Ergod. Th. & Dynam. Sys. (2019), **39**, 2968–2982 © *Cambridge University Press, 2018*
doi:10.1017/etds.2018.4

Minimality for actions of abelian semigroups on compact spaces with a free interval

MATÚŠ DIRBÁK, ROMAN HRIC, PETER MALIČKÝ, L'UBOMÍR SNOHA and VLADIMÍR ŠPITALSKÝ

Department of Mathematics, Faculty of Natural Sciences, Matej Bel University, Tajovského 40, 974 01 Banská Bystrica, Slovakia
(e-mail: Matus.Dirbak@umb.sk, Roman.Hric@umb.sk, Peter.Malicky@umb.sk, Lubomir.Snoha@umb.sk, Vladimir.Spitalsky@umb.sk)

(*Received* 30 *March* 2017 *and accepted in revised form* 20 *December* 2017)

Dedicated to Sergii Kolyada on the occasion of his 60th birthday

Abstract. We study minimality for continuous actions of abelian semigroups on compact Hausdorff spaces with a free interval. First, we give a necessary and sufficient condition for such a space to admit a minimal action of a given abelian semigroup. Further, for actions of abelian semigroups we provide a trichotomy for the topological structure of minimal sets intersecting a free interval.

1. *Introduction*

Various aspects of the dynamics of group actions on the circle $\mathbb{S}^1$ by homeomorphisms are well understood. By the Ghys–Margulis alternative [**8**], every (effective) action of a group G on the circle either has an invariant probability measure or contains a free subgroup with two generators. Moreover, by Malyutin's theorem [**7**], if the action of G is minimal then either G is conjugate to a group of rotations or it is a finite cover of a proximal action (see also Glasner's paper [**5**] for a shortened proof). As far as minimal sets of group actions on $\mathbb{S}^1$ are concerned, the following trichotomy holds (see [**4**, Proposition 5.6], [**1**, Theorem 3.7], [**5**, Theorem 3.3]):

(1) every minimal set is finite;
(2) the whole circle $\mathbb{S}^1$ is a (unique) minimal set;
(3) the action has a unique minimal set, which is a Cantor set.

For actions of general semigroups S on the circle by continuous maps, much less is known about minimality and minimal sets, an obvious exception being the completely understood case $S = \mathbb{N}$. Recall that every minimal map on $\mathbb{S}^1$ is conjugate to an irrational rotation. Moreover, the following full topological characterization of minimal sets of continuous circle maps holds: a subset of the circle is a minimal set for some circle map

https://doi.org/10.1017/etds.2018.4 Published online by Cambridge University Press

if and only if it is either finite or a Cantor set or the whole circle. Contrary to the above trichotomy for group actions on $\mathbb{S}^1$, a non-invertible circle map may have both finite and Cantor minimal sets.

The minimality for continuous maps is well understood even on spaces with a free interval. Recall that a *free interval* J in a space X is an open subset of X homeomorphic to the open interval $(0, 1)$. By [**2**, Theorem A], every minimal continuous map on a metric continuum X with a free interval is conjugate to an irrational circle rotation. Further, if X is a compact metric space with a free interval J and $f\colon X \to X$ is a continuous map then, by [**2**, Theorem B], the following trichotomy holds for every minimal set M of f intersecting J:

(1) M is finite;
(2) M is a disjoint union of finitely many circles;
(3) M is a nowhere dense cantoroid,

where by a *cantoroid* we mean a compact metric space with a dense set of degenerate components and without isolated points.

In the present paper we study minimality for actions, by continuous maps, of abelian semigroups on compact spaces with a free interval. First, we show that in the class of such spaces only finite disjoint unions of circles admit minimal actions of abelian semigroups.

THEOREM A. *Let X be a compact Hausdorff space with a free interval and let Φ be a minimal action of an abelian semigroup S on X. Then X is a disjoint union of finitely many circles and all the acting maps of Φ are homeomorphisms of X.*

If X is connected then we have the following strengthening of Theorem A.

THEOREM B. *Let X be a compact connected Hausdorff space with a free interval and let Φ be a minimal action of an abelian semigroup S on X. Then Φ is conjugate to an action of S by rotations on the circle $\mathbb{S}^1$.*

Having Theorems A and B at our disposal, we may ask whether a given abelian semigroup S acts in a minimal way on a given disjoint union of circles X. Via Proposition 13, this can be reduced to an algebraic characterization of the abelian subgroups of the group $\mathcal{H}(X)$ of all homeomorphisms of X, whose natural actions on X are minimal. The latter problem is addressed in the following theorem.

