_{\mathbb{R}^N} \mu_n \left(u_n - \sum_{i=1}^{\ell}(\widetilde{\psi}_n v_i)(\cdot - q_{n,i})\right) (\widetilde{\psi}_n v_j)(\cdot - q_{n,j}) = o_n(1).
\end{aligned}$$

Thus, from the above results, we conclude

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left|\nabla \left(u_n - \sum_{i=1}^{\ell}(\widetilde{\psi}_n v_i)(\cdot - q_{n,i})\right)\right|^2 + (\lambda_n \varepsilon_n^2 V_{\varepsilon_n} + \mu_n) \left|u_n - \sum_{i=1}^{\ell}(\widetilde{\psi}_n v_i)(\cdot - q_{n,i})\right|^2 \\
&\le o_n(1).
\end{aligned}$$

This proves claim 4.

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Now we are ready to complete the proof. By claim 4, we have

$$\sum_{i=1}^{\ell}|v_i|_2^2=\lim_{n\to\infty}\left|\sum_{i=1}^{\ell}(\xi_{\varepsilon_n}v_i)(\cdot-q_{n,i})\right|_2^2=1$$

and together with Lemma 2.3,

$$\begin{aligned}\lim_{n\to\infty}J_{\lambda,\varepsilon}(u_n)=\lim_{n\to\infty}\Bigg(&\frac{1}{2}\left|\nabla\sum_{i=1}^{\ell}(\widetilde{\psi}_n v_i)(\cdot-q_{n,i})\right|_2^2-\frac{1}{2p}\\&\times\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\left(\sum_{i=1}^{\ell}(\widetilde{\psi}_n v_i)(\cdot-q_{n,i})\right)\right|^p\left|\left(\sum_{i=1}^{\ell}(\widetilde{\psi}_n v_i)(\cdot-q_{n,i})\right)\right|^p}{|x-y|^{N-\alpha}}\Bigg)\\=&\sum_{i=1}^{\ell}\left(\frac{1}{2}|\nabla v_i|_2^2-\frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_i|^p|v_i|^p}{|x-y|^{N-\alpha}}\right)\geq E(s^0).\end{aligned}$$

This indicates $v_i\in S_{\ell^{-1}}$ and the proof is completed. □

4. Deformation outside small balls

To find the critical point of $J_{\lambda,\varepsilon}$ in X_ε^d, it is crucial to control the size of u of outside of the set $\Omega_{L,\varepsilon}^{8r}$. We notice that the penalization technique described in [43] can only handle the case of $p\geq 2$, so we take a different approach to construct a special deformation flow outside a small ball, i.e., we construct a map τ_ε as shown below.

Define

$$Y_\varepsilon^d=\left\{u\in E_\varepsilon|\inf_{v\in X_\varepsilon}\|u-v\|_\varepsilon\leq d\right\}.$$

PROPOSITION 4.1. *For sufficiently small $\varepsilon>0$, there exists a map $\tau_\varepsilon: X_\varepsilon^d\to Y_\varepsilon^d$ such that*

(i) *τ_ε is continuous.*
(ii) *$\tau_\varepsilon(u)=u$ if $u(x)=0$ for all $(\Omega_{L,\varepsilon}^{8r})^c$.*
(iii) *For each $u\in X_\varepsilon^d$,*

$$\tau_\varepsilon(u)=u(x)\ \text{for}\ x\in\Omega_{L,\varepsilon}^{2r},\tag{4.1}$$

$$|\nabla\tau_\varepsilon(u)|^2_{L^2\left((\Omega_{L,\varepsilon}^{8r})^c\right)}\leq\sigma_\varepsilon,\tag{4.2}$$

$$J_{\lambda,\varepsilon}(\tau_\varepsilon(u))\leq J_{\lambda,\varepsilon}(u).\tag{4.3}$$

Here $\sigma_\varepsilon>0$ is independent of u *and satisfies $\lim_{\varepsilon\to0}\sigma_\varepsilon=0$.*

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In what follows, we construct $\Gamma_\varepsilon(u) : X_\varepsilon^d \to \mathbb{R}$ by

$$\Gamma_\varepsilon(u) = |\nabla u|^2_{L^2((\Omega^{8r}_{L,\varepsilon})^c)},$$

which will play an important role to build τ_ε.

LEMMA 4.2. *For any $u \in X_\varepsilon^d$, there exists a $\zeta_u \in H_0^1\left((\Omega^{2r}_{L,\varepsilon})^c\right)$ such that*

(i)

$$\|\zeta_u\| \le 2d. \tag{4.4}$$

(ii)

$$J'_{\lambda,\varepsilon}(u)\zeta_u \ge a\left(\Gamma_\varepsilon(u) - \sigma_\varepsilon\right), \tag{4.5}$$

where σ_ε is independent of p, u and satisfies $\lim\limits_{\varepsilon\to 0}\sigma_\varepsilon = 0$.

(iii)

$$\Gamma'_\varepsilon(u)\zeta_u \ge a\left(\Gamma_\varepsilon(u) - \sigma_\varepsilon\right). \tag{4.6}$$

*where $a, \sigma_\varepsilon > 0$ are as in (*ii*).*

Proof. (i) Denote

$$\zeta_u(x) = \sum_{i=1}^{\ell} \varphi_{\varepsilon,i}(x)u(x) \in H_0^1\left((\Omega^{2r}_{L,\varepsilon})^c\right),$$

where $\varphi_{\varepsilon,i}(x) \in C_0^\infty\left(\mathbb{R}^N, [0,1]\right)$ is a cut-off function satisfying $\varphi_{\varepsilon,i} = 1$ on $(\Omega^{8r}_{\varepsilon,i})^c$ and $\varphi_{\varepsilon,i} = 0$ on $\Omega^{2r}_{\varepsilon,i}$ for each $i = 1, \cdots, \ell$. Clearly, for $\varepsilon > 0$ small, it holds $\|\zeta_u\|_{H_0^1\left((\Omega^{2r}_{L,\varepsilon})^c\right)} \le 2d$.

(ii) From Propositions 2.1, we calculate that

$$\begin{aligned}
\int_{\mathbb{R}^N}\int_{(\Omega^{2r}_{L,\varepsilon})^c} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}} &= \int_{\mathbb{R}^N}\int_{(\Omega^{8r}_{L,\varepsilon})^c} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}} \\
&\quad + \int_{\mathbb{R}^N}\int_{\Omega^{8r}_{L,\varepsilon}\setminus\Omega^{2r}_{L,\varepsilon}} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}, \\
&\le \int_{(\Omega^{8r}_{L,\varepsilon})^c}\int_{(\Omega^{8r}_{L,\varepsilon})^c} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}} \\
&\quad + 2\int_{\mathbb{R}^N}\int_{\Omega^{8r}_{L,\varepsilon}\setminus\Omega^{2r}_{L,\varepsilon}} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}} \\
&\quad + \int_{\Omega^{2r}_{L,\varepsilon}}\int_{(\Omega^{8r}_{L,\varepsilon})^c} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}} \\
&\le C_\varepsilon + C\|u\|^{2p}_{H_0^1((\Omega^{8r}_{L,\varepsilon})^c)}.
\end{aligned}$$

