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Note on the generalized Mehler transform

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1. *The integral transform.* One result of recent studies of boundary-value problems of the wave and diffusion equations involving wedge- or conically-shaped boundaries has been the interest shown in integrals in which the variable of integration appears as the order of Bessel or Legendre functions. An integral of this type occurs as the inversion formula for the generalized Mehler transform which is defined by

$$\left.\begin{aligned} F(\mu,k) &= \int_1^\infty f(r)\,P^k_{-\frac{1}{2}+i\mu}(r)\,dr \quad (1<r<\infty), && (a)\\ f(r) &= \frac{1}{\pi}\int_0^\infty \mu\sinh(\mu\pi)\,\psi(\mu,k)\,F(\mu,k)\,P^k_{-\frac{1}{2}+i\mu}(r)\,d\mu, && (b)\end{aligned}\right\} \tag{1}$$

where $\psi(\mu,k)=\Gamma(\frac{1}{2}-k+i\mu)\,\Gamma(\frac{1}{2}-k-i\mu)$ and $P^k_{-\frac{1}{2}+i\mu}(r)$ is the associated Legendre function of the first kind. Oberhettinger and Higgins (4) have given a table of transform pairs corresponding to the above transform.

In this note a Parseval relation is obtained in a formal manner for transforms of this type and it is shown how it may be used to deduce the values of a number of integrals. This was done in a previous paper by the author (3) for the Mehler transform, which is a special case of (1) with $k=0$. A rigorous derivation of Parseval's formula for the Mehler transform has been given by Lebedev (2).

2. *The Parseval relation.* Let $G(\mu,k)$ and $H(\mu,k)$ be transforms of the type (1) of the functions $g(r)$ and $h(r)$ respectively; then

$$\begin{aligned}\frac{1}{\pi}\int_0^\infty \mu\sinh(\mu\pi)\,\psi(\mu,k)\,G(\mu,k)\,H(\mu,k)\,d\mu&\\ = \frac{1}{\pi}\int_0^\infty \mu\sinh(\mu\pi)\,\psi(\mu,k)\,H(\mu,k)\,d\mu&\int_1^\infty g(r)\,P^k_{-\frac{1}{2}+i\mu}(r)\,dr,\end{aligned}$$

by the definition of $G(\mu,k)$. Assuming that an interchange of the order of integration is permissible we find

$$\begin{aligned}\frac{1}{\pi}\int_0^\infty \mu\sinh(\mu\pi)\,\psi(\mu,k)\,G(\mu,k)\,H(\mu,k)\,d\mu&\\ = \frac{1}{\pi}\int_1^\infty g(r)\,dr&\int_0^\infty \mu\sinh(\mu\pi)\,\psi(\mu,k)\,H(\mu,k)\,P^k_{-\frac{1}{2}+i\mu}(r)\,d\mu\\ = \int_1^\infty g(r)\,h(r)\,dr,&\end{aligned}$$

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by equation (1 b). Therefore the Parseval relation is

$$\frac{1}{\pi}\int_0^\infty \mu \sinh(\mu\pi)\,\psi(\mu,k)\,G(\mu,k)\,H(\mu,k)\,d\mu = \int_1^\infty g(r)\,h(r)\,dr. \tag{2}$$

3. *Evaluation of integrals.* With the aid of the relation (2) and known transform pairs it is possible to deduce in a simple way the values of certain integrals. A few examples are given below. For convenience a number of results given in (4) are listed in Table 1, where $K_\lambda(a)$ is a modified Bessel function, $W_{\alpha,\beta}(b)$ is a Whittaker function and $S_{\alpha,\beta}(d)$ is a Lommel function.

Table 1

	$F(\mu,k)$	$f(r)$ $\quad \Re(k) \leqslant \frac{1}{2}$
a	$K_{i\mu}(a)$	$(\frac{1}{2}\pi)^{\frac{1}{2}} a^{\frac{1}{2}-k}(r^2-1)^{-\frac{1}{2}k} e^{-ar}$
b	$\dfrac{\Gamma(\frac{3}{4}-\lambda+\frac{1}{2}i\mu)\,\Gamma(\frac{3}{4}-\lambda-\frac{1}{2}i\mu)}{\Gamma(\frac{3}{4}-\frac{1}{2}k+\frac{1}{2}i\mu)\,\Gamma(\frac{3}{4}-\frac{1}{2}k-\frac{1}{2}i\mu)}$	$\dfrac{2^{1-k}\Gamma(1-\lambda-\frac{1}{2}k)}{\Gamma(\lambda-\frac{1}{2}k)}(r^2-1)^{\lambda-1}$ $\quad \frac{1}{4} < \Re(\lambda) \leqslant \frac{3}{4}$
c	$W_{k,i\mu}(2b)$	$b\left(\dfrac{r+1}{r-1}\right)^{\frac{1}{2}k} e^{-br}$
d	$W_{\frac{1}{2}k-\frac{1}{4},\frac{1}{2}i\mu}(b)$	$\dfrac{2^{1-k}b^{\frac{3}{4}-\frac{1}{2}k}}{(r^2-1)^{\frac{1}{4}k}}\exp(\frac{1}{2}b-br^2)$
e	$W_{\frac{1}{2}k+\frac{1}{4},\frac{1}{2}i\mu}(b)$	$\dfrac{2^{1-k}rb^{\frac{3}{4}-\frac{1}{2}k}}{(r^2-1)^{\frac{1}{4}k}}\exp(\frac{1}{2}b-br^2)$
f	$S_{k+\frac{1}{2},i\mu}(d)$	$rd^{\frac{1}{2}}K_k[d(r^2-1)^{\frac{1}{2}}]$