THEOREM C. *Let ℓ be a positive integer and X be a disjoint union of ℓ circles. Given an abelian group G, the following conditions are equivalent:*

(1) *G is isomorphic to a subgroup of $\mathcal{H}(X)$, whose natural action on X is minimal;*
(2) *there is a short exact sequence of abelian groups*

$$0 \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 0$$

such that H is (isomorphic to) a dense subgroup of $\mathbb{S}^1$ and K has cardinality ℓ.

Finally, we study minimal sets of abelian semigroup actions which intersect a free interval and prove the following trichotomy analogous to that from [**2**, Theorem B]. Let us mention that for a subset B of a free interval J, we denote by $\operatorname{conv}(B)$ the convex hull of B in J (see also §3).

https://doi.org/10.1017/etds.2018.4 Published online by Cambridge University Press

THEOREM D. *Let X be a compact Hausdorff space with a free interval J and let Φ be an action of an abelian semigroup S on X. Assume that $M \subseteq X$ is a minimal set for Φ, which intersects J. Then M is contained in a closed metrizable locally connected subspace of X and exactly one of the following conditions holds:*

(1) *M is finite;*
(2) *M is a disjoint union of finitely many circles;*
(3) *M is nowhere dense in X, $A = \overline{\mathrm{conv}(M \cap J)}$ is an arc or a circle and $M \cap A$ is a Cantor set.*

If all the acting maps of Φ are homeomorphisms, this trichotomy can be slightly strengthened; see Corollary 19.

These results do not generalize to actions of arbitrary semigroups. For instance, by Duminy's theorem, the free semigroup on two generators has a minimal action on the interval [0, 1]; see, for example, [**9**, §3.3] and [**10**]. It would be interesting to know to what extent our results generalize to actions of non-abelian semigroups.

2. *Minimal abelian semigroup actions on topological spaces*

2.1. *Definitions.* The set of positive integers is denoted by $\mathbb{N}$. By a space we mean a topological space. For the composition of two selfmaps f, g of a space X, we write both $f \circ g$ and fg.

Let S be a (non-empty) abelian semigroup (equipped with the discrete topology) and X be a space. By an *action* of S on X we mean a continuous map $\Phi : S \times X \to X$ such that $\Phi(s, \Phi(t, x)) = \Phi(s + t, x)$ for all $s, t \in S$ and $x \in X$. The acting maps $\Phi(s, \cdot)$ of Φ will be denoted by $\varphi_s : X \to X$. If the semigroup S is a group with zero element 0 and we explicitly speak about a group action, then it is additionally assumed that $\Phi(0, x) = x$ for every $x \in X$; in such a case, all the acting maps φ_s of the group action Φ are homeomorphisms of X.

An action Φ of S on X is *minimal* if for every $x \in X$, the Φ-*orbit* $\mathrm{Orb}_\Phi(x) := \{\varphi_s(x) : s \in S\}$ of x is dense in X. An equivalent definition is that X is the unique minimal set of Φ. Recall that a subset M of X is a *minimal set* of an action Φ if it is non-empty, closed, Φ-invariant (that is, $\varphi_s(M) \subseteq M$ for every $s \in S$), and there is no proper subset with these three properties.

We say that an action Φ of S on X is *free* or that S *acts freely* on X if, for every $s \in S$, the acting map φ_s either is the identity or has no fixed point. If S is a group then the action of S on X is called *effective* if $\varphi_s = \mathrm{Id}_X$ implies $s = 0$.

For a space X, denote by $\mathcal{C}(X)$ the semigroup of all continuous maps $f : X \to X$, and by $\mathcal{H}(X)$ the group of all homeomorphisms of X. In the case where S is a subsemigroup of $\mathcal{C}(X)$, we shall often consider the *natural action* Φ of S on X with $\varphi_s = s$ for every $s \in S$.

Two actions Φ, Ψ of S on X, Y with acting maps φ_s, ψ_s, respectively, are said to be *(topologically) conjugate* if there is a homeomorphism $h : X \to Y$ such that $h \circ \varphi_s = \psi_s \circ h$ for every $s \in S$. Clearly, if Φ, Ψ are conjugate and one of them is minimal then so is the other one.

https://doi.org/10.1017/etds.2018.4 Published online by Cambridge University Press

2.2. *Basic facts on minimal semigroup actions.* If (X, f) is a dynamical system given by a space X and a continuous map $f: X \to X$, and a (not necessarily closed) subset A of X is *f-invariant*, that is, $f(A) \subseteq A$, then we also use a less precise notation (A, f) for the subsystem $(A, f|_A)$ of (X, f). By $\mathrm{Fix}(f)$, $\mathrm{Per}(f)$, $\mathrm{Min}(f)$ and $\mathrm{Rec}(f)$ we denote the set of fixed, periodic, minimal and recurrent points of f, respectively (recall that a *minimal point* is an element of a minimal set).