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where $\lim_{\varepsilon\to 0} C_\varepsilon = 0$. Combining with the above inequality, a direct calculation gives

$$\begin{aligned} J'_{\lambda,\varepsilon}(u)\zeta_u &= \int_{\mathbb{R}^N} \nabla u \nabla \sum_{i=1}^{\ell} \varphi_{\varepsilon,i}(x)u(x)dx + \int_{\mathbb{R}^N} \lambda V_\varepsilon(x) u \sum_{i=1}^{\ell} \varphi_{\varepsilon,i}(x)u(x)dx \\ &\quad - \int_{\mathbb{R}^N}\int_{(\Omega^{2r}_{L,\varepsilon})^c} \frac{|u(x)|^p|u(y)|^p \sum_{i=1}^{\ell}\varphi_{\varepsilon,i}(y)}{|x-y|^{N-\alpha}} \\ &\geq \int_{(\Omega^{8r}_{L,\varepsilon})^c} |\nabla u|^2 dx + \int_{\Omega^{8r}_{L,\varepsilon}\setminus\Omega_{L,\varepsilon}} u\nabla u \nabla \sum_{i=1}^{\ell}\varphi_{\varepsilon,i}(x)dx \\ &\quad - \int_{\mathbb{R}^N}\int_{(\Omega^{2r}_{L,\varepsilon})^c} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}} \\ &\geq \int_{(\Omega^{8r}_{L,\varepsilon})^c} |\nabla u|^2 dx - C_\varepsilon - D_\varepsilon - C\|u\|^{2p}_{H^1_0((\Omega^{8r}_{L,\varepsilon})^c)}, \end{aligned} \tag{4.7}$$

where $\lim_{\varepsilon\to 0} D_\varepsilon = 0$. Controlling d small, there is $a>0$ such that

$$\int_{(\Omega^{8r}_{L,\varepsilon})^c} |\nabla u|^2 dx - C\|u\|^{2p}_{H^1_0((\Omega^{8r}_{L,\varepsilon})^c)} \geq a \int_{(\Omega^{8r}_{L,\varepsilon})^c} |\nabla u|^2 dx. \tag{4.8}$$

Let $\sigma_\varepsilon = (C_\varepsilon + D_\varepsilon)/a$ and combining with (4.7)–(4.8), (4.5) is proved.

(*iii*) (4.6) can be shown as in (*ii*). □

Let $u \in X^d_\varepsilon$ and $X(u) = \frac{\zeta_u}{\|\zeta_u\|}$. By Lemma 4.2, we have the following corollary.

COROLLARY 4.3. *There is a locally Lipschitz vector field*

$$X(u) : X^d_\varepsilon \to H^1_0\left((\Omega^{2r}_{L,\varepsilon})^c\right)$$

such that, for all $u \in X^d_\varepsilon$,

(i) $\|X(u)\|_{H^1} \leq 1$.
(ii) $J'_{\lambda,\varepsilon}(u)X(u), \Gamma'_\varepsilon(u)X(u) > 0$ *for* $\Gamma_\varepsilon(u) > \sigma_\varepsilon$. *Here* $\sigma_\varepsilon > 0$ *is as in Lemma 4.2.*

We consider the following equation in H: for $u \in X^d_\varepsilon$,

$$\begin{cases} \frac{dW}{dT} = -\rho(\Gamma_\varepsilon(W))X(W), \\ W(0,u) = u, \end{cases} \tag{4.9}$$

where the function $\rho(r) \in C^\infty([0,\infty),[0,1])$ satisfies

$$\rho(r) = \begin{cases} 1, & \text{for } r \geq 2\sigma_\varepsilon, \\ 0, & \text{for } r \leq \sigma_\varepsilon. \end{cases}$$

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Obviously, problem (4.9) has a unique solution, which we may assume to be $W(T,u)$. We next investigate some properties of $W(T,u)$.

LEMMA 4.4. *Let $W(T,u)$ be the unique solution of (4.9). Then we have the following conclusions:*

(i) *$W(T,u) \in Y_\varepsilon^d$ for all $\mathrm{T} > 0$ and $u \in X_\varepsilon^d$.*
(ii) *There holds*

$$\frac{d}{dT}\Gamma_\varepsilon(W(T,u)) \le 0, \text{ for all } u \in X_\varepsilon^d,$$
$$\frac{d}{dT}\Gamma_\varepsilon(W(T,u)) < 0, \text{ if } \Gamma_\varepsilon(W) > \sigma_\varepsilon.$$

(iii) *There holds*

$$\frac{d}{dT}J_{\lambda,\varepsilon}(\widetilde{W}(T,u)(x)) \le 0,$$

where

$$\widetilde{W}(T,u)(x) = \begin{cases} u(x) & \text{for } x \in \Omega_{L,\varepsilon}^{2r}, \\ W(T,u) & \text{for } x \in \left(\Omega_{L,\varepsilon}^{2r}\right)^c. \end{cases}$$

(iv) There exists $T_\varepsilon > 0$ such that for all $u \in X_\varepsilon^d$,

$$\Gamma_\varepsilon(W(T_\varepsilon,u)) \le \sigma_\varepsilon.$$

Proof. First, we show (ii). Obviously, there holds $\frac{d}{dT}\Gamma_\varepsilon(W(T,u)) \le 0$ for all $u \in X_\varepsilon^d$. It follows from $\Gamma_\varepsilon(W) > \sigma_\varepsilon$ that $\rho(\Gamma_\varepsilon(W)) > 0$. Hence, combining with Lemma 4.2,

$$\frac{d}{dT}\Gamma_\varepsilon(W(T,u)) = -\rho(\Gamma_\varepsilon(W))\Gamma_\varepsilon'(W)X(W) < 0.$$

By (ii), we get that $\Gamma_\varepsilon(W(T,u))$ is non-increasing. This suggests that $W(T,u) \in Y_\varepsilon^d$ for $T \in [0,\infty)$. Hence, (i) follows. (iv) is easily inferred from (ii). Since $W(T,u)|_{x\in\cup_{i=1}^{\ell}\partial\Omega_{\varepsilon,i}^{2r}}$ is independent of T, one has $\widetilde{W}(T,u)(x) \in H_0^1(\mathbb{R}^N)$. Hence, from Proposition 4.3 (ii), we can see that (iii) holds. □

The proof of Proposition 4.1: For any $u \in X_\varepsilon^d$, we define $\tau_\varepsilon = \widetilde{W}(T_\varepsilon,u)$. It follows from Lemma 4.4 that τ_ε satisfies (4.1)–(4.3) mentioned in Proposition 4.1.

5. Existence of ℓ-bump critical points

In this section, we complete the proof of Theorems 1.1 and 1.2.

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PROPOSITION 5.1. *There is $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and $\lambda > \overline{\lambda}_\varepsilon$, $J_{\lambda,\varepsilon}|_{M_\varepsilon}$ has a critical point $u_{\lambda,\varepsilon} \in X_\varepsilon^d$ with an associated Lagrange multiplier $\mu_{\lambda,\varepsilon}$ satisfying*

$$\lim_{\varepsilon\to 0} J_{\lambda,\varepsilon}(u_{\lambda,\varepsilon}) = E(s^0),$$
$$\lim_{\varepsilon\to 0} \sup_{\lambda>\overline{\lambda}_\varepsilon} |\mu_{\lambda,\varepsilon} - \mu_{\ell^{-1}}| = 0.$$

Proof of proposition 5.1. Choosing $d>0$ smaller if necessary, as in Proposition 3.3 we can see for some $\nu_d, k_\ell > 0$ independent of ε that for $\varepsilon>0$ small,

$$\|J'_{\lambda,\varepsilon}|_{M_\varepsilon}(u)\|_{T_u^*(M_\varepsilon)} \geq \nu_d \quad \text{for } u \in \left(X_\varepsilon^d \setminus X_\varepsilon^{d/4}\right) \cap \left[J_{\lambda,\varepsilon} \leq E(s^0) + k_\ell\right]. \tag{5.1}$$

If $J_{\lambda,\varepsilon}(u)$ does not have any critical point with $J_{\lambda,\varepsilon} \leq E(s^0) + k_\ell$ in X_ε^d, then from Proposition 3.2, there exists $\kappa_{\lambda,\varepsilon} > 0$ such that

$$\|J'_{\lambda,\varepsilon}\|_{M_\varepsilon}(u)\|_{T_u^*(M_\varepsilon)} \geq \kappa_{\lambda,\varepsilon} \quad \text{for } u \in X_\varepsilon^d \cap \left[J_{\lambda,\varepsilon} \leq E(s^0) + k_\ell\right]. \tag{5.2}$$

For convenience, we set

$$[J_{\lambda,\varepsilon} \leq b_1] := \{u \in E_\varepsilon | J_{\lambda,\varepsilon}(u) \leq b_1\},$$
$$[b_2 \leq J_{\lambda,\varepsilon}] := \{u \in E_\varepsilon | J_{\lambda,\varepsilon}(u) \geq b_2\},$$
$$[b_2 \leq J_{\lambda,\varepsilon} \leq b_1] := \{u \in E_\varepsilon | b_2 \leq J_{\lambda,\varepsilon}(u) \leq b_1\}$$

for any $b_1, b_2 \in \mathbb{R}$.