The first integral we shall evaluate is

$$I_1 = \frac{1}{\pi}\int_0^\infty \mu \sinh(\mu\pi)\,\psi(\mu,k)\,K_{i\mu}(a)\,K_{i\mu}(b)\,d\mu, \quad \Re(k) \leqslant \tfrac{1}{2}. \tag{3}$$

Taking $G(\mu,k) = K_{i\mu}(a)$, $H(\mu,k) = K_{i\mu}(b)$, and $g(r)$, $h(r)$ as given by the tabulated result (a), we have, from the Parseval relation (2),

$$I_1 = \tfrac{1}{2}\pi\,(ab)^{\frac{1}{2}-k}\int_1^\infty \frac{e^{-(a+b)r}}{(r^2-1)^k}dr.$$

Using a result given in (1), p. 18, we find that

$$I_1 = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} 2^{-k}\left(\frac{1}{a}+\frac{1}{b}\right)^{k-\frac{1}{2}}\Gamma(1-k)\,K_{\frac{1}{2}-k}(a+b), \quad \Re(k) \leqslant \tfrac{1}{2}, \tag{4}$$

which agrees with the value given in (4), p. 14.

Setting

$$G(\mu,k) = K_{i\mu}(a), \quad H(\mu,k) = \frac{\Gamma(\frac{3}{4}-\lambda+\frac{1}{2}i\mu)\,\Gamma(\frac{3}{4}-\lambda-\frac{1}{2}i\mu)}{\Gamma(\frac{3}{4}-\frac{1}{2}k+\frac{1}{2}i\mu)\,\Gamma(\frac{3}{4}-\frac{1}{2}k-\frac{1}{2}i\mu)},$$

and $g(r)$, $h(r)$ as given by the tabulated results (a) and (b), we find from (2) that

$$\begin{aligned} I_2 &= \frac{1}{\pi}\int_0^\infty \mu \sinh(\mu\pi)\,\psi(\mu,k)\,\frac{\Gamma(\frac{3}{4}-\lambda+\frac{1}{2}i\mu)\,\Gamma(\frac{3}{4}-\lambda-\frac{1}{2}i\mu)}{\Gamma(\frac{3}{4}-\frac{1}{2}k+\frac{1}{2}i\mu)\,\Gamma(\frac{3}{4}-\frac{1}{2}k-\frac{1}{2}i\mu)}K_{i\mu}(a)\,d\mu \\ &= 2^{\lambda-\frac{3}{2}k}a^{1-\lambda-\frac{1}{2}k}\Gamma(1-\lambda-\tfrac{1}{2}k)\,K_{\lambda-\frac{1}{2}k-\frac{1}{2}}(a), \end{aligned} \tag{5}$$

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provided $\Re(k) \leqslant \frac{1}{2}, \frac{1}{4} < \Re(\lambda) \leqslant \frac{3}{4}$ and $\Re(\lambda - \frac{1}{2}k) > 0$. The conditions on λ may be relaxed in certain cases. For example, taking $2\lambda = -k = \frac{1}{2}$, we have, as a special case of (5),

$$\int_0^\infty \mu \tanh(\tfrac{1}{2}\mu\pi) K_{i\mu}(a)\, d\mu = aK_0(a), \tag{6}$$

which is given in (4), p. 9.

Other integrals which may be evaluated in a similar manner are as follows.

$$\begin{aligned} I_3 &= \frac{1}{\pi}\int_0^\infty \mu \sinh(\mu\pi)\, \psi(\mu, k)\, W_{k, i\mu}(2b)\, K_{i\mu}(a)\, d\mu \\ &= (\tfrac{1}{2}\pi)^{\frac{1}{2}} a^{\frac{1}{2}-k} b(a+b)^{k-1}\, \Gamma(1-k)\, e^{-(a+b)}, \quad \Re(k) \leqslant \tfrac{1}{2}, \end{aligned} \tag{7}$$

which is listed in (4), p. 17.

$$\begin{aligned} I_4 &= \frac{1}{\pi}\int_0^\infty \mu \sinh(\mu\pi)\, \psi(\mu, k)\, W_{\frac{1}{2}k-\frac{1}{4}, \frac{1}{2}i\mu}(a)\, W_{\frac{1}{2}k+\frac{1}{4}, \frac{1}{2}i\mu}(b)\, d\mu \\ &= 2^{1-2k} b^{\frac{1}{2}} (ab)^{\frac{3}{4}-\frac{1}{2}k} (a+b)^{k-1}\, \Gamma(1-k)\, e^{-\frac{1}{2}(a+b)}, \quad \Re(k) \leqslant \tfrac{1}{2}. \end{aligned} \tag{8}$$

$$\begin{aligned} I_5 &= \frac{1}{\pi}\int_0^\infty \mu \sinh(\mu\pi)\, \psi(\mu, k)\, W_{\frac{1}{2}k-\frac{1}{4}, \frac{1}{2}i\mu}(b)\, S_{k+\frac{1}{2}, i\mu}(d)\, d\mu \\ &= \frac{d}{2}\left(\frac{b}{d^2}\right)^{\frac{1}{2}k-\frac{1}{4}} \Gamma(1-k) \exp\left(-\frac{b}{2} + \frac{d^2}{4b}\right) \Gamma\left(k, \frac{d^2}{4b}\right), \quad \Re(k) \leqslant \tfrac{1}{2}, \end{aligned} \tag{9}$$

where

$$\Gamma(k, x) = \int_x^\infty t^{k-1} e^{-t}\, dt.$$

The integral I_3 is evaluated using the tabulated pairs (a) and (c), I_4 using the pairs (d) and (e), and I_5 using the pairs (d) and (f).

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