LEMMA 1. *Let Φ be a minimal action of an abelian semigroup S on a topological space X and f, g be two of the acting maps of Φ. Then the following statements hold:*

(1) *if $\emptyset \neq A \subseteq X$ is Φ-invariant then it is dense in X;*
(2) *$(g(X), f)$ is both a subsystem of (X, f) and a factor of (X, f);*
(3) *if $A \subseteq X$ is f-invariant then $(g(A), f)$ is both a subsystem of (X, f) and a factor of (A, f); in particular, if A is a minimal set of f then so is $g(A)$;*
(4) *the set $g(X)$ is Φ-invariant and dense in X;*
(5) *the sets $\mathrm{Fix}(f)$, $\mathrm{Per}(f)$, $\mathrm{Min}(f)$ and $\mathrm{Rec}(f)$ are Φ-invariant, and if any of them is non-empty then it is dense in X.*

Proof. Statement (1) follows from definition of minimality, and (2) and (3) follow from the identity $f \circ g = g \circ f$. Further, (4) follows from (2) and (1). We check (5). By virtue of (3), if a point x belongs to one of these sets, then so does its image under every acting map of Φ. Consequently, all these sets are Φ-invariant and so the density part of statement (5) follows from (1). □

LEMMA 2. *Let Φ be a minimal action of an abelian semigroup S on a Hausdorff space X. Then Φ is free.*

Proof. Let $s \in S$ and assume that φ_s has a fixed point. Then, by Lemma 1(5), $\mathrm{Fix}(\varphi_s)$ is dense in X. Since X is Hausdorff, the continuity of φ_s gives $\mathrm{Fix}(\varphi_s) = X$ and hence $\varphi_s = \mathrm{Id}_X$. □

COROLLARY 3. *If a non-degenerate Hausdorff space X has the fixed point property, then it does not admit a minimal action of any abelian semigroup S.*

We shall have an occasion to use the following well-known lemma. We include a proof for completeness.

LEMMA 4. *Let Φ be a minimal action of an abelian semigroup S on a compact Hausdorff space X. Then, for every non-empty open set $V \subseteq X$, there exist finite sets E, $F \subseteq S$ with $X = \bigcup_{t\in E} \varphi_t^{-1}(V)$ and $X = \bigcup_{r\in F} \varphi_r(V)$.*

Proof. By minimality of Φ, $X = \bigcup_{t\in S} \varphi_t^{-1}(V)$ and so the existence of the required set E follows by compactness of X. Further, by virtue of Lemma 1(4) and our assumptions on X, all the acting maps of Φ are surjective. Now we may assume that V is a proper subset of X, otherwise the rest of the proof is trivial. Thus, the set E has at least two elements. Set $p = \sum_{s\in E} s$, $r_t = \sum_{s\in E\setminus t} s$ for $t \in E$ and $F = \{r_t : t \in E\}$. Since $p = r_t + t$ for every $t \in E$, we obtain

$$X = \varphi_p(X) = \bigcup_{t\in E} \varphi_p(\varphi_t^{-1}(V)) = \bigcup_{t\in E} \varphi_{r_t}(V) = \bigcup_{r\in F} \varphi_r(V),$$

for $\varphi_t(\varphi_t^{-1}(V)) = V$ by surjectivity of φ_t. □

https://doi.org/10.1017/etds.2018.4 Published online by Cambridge University Press

LEMMA 5. *Let Φ be a minimal action of an abelian semigroup S on a compact Hausdorff space X. Then the following statements hold:*

(1) *if $\varphi_s(V)$ is a singleton for a non-empty open set $V \subseteq X$ and some $s \in S$ then the space X is finite;*
(2) *if the space X has an isolated point then it is finite.*

Proof. Fix V and s as in (1) and write $\varphi_s(V) = \{x\}$. By Lemma 1(4), the open set $\varphi_s^{-1}(V)$ is non-empty. Since Φ is minimal, we have $\varphi_t(x) \in \varphi_s^{-1}(V)$ for some $t \in S$. Then $x = \varphi_s\varphi_s\varphi_t(x) = \varphi_{2s+t}(x)$ and so $\varphi_{2s+t} = \varphi_{s+t}\varphi_s$ is the identity on X by virtue of Lemma 2. This means that φ_s is an injection, whence it follows that V is a singleton. Now, by Lemma 4, there is a finite set $F \subseteq S$ with $X = \bigcup_{r\in F} \varphi_r(V)$. Thus, the space X is finite, which verifies statement (1). Statement (2) follows immediately from statement (1). □

Let Φ be a minimal action of an abelian semigroup S on a space X. Assume that the acting maps φ_s are homeomorphisms of X, thus forming an (abelian) *subsemigroup* $S_\Phi = \{\varphi_s : s \in S\}$ of the group $\mathcal{H}(X)$ of all homeomorphisms of X. Define

$$G_\Phi = \{\varphi_s \circ \varphi_t^{-1} : s, t \in S\}. \tag{2.1}$$

Then

- G_Φ is an *abelian subgroup* of $\mathcal{H}(X)$;
- $G_\Phi \supseteq S_\Phi$—indeed, given $s \in S$, we have $\varphi_s = \varphi_{s+s} \circ \varphi_s^{-1} \in G_\Phi$.