If $J_{\lambda,\varepsilon}(u)$ does not have any critical point with $J_{\lambda,\varepsilon} \leq E(s^0) + k_\ell$ in X_ε^d, by (5.1)–(5.2), we can find a deformation flow $\eta : M_\varepsilon \to M_\varepsilon$ be defined as the following lemma. □

LEMMA 5.2. *Under the assumption (5.1) and (5.2), for any $k \in \left(0, \min\left\{k_\ell, \frac{d\nu_d}{16}\right\}\right)$, there exists an $\varepsilon_k > 0$ such that for any $\varepsilon \in (0, \varepsilon_k)$ and $\lambda \in (\overline{\lambda}_\varepsilon, +\infty)$, there is a deformation $\eta : M_\varepsilon \to M_\varepsilon$ satisfying*

(1) $\eta(u) = u$ *if* $u \in M_\varepsilon \setminus \left(X_\varepsilon^d \cap \left[E(s^0) - 2k_\ell \leq J_{\lambda,\varepsilon} \leq E(s^0) + k_\ell\right]\right)$.
(2) $J_{\lambda,\varepsilon}(\eta(u)) \leq J_{\lambda,\varepsilon}(u)$ *if* $u \in M_\varepsilon$.
(3) $\eta(u) \in X_\varepsilon^d \cap \left[J_{\lambda,\varepsilon} \leq E(s^0) - k\right]$ *if* $u \in X_\varepsilon^{d/4} \cap \left[J_{\lambda,\varepsilon} \leq E(s^0) + k\right]$.

Proof. Set $\nu_\varepsilon : M_\varepsilon \to E_\varepsilon$ is locally Lipschitz continuous tangent vector field satisfying $\nu_\varepsilon(u) \in T_u M_\varepsilon$ for every $u \in M_\varepsilon$ and

$$\|\nu_\varepsilon\| \leq 1 \text{ and } \langle J'_{\lambda,\varepsilon}|_{M_\varepsilon}(u), \nu_\varepsilon(u)\rangle \geq 0 \text{ for } u \in M_\varepsilon,$$
$$\langle J'_{\lambda,\varepsilon}|_{M_\varepsilon}(u), \nu_\varepsilon(u)\rangle \geq \frac{\kappa_{\lambda,\varepsilon}}{2} \text{ for } u \in X_\varepsilon^d \cap \left[J_{\lambda,\varepsilon} \leq E(s^0) + k_\ell\right],$$
$$\langle J'_{\lambda,\varepsilon}|_{M_\varepsilon}(u), \nu_\varepsilon(u)\rangle \geq \frac{\nu_d}{2} \text{ for } u \in (X_\varepsilon^d \setminus X_\varepsilon^{d/4}) \cap \left[J_{\lambda,\varepsilon} \leq E(s^0) + k_\ell\right]. \tag{5.3}$$

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Then for every $u \in M_\varepsilon$, the flow $\overline{\eta}(t,u)$ is defined as a solution of

$$\begin{cases} \frac{\mathrm{d}\overline{\eta}}{\mathrm{d}t}(t,u) = -\zeta_1(\overline{\eta}(t,u))\zeta_2(\overline{\eta}(t,u))\nu_\varepsilon(\overline{\eta}(t,u)), \\ \overline{\eta}(0,u) = u, \end{cases}$$

where locally Lipschitz functions $\zeta_1, \zeta_2 : M_\varepsilon \to [0,1]$ are defined as

$$\zeta_1(u) = \begin{cases} 1, & u \in X_\varepsilon^{d/2}, \\ 0, & u \notin X_\varepsilon^d, \end{cases} \quad \zeta_2(u) = \begin{cases} 1, & u \in \left[E(s^0) - k_\ell \le J_{\lambda,\varepsilon}\right], \\ 0, & u \notin \left[E(s^0) - 2k_\ell \le J_{\lambda,\varepsilon}\right]. \end{cases}$$

It is easy to obtain that $\overline{\eta} \in C([0,+\infty) \times M_\varepsilon, M_\varepsilon)$ satisfies

$$\begin{cases} \overline{\eta}(\cdot,u) \in X_\varepsilon^d \quad \text{if } u \in X_\varepsilon^d, \\ \overline{\eta}(\cdot,u) \equiv u \quad \text{if } u \notin \left[E(s^0) - 2k_\ell \le J_{\lambda,\varepsilon}\right] \cap X_\varepsilon^d, \end{cases}$$

For $u \in M_\varepsilon$ and $t \ge 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} J_{\lambda,\varepsilon}(\overline{\eta}(t,u)) = -\zeta_1(\overline{\eta}(t,u))\zeta_2(\overline{\eta}(t,u))\langle J'_{\lambda,\varepsilon}|_{M_\varepsilon}(\overline{\eta}(t,u)), \nu_\varepsilon(t,u))\rangle \le 0. \tag{5.4}$$

We set $\eta := \overline{\eta}\left(\frac{4k}{\kappa_{\lambda,\varepsilon}}, u\right)$ and note that it satisfies the desired first two conclusions. To show (3), we argue by contradiction and assume $u_\varepsilon \in X_\varepsilon^{d/4} \cap \left[J_{\lambda,\varepsilon} \le E(s^0) + k\right]$ exists such that

$$J_{\lambda,\varepsilon}(\overline{\eta}(t,u)) \in \left(E(s^0) - k, E(s^0) + k\right] \quad \text{for all } t \in \left[0, \frac{4k}{\kappa_{\lambda,\varepsilon}}\right]. \tag{5.5}$$

If $\overline{\eta} \in X_\varepsilon^{d/2}$ for every $t \in \left[0, \frac{4k}{\kappa_{\lambda,\varepsilon}}\right]$, it follows from (5.3) and (5.4) that

$$\begin{aligned} J_{\lambda,\varepsilon}(\eta(u_\varepsilon)) =& J_{\lambda,\varepsilon}\left(\overline{\eta}\left(\frac{4k}{\kappa_{\lambda,\varepsilon}}, u_\varepsilon\right)\right) \\ =& \int_0^{\frac{4k}{\kappa_{\lambda,\varepsilon}}} \frac{\mathrm{d}}{\mathrm{d}t} J_{\lambda,\varepsilon}(\overline{\eta}(t,u_\varepsilon))dt + J_{\lambda,\varepsilon}(u_\varepsilon) \\ \le& -\int_0^{\frac{4k}{\kappa_{\lambda,\varepsilon}}} \frac{\kappa_{\lambda,\varepsilon}}{2} dt + E(s^0) + k \\ =& E(s^0) - k. \end{aligned}$$