Thus, G_Φ is the subgroup of $\mathcal{H}(X)$ generated by S_Φ.

3. *Minimal abelian semigroup actions on compact spaces with a free interval*

While in the statements of our main theorems we use the notion of a free interval, now it will be convenient to use also the notion of a free arc. Recall that a *free arc* A in a space X is a subset of X homeomorphic to the compact unit interval $[0, 1]$, which becomes an open set in X after removing its two end points. We shall often use an identification of a free arc (free interval) with a genuine compact (open) interval in the real line, together with the natural order. In particular, the notion of the convex hull $\mathrm{conv}(M)$ is naturally defined for a subset M of a free interval or a free arc.

Assume that a map f sends a closed arc $[a, b]$ onto a closed arc $[c, d]$. If $f(a) = c$ and $f(b) = d$ then we write $f : [a, b] \nearrow [c, d]$. Similarly, if $f(a) = d$ and $f(b) = c$ then we write $f : [a, b] \searrow [c, d]$.

LEMMA 6. *Let X be a Hausdorff space with a free arc $A = [0, 1]$ and let $f : X \to X$ be a continuous map without fixed points in A. Assume that $a, b \in [0, 1]$ are such that $f(a) \in (0, 1)$ and $f(b) \notin [0, 1]$. Then the following statements hold:*

(1) *if $a < b$ and $f(a) > a$ then there is an arc $I \subseteq [a, b]$ with $f : I \nearrow [f(a), 1]$;*
(2) *if $a < b$ and $f(a) < a$ then there is an arc $I \subseteq [a, b]$ with $f : I \searrow [0, f(a)]$;*
(3) *if $b < a$ and $f(a) > a$ then there is an arc $I \subseteq [b, a]$ with $f : I \searrow [f(a), 1]$;*
(4) *if $b < a$ and $f(a) < a$ then there is an arc $I \subseteq [b, a]$ with $f : I \nearrow [0, f(a)]$.*

https://doi.org/10.1017/etds.2018.4 Published online by Cambridge University Press

Proof. Before turning to the proof, notice that the set $(0, 1)$ is open in X by definition of a free arc and the set $[0, 1]$ is (compact hence) closed in X by the Hausdorff property of X.

Now, without loss of generality, we may assume that $a < b$. (The other case $b < a$ then follows by an obvious symmetry argument.) The set $f^{-1}(0, 1)$ is open in X and so $U = [a, b] \cap f^{-1}(0, 1)$ is open in $[a, b]$. Let C be the component of U containing $a \in U$. Since $b \notin U$, the set C is of the form $C = [a, z)$ for some $a < z < b$. Further, as observed above, the sets $(0, 1)$ and $X \setminus [0, 1]$ are open in X and so $f(z) \in \{0, 1\}$. Finally, using the fact that $A \cap \mathrm{Fix}(f) = \emptyset$, it is sufficient to set $u = \max\{t \in [a, z] : f(t) = f(a)\}$ and $I = [u, z]$. □

LEMMA 7. *Let X be a Hausdorff space with a free arc $A = [0, 1]$ and $h : X \to X$ be a continuous map. Let $0 \le \alpha < \beta \le \gamma < \delta \le 1$ be such that $h([\alpha, \beta] \cup [\gamma, \delta]) \subseteq [0, 1]$, $h(\alpha) \ge \delta$, $h(\gamma) \le \alpha$ and $h(\delta) \in h([\alpha, \beta])$. Then*

$$\mathrm{Fix}(h) \cap [\gamma, \delta] \neq \emptyset \quad \textit{or} \quad \mathrm{Per}(h) \cap [\alpha, \beta] \neq \emptyset.$$

Proof. Suppose that $\mathrm{Fix}(h) \cap ([\alpha, \beta] \cup [\gamma, \delta]) = \emptyset$. Then $h(x) > x$ on $[\alpha, \beta]$ and $h(x) < x$ on $[\gamma, \delta]$. By replacing β with an appropriate point from $(\alpha, \beta]$, if necessary, we may assume that $\beta < h(\beta) = h(\delta) < \delta$ and $h(x) \ge h(\delta)$ for every $x \in [\alpha, \beta]$. Set $\tilde{\alpha} = \max\{t \in [\alpha, \beta] : h(t) = \delta\}$ and $\tilde{\gamma} = \max\{t \in [\gamma, \delta] : h(t) = \tilde{\alpha}\}$. By applying [**2**, Lemma A.1] to $f = h|_{[\tilde{\alpha}, \beta]} : [\tilde{\alpha}, \beta] \to [\tilde{\alpha}, \delta]$ and $g = h|_{[\tilde{\gamma}, \delta]} : [\tilde{\gamma}, \delta] \to [\tilde{\alpha}, \delta]$, we obtain $\mathrm{Per}(h) \cap [\tilde{\alpha}, \beta] \neq \emptyset$. □

LEMMA 8. *Let X be a Hausdorff space with a free arc $A = [0, 1]$ and $f : X \to X$ be a continuous map without periodic points in A. Let $a, b \in A$ be such that $0 < a < f(a) \le f(b) < b < 1$. Then for every $x \in A$:*

(1) *if $x < a$ then $x < f(x) \le b$;*
(2) *if $x > b$ then $x > f(x) \ge a$.*

Proof. By a simple symmetry argument it is sufficient to verify (1). Assume that $f(x) \notin (x, b]$ for some $0 \le x < a$. Then in fact $f(x) \notin [x, b]$, for f has no fixed points in A. Since $f(a) \in (x, b)$, Lemma 6(3) yields the existence of an arc $I' \subseteq [x, a]$ such that $f : I' \searrow [f(a), b]$. Further, since f has no fixed point in $[a, b]$, there is $y \in (a, b)$ such that $f(y) \notin [0, 1]$. As $f(y) \notin [x, b]$ and $f(b) \in (x, b)$, by Lemma 6(4) there is an arc $I'' \subseteq [y, b]$ such that $f : I'' \nearrow [x, f(b)]$. Now applying Lemma 7 to $[\alpha, \beta] = I'$ and $[\gamma, \delta] = I''$ yields a periodic point of f in $[x, b]$, a contradiction. □

LEMMA 9. *Let X be a Hausdorff space with a free arc $A = [0, 1]$ and $f : X \to X$ be a continuous map without periodic points in A. Assume that $0 < f(0) = c = f(1) < 1$. Then $\mathrm{Orb}_f(c) \subseteq (0, 1)$.*

Proof. The proof is by contradiction. Let n be the smallest positive integer with $f^n(c) \notin (0, 1)$. Then $f^n(c) \notin [0, 1]$, for otherwise 0 or 1 would be a periodic point of f. Put $d = f^{n-1}(c) = f^n(0) = f^n(1) \in (0, 1)$. Without loss of generality, we may assume that $d \le c$. By applying Lemma 6(4) to f^n, $a = 1$ and $b = c$, we find an arc $I' \subseteq [c, 1]$ such that $f^n : I' \nearrow [0, d]$. Applying Lemma 6(1) to f, $a = 0$ and $b = d$ yields an arc $I'' \subseteq [0, d]$

https://doi.org/10.1017/etds.2018.4 Published online by Cambridge University Press

with $f\colon I'' \nearrow [c, 1]$. Consequently, I' contains an arc K such that $f^{n+1}(K) = [c, 1] \supseteq K$. Thus f^{n+1} has a fixed point in K, a contradiction. □

LEMMA 10. *Let X be a Hausdorff space with a free interval J and let Φ be a minimal action of an abelian semigroup S on X. Then all the acting maps φ_s of Φ are injective on J.*

Proof. We shall proceed by contradiction. Assume that $s \in S$ and $a, b \in J$ are such that $a < b$ and $\varphi_s(a) = \varphi_s(b)$. By minimality of Φ, there is $r \in S$ with $c := \varphi_r\varphi_s(a) = \varphi_r\varphi_s(b) \in (a, b)$. Set $t = r + s \in S$. Notice that φ_t has no periodic points. (Indeed, if $(\varphi_t)^k(y) = y$ for some $k \geq 1$ and $y \in X$ then $(\varphi_t)^k$ would be the identity by Lemma 2, hence φ_t would be injective.) Thus, we may apply Lemma 9 to $f = \varphi_t$ and $A = [a, b]$ to obtain $\mathrm{Orb}_{\varphi_t}(c) \subseteq (a, b)$.