This contradicts (5.5). If $\overline{\eta}(t,u_\varepsilon) \notin X_\varepsilon^{d/2}$ for some $t \in \left[0, \frac{4k}{\kappa_{\lambda,\varepsilon}}\right]$, then we can find an interval $[t_1,t_2] \subset \left[0, \frac{4k}{\kappa_{\lambda,\varepsilon}}\right]$ such that $\overline{\eta}(t_1,\mu_\varepsilon) \in \partial X_\varepsilon^{d/4}$, $\overline{\eta}(t_2,u_\varepsilon) \in \partial X_\varepsilon^{d/2}$ and

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$\overline{\eta}(t,u_\varepsilon)\subset X_\varepsilon^{d/2}\setminus X_\varepsilon^{d/4}$ for all $t\in(t_1,t_2)$. Since

$$t_2-t_1\geq\int_{t_1}^{t_2}\|\nu_\varepsilon(\overline{\eta}(t,u_\varepsilon))\|dt\geq\left\|\int_{t_1}^{t_2}\frac{\mathrm{d}\overline{\eta}}{\mathrm{dt}}(t,u_\varepsilon)dt\right\|=\overline{\eta}(t_2,u_\varepsilon)-\overline{\eta}(t_1,u_\varepsilon)\geq d/4,$$

according to (5.3) and (5.4), there holds

$$\begin{aligned}J_{\lambda,\varepsilon}(\eta(u_\varepsilon))=J_{\lambda,\varepsilon}\left(\overline{\eta}\left(\frac{4k}{\kappa_{\lambda,\varepsilon}},u_\varepsilon\right)\right)\leq&J_{\lambda,\varepsilon}(\overline{\eta}(t_2,u_\varepsilon))\leq\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{dt}}J_{\lambda,\varepsilon}(\overline{\eta}(t,u_\varepsilon))\\&+J_{\lambda,\varepsilon}(\overline{\eta}(t_1,u_\varepsilon))\\\leq&-\frac{\nu_d}{2}(t_2-t_1)+E(s^0)+k\\\leq&-2k+E(s^0)+k\\\leq&E(s^0)-k,\end{aligned}$$

which also contradicts (5.5). Therefore (c) holds. □

As shown by (2.3) and (2.4), the geometric structure of the energy functional $I(u)$ depends on p. Moreover, from Lemma 2.2, we also see that the magnitude relation between $E(s)$ and $E(s^0)$ depends on p. Thus, we study it in two parts.

Step 1: L^2-subcritical case $p\in\left(\frac{N+\alpha}{N},\frac{N+\alpha+2}{N}\right)$

We define the $(\ell-1)$-dimensional initial path by

$$\widetilde{\chi}_\varepsilon(s)=C_\varepsilon\sum_{i=1}^{\ell}(\widetilde{\varphi}_\varepsilon w_{s_i})(\cdot-q_{i,\varepsilon})\quad\text{for } s=(s_1,\cdots,s_\ell)\in\mathcal{S}_\delta,$$

where $\mathcal{S}_\delta:=\{s\in\mathcal{S}_+^\ell|\ |s-s^0|\leq\delta,\delta>0\}$, $\theta:=\frac{1}{4}\min\limits_{1\leq i\leq\ell}dist(q_i,\partial\Omega_i)$, $w_{s_i}\in S_{s_i}$, $\widetilde{\varphi}_\varepsilon(\cdot)=\widetilde{\varphi}(\varepsilon\cdot)$ is defined for some $\widetilde{\varphi}\in C_0^1(B_{2\theta}(0),[0,1])$ satisfying $\widetilde{\varphi}=1$ in $B_\theta(0)$ and $|\nabla\widetilde{\varphi}|\leq 2/\theta$, and

$$C_\varepsilon:=\left|\sum_{i=1}^{\ell}(\widetilde{\varphi}_\varepsilon w_{s_i}(\cdot-q_{i,\varepsilon}))\right|_2^{-1}.$$

Fix $\delta>0$ small enough (independent of ε small) such that

$$\widetilde{\chi}_\varepsilon(s)\in X_\varepsilon^{d/4}\quad\text{for } s\in\mathcal{S}_\delta$$

and

$$C_\varepsilon\to 1\quad\text{as } \varepsilon\to 0 \text{ uniformly in } \mathcal{S}_\delta.$$

According to $V_\varepsilon=0$ in $supp\ \widetilde{\varphi}_\varepsilon(\cdot-q_{i,\varepsilon})\subset\Omega_{i,\varepsilon}$ for each $i=1,\cdots,\ell$, we obtain that $J_{\lambda,\varepsilon}(\widetilde{\chi}_\varepsilon(s))$ is independent of λ and

$$J_{\lambda,\varepsilon}(\widetilde{\chi}_\varepsilon(s))\to E(s)\text{ as }\varepsilon\to 0\text{ uniformly for }s\in\mathcal{S}_\delta$$

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Moreover, using $E(s) < E(s^0)$ introduced in Lemma 2.2, we have

$$\lim_{\varepsilon\to 0}\max_{s\in\mathcal{S}_\delta} J_{\lambda,\varepsilon}(\widetilde{\chi}_\varepsilon(s)) = E(s^0). \tag{5.6}$$

Fix $k \in (0, \min\{k_\ell, , \frac{d\nu_d}{16}\})$ such that

$$\lim_{\varepsilon\to 0}\max_{s\in S_{\partial\mathcal{S}_\delta}} J_{\lambda,\varepsilon}(\widetilde{\chi}_\varepsilon(s)) = \max_{s\in S_{\partial\mathcal{S}_\delta}} E(s) < E(s^0) - 2k. \tag{5.7}$$

Next, we will have the contradiction in the L^2-subcritical case. By (5.6) and Lemma 5.2 (3), for small $\varepsilon > 0$, we have

$$J_{\lambda,\varepsilon}(\eta(\widetilde{\chi}_\varepsilon(s))) \le E(s^0) - k \quad \text{and } \eta(\widetilde{\chi}_\varepsilon(s)) \in X_\varepsilon^d \text{ for } s \in \mathcal{S}_\delta. \tag{5.8}$$

When $\ell \ge 2$, it follows from (5.8) and lemma 5.2 (1) that for smaller $\varepsilon > 0$, there holds

$$\eta(\widetilde{\chi}_\varepsilon(s)) = \widetilde{\chi}_\varepsilon(s) \quad \text{if } s \in \partial\mathcal{S}_\delta.$$

Set

$$\Psi_{\varepsilon,i}(u) = \left(\int_{\Omega_{i,\varepsilon}^{8r}} |u|^2\right)\left(\int_{\Omega_{L,\varepsilon}^{8r}} |u|^2\right)^{-1} \quad \forall\ u \in M_\varepsilon,\ i = 1, \cdots, \ell.$$

To get a contradiction, we now claim that there exists $s^1 \in \mathcal{S}_\delta$ such that

$$\Psi_{\varepsilon,i}(\tau_\varepsilon(\eta(\widetilde{\chi}_\varepsilon(s^1)))) = \ell^{-1} \quad \text{for each } i = 1, \cdots, \ell. \tag{5.9}$$