Since the closure of $\mathrm{Orb}_{\varphi_t}(c)$ is compact, the set $\mathrm{Min}(\varphi_t)$ is non-empty and, by Lemma 1(5), it is dense in X. Consequently, φ_t has a minimal set M, which intersects J on the left of a. As M is not a periodic orbit for φ_t, it has no isolated points. Therefore, there exist $z, x \in M \cap J$ with $z < x < a$. Now, given $n \in \mathbb{N}$, we have $a < (\varphi_t)^n(a) = (\varphi_t)^n(b) < b$. Thus, by Lemma 8(1) applied to $f = (\varphi_t)^n$, we obtain $x < (\varphi_t)^n(x) \leq b$. Since the latter holds for every $n \in \mathbb{N}$ and since $z < x$, we infer that z is not contained in the orbit closure of x under the action of φ_t. This contradicts the fact that M is a minimal set of φ_t, which finishes the proof. □

LEMMA 11. *Let X be a Hausdorff space, which can be expressed as a union of finitely many arcs. Then the union of the free intervals of X is dense in X.*

Proof. Obviously, it suffices to prove the following claim:

- Let $X = Y \cup Z$ be a compact Hausdorff space, where Y and Z are closed subspaces of X, each of them having a dense union of free intervals in the relative topology. Then X also has a dense union of free intervals.

To prove this claim, let U be a non-empty open subset of X. We want to find a free interval J of X contained in U. We distinguish three cases.

First assume that $U \cap (Y \setminus Z) \neq \emptyset$. The set $U \setminus Z$ is non-empty and open in X and is contained in Y. By the assumption on Y, $U \setminus Z$ contains a free interval J of Y. Clearly, J is open in $U \setminus Z$ and hence also in X. Thus J is a free interval of X contained in U. The case $U \cap (Z \setminus Y) \neq \emptyset$ is handled analogously.

It remains to consider the case $U \subseteq Y \cap Z$. Since $U \subseteq Y$, there is a free interval J of Y in U. The set J is open in U, hence in X, and so it is a free interval of X contained in U. □

THEOREM A. *Let X be a compact Hausdorff space with a free interval and let Φ be a minimal action of an abelian semigroup S on X. Then X is a disjoint union of finitely many circles and all the acting maps of Φ are homeomorphisms of X.*

Proof. First observe that X is a union of finitely many arcs by Lemmas 4 and 10 and so, by Lemma 11, the union of the free intervals of X is dense in X.

https://doi.org/10.1017/etds.2018.4 Published online by Cambridge University Press

We begin the proof by showing that the acting maps φ_s of Φ are homeomorphisms of X. Since all φ_s are surjective by Lemma 1(4) and by compactness of the Hausdorff space X, it is sufficient to show that they are injective. So assume that $\varphi_s(a) = \varphi_s(b)$ for some $s \in S$ and some distinct points $a, b \in X$. Use Lemma 4 to find $r \in S$, a free interval $J \subseteq X$ and $x \in J$ with $\varphi_r(x) = a$. Also, fix $y \in X$ with $\varphi_r(y) = b$ and notice that $y \neq x$. By minimality of Φ, there are $t \in S$ and a free interval $J' \subseteq X$ with $\varphi_t\varphi_s(a) = \varphi_t\varphi_s(b) \in J'$. Set $p = r + s + t$.

Fix a neighbourhood W of y in X with $\varphi_p(W) \subseteq J'$ and choose a free interval $I \subseteq W$ of X. Since φ_p has no fixed points by Lemma 2, the points x, y and $\varphi_p(x) = \varphi_p(y)$ are mutually distinct and so we may assume that J, I, J' are mutually disjoint. Moreover, due to Lemma 10, we may also suppose that $\varphi_p\colon J \to J'$ is a homeomorphism and $\varphi_p\colon I \to J'$ is an open map (in fact, a homeomorphism onto the open subset $\varphi_p(I)$ of J').

By Lemma 1(5) and compactness of X, I intersects a minimal set M for φ_p. We claim that

$(*)$ $K \cap \varphi_p^{-1}(M) \subseteq M$ for every free interval K of X.

Indeed, since the map φ_p has no periodic points and it has dense minimal (hence recurrent) points, [**2**, Theorem 20] yields $K \cap \varphi_p^{-1}(M) \subseteq \mathrm{Rec}(\varphi_p) \cap \varphi_p^{-1}(M) \subseteq M$, where the last inclusion follows from minimality of M for φ_p.

Now set $U = \varphi_p(I \cap M)$ and $V = J \cap \varphi_p^{-1}(U)$. Clearly, both U and V are non-empty; we want to verify that they are open subsets of M. First, by applying $(*)$ to $K = I$, we get $I \cap \varphi_p^{-1}(M) \subseteq M$, whence $U = \varphi_p(I \cap M) = \varphi_p(I) \cap M$ is open in M. Further, by applying $(*)$ to J, we obtain $V = J \cap \varphi_p^{-1}(U) \subseteq M$ and hence $V = J \cap (\varphi_p|_M)^{-1}(U)$ is also an open subset of M.