By Proposition 4.1, we have

$$\begin{cases} J_{\lambda,\varepsilon} \circ \tau_\varepsilon \circ \eta \circ \widetilde{\chi}_\varepsilon(s) \le E(s^0) - k, \text{ if } s \in \mathcal{S}_\delta, \\ \|\nabla\tau_\varepsilon(\eta(\widetilde{\chi}_\varepsilon(s)))\|_{L^2((\Omega_{L,\varepsilon}^{8r})^c)} = o_\varepsilon(1), \\ \tau_\varepsilon \circ \eta \circ \widetilde{\chi}_\varepsilon(s)(x) = \eta \circ \widetilde{\chi}_\varepsilon(s)(x), \text{ if } x \in \Omega_{L,\varepsilon}^{2r}, \\ \tau_\varepsilon \circ \eta \circ \widetilde{\chi}_\varepsilon(s)(x) = \widetilde{\chi}_\varepsilon(s)(x), \text{ if } s \in \partial\mathcal{S}_\delta \text{ and } x \in \Omega_{L,\varepsilon}^{2r}. \end{cases} \tag{5.10}$$

We use the Brouwer degree to prove (5.9) holds. We consider the map $f_\varepsilon : \mathcal{S}_\delta \to \mathbb{R}^\ell$ defined by

$$f_\varepsilon(s) = (\Psi_{\varepsilon,1} \circ \widetilde{\chi}_\varepsilon(s), \cdots, \Psi_{\varepsilon,\ell} \circ \widetilde{\chi}_\varepsilon(s)).$$

Clearly,

$$\Psi_{\varepsilon,i} \circ \widetilde{\chi}_\varepsilon(s) \to |w_{s_i}|_2^2 = s_i \quad \text{as } \varepsilon \to 0 \text{ uniformly for every } s \in \mathcal{S}_\delta.$$

Hence $s^0 \notin f_\varepsilon(\partial\mathcal{S}_\delta)$ for ε small and the Brouwer degree

$$\deg(f_\varepsilon, \mathcal{S}_\delta, s^0) = \deg(id, \mathcal{S}_\delta, s^0) = 1 \quad \text{for } \varepsilon \text{ small.}$$

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Then we set

$$h_\varepsilon(s) := (\Psi_{\varepsilon,1} \circ \tau_\varepsilon \circ \eta \circ \widetilde{\chi}_\varepsilon(s), \cdots, \Psi_{\varepsilon,\ell} \circ \tau_\varepsilon \circ \eta \circ \widetilde{\chi}_\varepsilon(s)).$$

Combining with (5.10), we deduce $s^1 \notin g_\varepsilon(\partial\mathcal{S}_\delta)$ for ε small and the Brower degree

$$\deg(h_\varepsilon, \mathcal{S}_\delta, s^0) = \deg(f_\varepsilon, \mathcal{S}_\delta, s^0) = 1 \quad \text{for } \varepsilon \text{ small},$$

and the claim is verified. Now we denote

$$U_\varepsilon := \begin{cases} \tau_\varepsilon(\eta(\widetilde{\chi}_\varepsilon(1))) & \text{if } \ell = 1, \\ \tau_\varepsilon(\eta(\widetilde{\chi}_\varepsilon(s^1))) & \text{if } \ell \geq 2, \end{cases} \qquad U_{i,\varepsilon} := \xi_{i,\varepsilon} U_\varepsilon,$$

where $\xi_{i,\varepsilon} \in C_0^\infty(\Omega_{i,\varepsilon}^{10r}, [0,1])$ is a cut-off function such that $\xi_{i,\varepsilon} = 1$ on $\Omega_{i,\varepsilon}^{8r}$ and $|\nabla \xi_{i,\varepsilon}| \leq \varepsilon/r$, for each $i = 1, 2, \cdots, \ell$. It follows from $\|\nabla U_\varepsilon\|_{L^2(\Omega_{L,\varepsilon}^{8r})^c} = o_\varepsilon(1)$ and the fact that $\Omega_{i,\varepsilon}^{8r} \cap \Omega_{j,\varepsilon}^{10r} = \emptyset$ for each $i \not\equiv j$, we conclude that

$$|U_{i,\varepsilon}|_2^2 = \Psi_{\varepsilon,i}(U_\varepsilon) + o_\varepsilon(1) = \ell^{-1} + o_\varepsilon(1). \tag{5.11}$$

Thus

$$I_{\ell^{-1}} \leq I\left(\ell^{-1/2} U_{i,\varepsilon} |U_{i,\varepsilon}|_2^{-1}\right) = I(U_{i,\varepsilon}) + o_\varepsilon(1). \tag{5.12}$$

Using (5.8), (5.11) and (5.12), we calculate that

$$\begin{aligned} E(s^0) - k \geq & J_{\lambda,\varepsilon}(U_\varepsilon) \geq I(U_\varepsilon) \\ \geq & \sum_{i=1}^{\ell} \frac{1}{2} \int_{\Omega_{i,\varepsilon}^{8r}} |\nabla U_\varepsilon|^2 - \frac{1}{2p} \int_{\Omega_{i,\varepsilon}^{8r}} \int_{\Omega_{i,\varepsilon}^{8r}} \frac{|U_\varepsilon|^p |U_\varepsilon|^p}{|x-y|^{N-\alpha}} + o_\varepsilon(1) \\ = & \sum_{i=1}^{\ell} I(U_{i,\varepsilon}) + o_\varepsilon(1) \\ \geq & \ell I_{\ell^{-1}} + o_\varepsilon(1) \\ = & E(s^0) + o_\varepsilon(1), \end{aligned}$$

which is a contradiction.

Claim 2: L^2-supercritical case $p \in \left(\frac{N+\alpha+2}{N}, \frac{N+\alpha}{N-2}\right)$

Similar to Claim 1, according to the geometry of the energy functional $I(u)$, we set the initial path by

$$\vartheta_\varepsilon(t) := \widetilde{C_\varepsilon} \sum_{i=1}^{\ell} \left(t_i^{N/2} \widetilde{\varphi}_\varepsilon(t_i(\cdot - q_{i,\varepsilon})) u_i\left(t_i(\cdot - q_{i,\varepsilon})\right)\right) \quad \text{for } t = (t_1, \cdots, t_\ell) \in \mathbb{R}_+^\ell,$$

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where $u_i \in S_{\ell^{-1}}$, $q_i \in \Omega_i$ and $q_{i,\varepsilon} = \frac{1}{\varepsilon} q_i$ for $i = 1, 2, \cdots, \ell$, $\widetilde{\varphi}_\varepsilon(\cdot) = \widetilde{\varphi}(\varepsilon \cdot)$ with $\widetilde{\varphi} \in C_0^1(B_{2\theta}(0), [0,1])$ satisfying $\widetilde{\varphi} = 1$ in $B_\theta(0)$ and $|\nabla \widetilde{\varphi}| \leq 2/\theta$,

$$\widetilde{C_\varepsilon} := \left| \sum_{i=1}^{\ell} \left(t_i^{N/2} \widetilde{\varphi}_\varepsilon(t_i(\cdot - q_{i,\varepsilon})) u_i\left(t_i(\cdot - q_{i,\varepsilon})\right) \right) \right|_2^{-1}.$$

Then we fix $\sigma^* > 0$ small enough (independent of ε small) such that

$$\vartheta_\varepsilon(t) \in X_\varepsilon^{d/4} \quad \text{for } t \in [1-\sigma^*, 1+\sigma^*]^\ell.$$

Obviously, there holds $\lim\limits_{\varepsilon \to 0} \widetilde{C_\varepsilon} = 1$ and

$$\begin{aligned} \lim_{\varepsilon \to 0} J_{\lambda,\varepsilon}(\vartheta_\varepsilon(t)) &= \sum_{i=1}^{\ell} I\left(t_i^{N/2} u_i(t_i \cdot) \right) \\ &= \sum_{i=1}^{\ell} \left(\frac{1}{2} t_i^2 |\nabla u_i|_2^2 - \frac{1}{2p} t_i^{\alpha-1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_i(x)|^p |u_i(y)|^p}{|x-y|^{N-\alpha}} \right) \end{aligned}$$

for any $t \in [1-\sigma^*, 1+\sigma^*]^\ell$. Since $I(u_i) > I\left(t_i^{N/2} u_i(t_i \cdot) \right)$ for $t_i \in [1-\sigma^*, 1+\sigma^*] \backslash \{1\}$, we can fix $k \in (0, \min\{k_\ell, \frac{d\nu_d}{16}\})$ such that

$$\lim_{\varepsilon \to 0} \max_{\partial [1-\sigma^*, 1+\sigma^*]^\ell} J_{\lambda,\varepsilon} \circ \vartheta_\varepsilon < E(s^0) - 2k < \lim_{\varepsilon \to 0} \max_{[1-\sigma^*, 1+\sigma^*]^\ell} J_{\lambda,\varepsilon} \circ \vartheta_\varepsilon = E(s^0). \tag{5.13}$$

It follows from (5.13) and Lemma 5.2 that

$$J_{\lambda,\varepsilon}(\eta(\vartheta_\varepsilon(t))) \leq E(s^0) - k, \ \eta(\vartheta_\varepsilon(t)) \in X_\varepsilon^d \text{ for } t \in [1-\sigma^*, 1+\sigma^*]^\ell \tag{5.14}$$

and

$$\eta(\vartheta_\varepsilon(t)) = \vartheta_\varepsilon(t) \quad \text{if } t \in \partial [1-\sigma^*, 1+\sigma^*]^\ell.$$

For each $i = 1, \cdots, \ell$, define

$$\overline{\Psi}_{\varepsilon,i}(u) = \left(\int_{\Omega_{i,\varepsilon}^{8r}} |\nabla u|^2 \right)^{\frac{1}{-Np+2+N+\alpha}} \left(\frac{1}{2p} \int_{\Omega_{i,\varepsilon}^{8r}} \int_{\Omega_{i,\varepsilon}^{8r}} \frac{|\nabla u|^p |\nabla u|^p}{|x-y|^{N-\alpha}} \right)^{\frac{-1}{-Np+2+N+\alpha}}$$

Now we claim that there exists $t^1 \in [1-\sigma^*, 1+\sigma^*]^\ell$ such that

$$\overline{\Psi}_{\varepsilon,i}(\chi(\eta(\vartheta_\varepsilon(t^1)))) = 1 \quad \text{for each } i = 1, \cdots, \ell. \tag{5.15}$$

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From Proposition 4.1, we obtain that

$$\begin{cases} J_{\lambda,\varepsilon}\circ\tau_\varepsilon\circ\eta\circ\vartheta_\varepsilon(t)\le E(s^0)-k, \text{ if } t\in[1-\sigma^*,1+\sigma^*]^\ell,\\ \|\nabla\tau_\varepsilon(\eta(\vartheta_\varepsilon(t)))\|_{L^2((\Omega^{8r}_{L,\varepsilon})^c)}=o_\varepsilon(1),\\ \tau_\varepsilon\circ\eta\circ\vartheta_\varepsilon(t)(x)=\eta\circ\vartheta_\varepsilon(t)(x), \text{ if } x\in\Omega^{2r}_{L,\varepsilon},\\ \tau_\varepsilon\circ\eta\circ\vartheta_\varepsilon(t)(x)=\vartheta_\varepsilon(t)(x), \text{ if } t\in\partial[1-\sigma^*,1+\sigma^*]^\ell \text{ and } x\in\Omega^{2r}_{L,\varepsilon}. \end{cases} \tag{5.16}$$

Define the map $\widetilde{f}_\varepsilon:[1-\sigma^*,1+\sigma^*]^\ell\to\mathbb{R}^\ell$:

$$\widetilde{f}_\varepsilon(t)=\left(\overline{\Psi}_{\varepsilon,1}\circ\vartheta_\varepsilon(t),\cdots,\overline{\Psi}_{\varepsilon,\ell}\circ\vartheta_\varepsilon(t)\right).$$

Since $supp\widetilde{\varphi}_\varepsilon(\cdot-q_{i,\varepsilon})\subset\Omega_{i,\varepsilon}\subset\Omega^{8r}_{i,\varepsilon}$ and $\Omega^{8r}_{i,\varepsilon}\cap\Omega^{8r}_{j,\varepsilon}=\emptyset$ for each $i\neq j$, we deduce

$$\lim_{\varepsilon\to0}\overline{\Psi}_{\varepsilon,i}\circ\vartheta_\varepsilon(t)=t_i$$

uniformly for $t\in[1-\sigma^*,1+\sigma^*]^\ell$. Then $t^0:=(1,\cdots,1)\in\mathbb{R}^\ell\setminus\widetilde{f}_\varepsilon(\partial[1-\sigma^*,1+\sigma^*]^\ell)$ for ε small and the Brower degree

$$\deg(\widetilde{f}_\varepsilon,[1-\sigma^*,1+\sigma^*]^\ell,t^0)=\deg(id,[1-\sigma^*,1+\sigma^*]^\ell,t^0)=1\quad\text{for }\varepsilon\text{ small.}$$

Then we denote

$$\widetilde{h}_\varepsilon:=\left(\overline{\Psi}_{\varepsilon,1}\circ\tau_\varepsilon\circ\eta\circ\vartheta_\varepsilon(t),\cdots,\overline{\Psi}_{\varepsilon,\ell}\circ\tau_\varepsilon\circ\eta\circ\vartheta_\varepsilon(t)\right).$$

It follows from (5.16) that $t^0:=(1,\cdots,1)\in\mathbb{R}^\ell\setminus\widetilde{f}_\varepsilon(\partial[1-\sigma^*,1+\sigma^*]^\ell)$ for ε small and the Brower degree

$$\deg(\widetilde{h}_\varepsilon,[1-\sigma^*,1+\sigma^*]^\ell,t^0)=\deg(\widetilde{f}_\varepsilon,[1-\sigma^*,1+\sigma^*]^\ell,t^0)=1\quad\text{for }\varepsilon\text{ small.}$$

Hence, the claim is proved. Now denote

$$\widetilde{U}_\varepsilon:=\tau_\varepsilon\left(\eta(\vartheta_\varepsilon(t^1))\right),\quad \widetilde{U}_{i,\varepsilon}:=\xi_{i,\varepsilon}\widetilde{U}_\varepsilon\left(\sum_{i=1}^\ell|\xi_{i,\varepsilon}\widetilde{U}_\varepsilon|_2^2\right)^{-\frac{1}{2}}$$

where $\xi_{i,\varepsilon}\in C_0^\infty(\Omega^{10r}_{i,\varepsilon}[0,1])$ is a cut-off function such that $\xi_{i,\varepsilon}=1$ on $\Omega^{8r}_{i,\varepsilon}$ and $|\nabla\xi_{i,\varepsilon}|\le\varepsilon/r$, for each $i=1,\cdots,\ell$. Since $\|\nabla\widetilde{U}_\varepsilon\|_{L^2((\Omega^{8r}_{L,\varepsilon})^c)}=o_\varepsilon(1)$, we deduce that

$$\sum_{i=1}^\ell|\xi_{i,\varepsilon}\widetilde{U}_\varepsilon|_2^2=1+o_\varepsilon(1). \tag{5.17}$$

Combining with (5.15) and (5.17), we obtain that

$$t_{i,\varepsilon}:=\overline{\Psi}_{\varepsilon,i}(\widetilde{U}_\varepsilon)+o_\varepsilon(1)=1+o_\varepsilon(1). \tag{5.18}$$