We want to arrive at a contradiction by showing that V is a redundant open set for the minimal system (M, φ_p), meaning that $\varphi_p(M \setminus V) \supseteq \varphi_p(V)$ (see [**6**, Lemma 2.1]). Indeed, since V is disjoint with I, we have $\varphi_p(M \setminus V) \supseteq \varphi_p(I \cap M) = U$. Moreover, since the restriction $\varphi_p\colon J \to J'$ is a homeomorphism, $U = \varphi_p(V)$ by definition of V. Thus, indeed, $\varphi_p(M \setminus V) \supseteq \varphi_p(V)$ and this contradiction shows that all the acting maps φ_s of Φ are homeomorphisms of X.

Now we finish the proof of the theorem. Fix a free interval J of X. By compactness of X and minimality of Φ, there is a finite set $F \subseteq S$ with $X = \bigcup_{r \in F} \varphi_r(J)$ (see Lemma 4) and so X is a union of finitely many free intervals. Thus, X is a compact Hausdorff second countable one-dimensional manifold, and hence it is homeomorphic to a disjoint union of finitely many circles. □

So, for instance, the Warsaw circle does not admit a minimal action of an abelian semigroup, although this follows also from Corollary 3 and the fact that the Warsaw circle has the fixed point property. For an example which does not follow from Corollary 3, take any compact Hausdorff space with a free interval having a circle as its retract. (Notice that such a space does not have the fixed point property since the fixed point property is preserved by passing to a retract.)

THEOREM B. *Let X be a compact connected Hausdorff space with a free interval and let Φ be a minimal action of an abelian semigroup S on X. Then Φ is conjugate to an action of S by rotations on the circle $\mathbb{S}^1$.*

https://doi.org/10.1017/etds.2018.4 Published online by Cambridge University Press

Proof. By Theorem A, X is a circle (so we may assume that $X = \mathbb{S}^1$) and all the acting maps of Φ are homeomorphisms. Moreover, by Lemma 2, the action Φ is free. Since every orientation-reversing circle homeomorphism has a fixed point, we infer that all the acting homeomorphisms of Φ are orientation-preserving. Consequently, the group G_Φ defined by (2.1) is a subgroup of the group $\mathcal{H}_+(\mathbb{S}^1)$ formed by the orientation-preserving circle homeomorphisms. Since the natural action of G_Φ on $\mathbb{S}^1$ is minimal, G_Φ is conjugate to a group of rotations by [**4**, Corollary 5.15]. Hence Φ is conjugate to an S-action on $\mathbb{S}^1$ by rotations. □

COROLLARY 12. *The following statements hold:*

(1) *every minimal action of an abelian semigroup S on the circle $\mathbb{S}^1$ is conjugate to an S-action by rotations;*

(2) *every abelian subgroup of $\mathcal{H}(\mathbb{S}^1)$ with a minimal natural action on $\mathbb{S}^1$ is isomorphic to a dense subgroup of $\mathbb{S}^1$.*

Proof. Part (1) follows immediately from Theorem B. To verify part (2), fix an abelian subgroup G of $\mathcal{H}(\mathbb{S}^1)$ with a minimal natural action on $\mathbb{S}^1$. By part (1), G is conjugate in $\mathcal{H}(\mathbb{S}^1)$ to a group of rotations G'. Clearly, the natural action of G' on $\mathbb{S}^1$ is also minimal. Let H consist of all $\alpha \in \mathbb{S}^1$ such that the rotation of $\mathbb{S}^1$ by α is an element of G'. Then H is a subgroup of $\mathbb{S}^1$ isomorphic to G' and hence also to G. Finally, since the action of G' on $\mathbb{S}^1$ is minimal, H must be dense in $\mathbb{S}^1$. □

4. *Existence of minimal actions of a given abelian semigroup*

Our aim in this section is to give a necessary and sufficient condition for a given abelian semigroup S to act in a minimal way on a disjoint union X of finitely many circles. Our first step towards this goal is to reduce the original problem to the description of abelian subgroups G of $\mathcal{H}(X)$ whose natural action on X is minimal. This is done in the following proposition.

PROPOSITION 13. *Let X be a disjoint union of finitely many circles. Given an abelian semigroup S, the following conditions are equivalent:*

(1) *the semigroup S acts in a minimal way on the space X;*

(2) *there is a morphism of semigroups $h\colon S \to \mathcal{H}(X)$ such that the (abelian) subgroup G of $\mathcal{H}(X)$ generated by the image $h(S)$ of h has a minimal natural action on X.*

Proof. First notice that, in an arbitrary group, the subgroup generated by an abelian subsemigroup is automatically abelian. In particular, the commutativity assumption on G in (2) is superfluous.

We show that (2) follows from (1). To this end, let Φ be a minimal action of S on X. By virtue of Theorem A, all the acting maps φ_s of Φ are homeomorphisms. Let $G = G_\Phi$ be the subgroup of $\mathcal{H}(X)$ generated by φ_s ($s \in S$). Then the map $h\colon S \ni s \mapsto \varphi_s \in G$ is a morphism of semigroups and its image $h(S)$ generates the group G by definition of G. Finally, minimality of the natural action of G on X is immediate by minimality of Φ and so condition (2) holds.