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A direct calculation shows that

$$s^* := \left(\left| t_{1,\varepsilon}^{-\frac{N}{2}} \widetilde{W}_{1,\lambda,\varepsilon}(t_{1,\varepsilon}^{-1}\cdot) \right|_2^2, \cdots, \left| t_{\ell,\varepsilon}^{-\frac{N}{2}} \widetilde{W}_{\ell,\lambda,\varepsilon}(t_{\ell,\varepsilon}^{-1}\cdot) \right|_2^2 \right) \in \mathcal{S}_+^\ell$$

Hence, using Lemma 2.2, equation (5.17) and (5.18), we deduce that

$$E(s^0) \le E(s^*) \le \sum_{i=1}^{\ell} I\left(t_{i,\varepsilon}^{-\frac{N}{2}} \widetilde{U}_{i,\varepsilon}(t_{i,\varepsilon}^{-1}\cdot) \right) = \sum_{i=1}^{\ell} I(\widetilde{U}_{i,\varepsilon}) + o_\varepsilon(1). \tag{5.19}$$

However, according to (5.13), (5.17) and (5.19) we conclude that

$$\begin{aligned} E(s^0) - k \ge J_{\lambda,\varepsilon}(\widetilde{U}_\varepsilon) \ge I(\widetilde{U}_\varepsilon) \ge & \sum_{i=1}^{\ell} \frac{1}{2} \int_{\Omega_{i,\varepsilon}^{8r}} |\nabla \widetilde{U}_\varepsilon|^2 - \frac{1}{2p} \int_{\Omega_{i,\varepsilon}^{8r}} \int_{\Omega_{i,\varepsilon}^{8r}} \frac{|\widetilde{U}_\varepsilon|^p |\widetilde{U}_\varepsilon|^p}{|x-y|^{N-\alpha}} \\ & + o_\varepsilon(1) \\ = & \sum_{i=1}^{\ell} I(\widetilde{U}_{i,\varepsilon}) + o_\varepsilon(1) \\ \ge & E(s^0) + o_\varepsilon(1), \end{aligned}$$

which is a contradiction.

This completes the proof of Proposition 5.1.

Proof of Theorems 1.1 and 1.2. Following the proof of Proposition 5.1, for ε sufficiently small and λ large enough, $J_{\lambda,\varepsilon}$ has a critical point $u_{\lambda,\varepsilon}$ with the associated Lagrange multipler $\mu_{\lambda,\varepsilon}$ in X_ε^d satisfying $\lim\limits_{\varepsilon\to 0} J_{\lambda,\varepsilon}(u_{\lambda,\varepsilon}) = E(s^0)$. According to the strong maximum principle, we obtain that $u_{\lambda,\varepsilon} > 0$. Then we complete the proof of Theorem 1.1 by setting

$$\overline{u}_{\lambda,\varepsilon} = \omega^{\frac{2+\alpha}{2(-Np+N+\alpha+2)}} u_{\lambda,\varepsilon}\left(\omega^{\frac{2(p-1)}{-Np+N+\alpha+2}}\right), \quad \overline{\mu}_{\lambda,\varepsilon} = \omega^{\frac{2(p-1)}{-Np+N+\alpha+2}} \mu_{\lambda,\varepsilon}$$

Next, we give the proof of Theorem 1.2. For fixed ε small, we consider a sequence $\lambda_n \to \infty$. Let $(\mu_n, u_n) := (\mu_{\lambda_n,\varepsilon}, u_{\lambda_n,\varepsilon})$ be the positive solution of problem (1.4). Similar to Proposition 3.2, we get $\|u_n\|_\varepsilon, \int_{\mathbb{R}^N} \lambda_n \varepsilon^2 V_\varepsilon(x) u_\varepsilon^2$ and μ_n are bounded. This shows that, up to a subsequence, we may assume that $u_n \rightharpoonup u_\varepsilon$ in E_ε and $\mu_n \to \mu_\varepsilon$ in $\mathbb{R}$. Using Fatou's lemma, we have

$$\int_{\mathbb{R}^N} \varepsilon^2 V_\varepsilon(x) u_\varepsilon^2 \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} \varepsilon^2 V_\varepsilon(x) u_n^2 \le \frac{\liminf\limits_{n\to\infty} \int_{\mathbb{R}^N} \lambda_n \varepsilon^2 V_\varepsilon(x) u_n^2}{\lambda_n} = 0,$$

which implies $u_\varepsilon = 0$ a.e. in $\mathbb{R}^N \setminus \Omega_\varepsilon$. Next, we claim $u_n \to u_\varepsilon$ in E_ε. Since u_n solves

$$-\Delta u_n + \lambda_n \varepsilon^2 V_\varepsilon(x) u_n + \mu_n u_n = \left(\int_{\mathbb{R}^N} \frac{|u_n|^p}{|x-y|^{N-\alpha}} \right) |u_n|^{p-2} u_n, \text{ in } \mathbb{R}^N,$$

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we deduce that $u_\varepsilon \in H_0^1(\Omega_\varepsilon)$ is the solution of

$$-\Delta u_\varepsilon + \mu_\varepsilon u_\varepsilon = \left(\int_{\mathbb{R}^N} \frac{|u_\varepsilon|^p}{|x-y|^{N-\alpha}}\right)|u_\varepsilon|^{p-2}u_\varepsilon, \text{ in } \Omega_\varepsilon \tag{5.20}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla(u_n - u_\varepsilon)|^2 + \int_{\mathbb{R}^N} \left(\lambda_n \varepsilon^2 V_\varepsilon(x) + \mu_n\right)(u_n - u_\varepsilon)^2 \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n - u_\varepsilon|^p |u_n - u_\varepsilon|^p}{|x-y|^{N-\alpha}} + o_n(1).\end{aligned} \tag{5.21}$$

By the Hardy-Littlewood-Sobolev inequality,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n - u_\varepsilon|^p |u_n - u_\varepsilon|^p}{|x-y|^{N-\alpha}} \le C\|u_n - u_\varepsilon\|^{2p} \le C\|u_n - u_\varepsilon\|^{2p-2}_{H^1(\mathbb{R}^N\setminus\Omega_\varepsilon)}\|u_n - u_\varepsilon\|^2.$$

Combining with (5.21), we have

$$\|u_n - u_\varepsilon\|^2 \le C\|u_n - u_\varepsilon\|^{2p-2}_{H^1(\mathbb{R}^N\setminus\Omega_\varepsilon)}\|u_n - u_\varepsilon\|^2.$$

Controlling ε small enough such that $C\|u_n - u_\varepsilon\|^{2p-2}_{H^1(\mathbb{R}^N\setminus\Omega_\varepsilon)} < 1$, then we have $\|u_n - u_\varepsilon\|^2 = o_n(1)$. Hence, we deduce $u_n \to u_\varepsilon$ in E_ε and $|u_\varepsilon|_2^2 = 1$. Clearly, it holds

$$\int_{\mathbb{R}^N} \lambda_n \varepsilon^2 V_\varepsilon(x) u_n^2 = \liminf_{n\to\infty} \int_{\mathbb{R}^N\setminus\Omega_\varepsilon} \lambda_n \varepsilon^2 V_\varepsilon (u_n - u_\varepsilon)^2 \to 0.$$

Similar to the proof of claim 2 in Proposition 3.3 to $u_\varepsilon|_{\Omega_\varepsilon\setminus\Omega_{L,\varepsilon}}$, we obtain $u_\varepsilon = 0$ in $\mathbb{R}^N \setminus \Omega_\varepsilon$.

Now, we study the properties of μ_ε and u_ε. Note that up to a subsequence $\mu_\varepsilon \to \mu_{\ell^{-1}}$. Set

$$Q_{R,\varepsilon} := \{x \in \Omega_{L,\varepsilon} : \ dist(x, \partial\Omega_\varepsilon) \le R\}$$

for every $R > 0$. Similar to the proof of Claim 1 in Proposition 3.3, we obtain

$$\lim_{\varepsilon\to 0} \int_{Q_{R,\varepsilon}} |u_\varepsilon|^q = 0 \tag{5.22}$$

for $1 < q < \frac{N+\alpha}{N-2}$. Let $\varphi_{\varepsilon,R} \in C^\infty(\mathbb{R}^N, [0,1])$ satisfying

$$\begin{cases} \varphi_{\varepsilon,R} = 1, \text{ in } \mathbb{R}^N \setminus \Omega_{L,\varepsilon}, \\ \varphi_{\varepsilon,R} = 0, \text{ in } \Omega_{L,\varepsilon} \setminus Q_{R,\varepsilon}, \\ |\nabla\varphi_{\varepsilon,R}| \le \frac{2}{R}. \end{cases}$$

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Clearly, it holds $\mu_\varepsilon > \frac{1}{2}\mu_{\ell^{-1}}$ for ε small. Multiplying (5.20) by $\varphi_{\varepsilon,R}^2 u_\varepsilon$ and integrating, we deduce for ε small enough,

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left[|\nabla(\varphi_{\varepsilon,R}u_\varepsilon)|^2+\frac{1}{2}\mu_{\ell^{-1}}(\varphi_{\varepsilon,R}u_\varepsilon)^2\right]\\
\le&\int_{\mathbb{R}^N}\left[\nabla u_\varepsilon\nabla(\varphi_{\varepsilon,R}^2u_\varepsilon)+\mu_\varepsilon(\varphi_{\varepsilon,R}u_\varepsilon)^2\right]+\int_{\mathbb{R}^N}|\nabla\varphi_{\varepsilon,R}|^2u_\varepsilon^2\\
=&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\varepsilon(y)|^p|u_\varepsilon(x)|^p\varphi_{\varepsilon,R}^2(x)}{|x-y|^{N-\alpha}}+\int_{\mathbb{R}^N}|\nabla\varphi_{\varepsilon,R}|^2u_\varepsilon^2\\
\le&\left(\int_{Q_{R,\varepsilon}}|u_\varepsilon|^{\frac{2Np}{N-\alpha}}\right)^{\frac{N-\alpha}{2Np}}\left(\int_{\mathbb{R}^N}|u_\varepsilon|^{\frac{2Np}{N-\alpha}}\right)^{\frac{N-\alpha}{2Np}}+\int_{Q_{R,\varepsilon}}|\nabla\varphi_{\varepsilon,R}|^2u_\varepsilon^2\\
\le&\left(\int_{Q_{R,\varepsilon}}|u_\varepsilon|^{\frac{2Np}{N-\alpha}}\right)^{\frac{N-\alpha}{2Np}}\left(\int_{\mathbb{R}^N}|u_\varepsilon|^{\frac{2Np}{N-\alpha}}\right)^{\frac{N-\alpha}{2Np}}+\int_{Q_{R,\varepsilon}}\frac{4}{R^2}u_\varepsilon^2.
\end{aligned}$$

Combining with the above inequality and (5.22), one has

$$\limsup_{\varepsilon\to0}\|\varphi_{\varepsilon,R}u_\varepsilon\|\le\frac{C}{R}. \tag{5.23}$$

For any sequence $\varepsilon_n\to0$, we define $\widetilde{\lambda}_n$ such that $\widetilde{\lambda}_n\varepsilon_n^2\alpha_{\varepsilon_n}\to+\infty$. Clearly, $(1-\varphi_{\varepsilon_n,R})v\in H_0^1(\Omega_{L,\varepsilon_n})$ for any $v\in E_{\varepsilon_n}$. By calculating,

$$\begin{aligned}
&\left|\left\langle J'_{\widetilde{\lambda}_n,\varepsilon_n}(u_{\varepsilon_n}),v\right\rangle+\mu_{\varepsilon_n}\langle G'(u_{\varepsilon_n}),v\rangle\right|\\
=&\left|\int_{\mathbb{R}^N}(\nabla u_{\varepsilon_n}\nabla v+\mu_{\varepsilon_n}u_{\varepsilon_n}v)-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_{\varepsilon_n}(y)|^p|u_{\varepsilon_n}|^{p-2}u_{\varepsilon_n}v}{|x-y|^{N-\alpha}}\right|\\
=&\left|\int_{\mathbb{R}^N}(\nabla u_{\varepsilon_n}\nabla(\varphi_{\varepsilon_n,R}v)+\mu_{\varepsilon_n}u_{\varepsilon_n}\varphi_{\varepsilon_n,R}v)\right.\\
&\left.-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_{\varepsilon_n}(y)|^p|u_{\varepsilon_n}(x)|^{p-2}u_{\varepsilon_n}(x)\varphi_{\varepsilon_n,R}v}{|x-y|^{N-\alpha}}\right|\\
=&\left|\int_{\mathbb{R}^N}[\nabla(\varphi_{\varepsilon_n,R}u_{\varepsilon_n})\nabla v+\nabla\varphi_{\varepsilon_n,R}(\nabla u_{\varepsilon_n}v-u_{\varepsilon_n}\nabla v)+\mu_{\varepsilon_n}u_{\varepsilon_n}\varphi_{\varepsilon_n,R}v]\right.\\
&\left.-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_{\varepsilon_n}(y)|^p|u_{\varepsilon_n}(x)|^{p-2}u_{\varepsilon_n}(x)\varphi_{\varepsilon_n,R}v}{|x-y|^{N-\alpha}}\right|\\
\le&C\left(\|\varphi_{\varepsilon_n,R}u_{\varepsilon_n}\|+\frac{1}{R}\|u_{\varepsilon_n}\|+\|u_{\varepsilon_n}\|^2\|\varphi_{\varepsilon_n,R}u_{\varepsilon_n}\|^2\right)\|v\|_{\varepsilon_n}.
\end{aligned}$$

Combining with (5.23) and passing to the limit as $R\to0$, it holds

$$\lim_{R\to\infty}\|J'_{\widetilde{\lambda}_n,\varepsilon_n}(u_{\varepsilon_n})+\mu_{\varepsilon_n}G'(u_{\varepsilon_n})\|_{E^*_{\varepsilon_n}}=0.$$

Since

$$\lim_{n\to\infty}J_{\widetilde{\lambda}_n,\varepsilon_n}(u_{\varepsilon_n})=E(s^0),$$

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by proposition 3.3 to μ_{ε_n} and u_{ε_n}, we obtain that there exist $q_{n,i} \in \Omega_{i,\varepsilon_n}$, $u_i \in S_{\ell^{-1}}$ such that

$$dist(q_{n,i}, \partial\Omega_{i,\varepsilon_n}) \to \infty \quad \text{and } \|u_{\varepsilon_n} - \sum_{i=1}^{\ell} (\xi_{\varepsilon_n} u_i)(\cdot - q_{n,i})\|_{\varepsilon_n} \to 0 \text{ as } n \to \infty.$$

By the following scaling,

$$\overline{u}_\varepsilon = \omega^{\frac{2+\alpha}{2(-Np+N+\alpha+2)}} u_\varepsilon \left(\omega^{\frac{2(p-1)}{-Np+N+\alpha+2}}\right), \quad \overline{\mu}_\varepsilon = \omega^{\frac{2(p-1)}{-Np+N+\alpha+2}} \mu_\varepsilon.$$

we complete the proof of Theorem 1.2. □

Acknowledgements

H. Zhang is supported by Hunan Provincial Innovation Foundation for Postgraduate (No. CX20230219) and the Fundamental Research Funds for the Central Universities of Central South University (No. 1053320220488). H. Chen is supported by the National Natural Science Foundation of China (Grant No. 12071486).

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