We show that (1) follows from (2). Assume that G is an (abelian) subgroup of $\mathcal{H}(X)$ with a minimal natural action on X and $h\colon S \to G$ is a morphism of semigroups whose

https://doi.org/10.1017/etds.2018.4 Published online by Cambridge University Press

image $h(S)$ generates G. Then $G = \{h(t)^{-1}h(s) : t, s \in S\}$ by commutativity of G. Now the semigroup S acts on the space X via homeomorphisms $\varphi_s = h(s)$ $(s \in S)$. We show that this action Φ of S on X is minimal. To this end, fix a non-empty open set $U \subseteq X$. Since the group G acts on X in a minimal way and the space X is compact, there exist $n \geq 2$ and $t_i, s_i \in S$ $(i = 1, \ldots, n)$ with $X = \bigcup_{i=1}^{n} (\varphi_{t_i}^{-1}\varphi_{s_i})(U)$ (see Lemma 4). Set $r = \sum_{j=1}^{n} s_j$ and $r_i = \sum_{j \neq i} s_j$ for $i = 1, \ldots, n$. Then

$$X = \varphi_r^{-1}(X) = \bigcup_{i=1}^{n} \varphi_{t_i}^{-1}\varphi_r^{-1}(\varphi_{s_i}(U)) = \bigcup_{i=1}^{n} (\varphi_{t_i+r_i})^{-1}(U) = \bigcup_{s \in S} \varphi_s^{-1}(U)$$

and the minimality of Φ thus follows. □

Now, in view of Proposition 13, it remains to find an algebraic characterization of those abelian subgroups of $\mathcal{H}(X)$ whose natural action on X is minimal. Such characterization is given in the following theorem. Before formulating it, let us recall some facts from the theory of abelian groups (the reader is referred to [**3**, Ch. IX] for details).

Let

$$0 \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 0$$

be a short exact sequence of abelian groups. Then there is a symmetric H-valued 2-*cocycle* f over K such that G is isomorphic to $K \times_f H$. This means that $f : K \times K \to H$ is a function satisfying the identities

- $f(k_1, k_2) = f(k_2, k_1)$,
- $f(k_1, k_2) + f(k_1 + k_2, k_3) = f(k_2, k_3) + f(k_2 + k_3, k_1)$,
- $f(0, k) = f(k, 0) = 0$,

and $K \times_f H$ is an abelian group whose elements are pairs (k, h) with $k \in K$, $h \in H$ and whose operation is given by the rule

$$(k_1, h_1) + (k_2, h_2) = (k_1 + k_2, h_1 + h_2 + f(k_1, k_2)).$$

We shall now use these facts in the proof of the following theorem.

THEOREM C. *Let ℓ be a positive integer and X be a disjoint union of ℓ circles. Given an abelian group G, the following conditions are equivalent:*

(1) *G is isomorphic to a subgroup of $\mathcal{H}(X)$, whose natural action on X is minimal;*

(2) *there is a short exact sequence of abelian groups*

$$0 \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 0$$

such that H is (isomorphic to) a dense subgroup of $\mathbb{S}^1$ and K has cardinality ℓ.

Remark 14. Condition (2) can be restated by saying that G is an extension of a group with ℓ elements by a dense subgroup of $\mathbb{S}^1$. Let us also recall that a subgroup H of $\mathbb{S}^1$ is dense if and only if it is infinite.

Proof. First we verify implication (1) ⇒ (2). Assume that G is a subgroup of $\mathcal{H}(X)$ with a minimal natural action on X. Write $\mathbb{S}^1$ for a chosen component of X and denote by H the stabilizer of $\mathbb{S}^1$:

$$H = \{\varphi \in G : \varphi(\mathbb{S}^1) = \mathbb{S}^1\} = \{\varphi \in G : \varphi(\mathbb{S}^1) \cap \mathbb{S}^1 \neq \emptyset\}.$$

https://doi.org/10.1017/etds.2018.4 Published online by Cambridge University Press

Clearly, H is a subgroup of G and the minimal action of G on X restricts to a minimal action of H on $\mathbb{S}^1$. Moreover, since G acts on X freely by Lemma 2, the restricted action of H on $\mathbb{S}^1$ is effective and so H can be identified with a subgroup of $\mathcal{H}(\mathbb{S}^1)$ with a minimal natural action on $\mathbb{S}^1$. Thus, H is isomorphic to a dense subgroup of $\mathbb{S}^1$ by Corollary 12. Let $K = G/H$ be the quotient group of G by H. By definition of H, the elements of K are in a one-to-one correspondence with the components of X via the map $G/H \ni \varphi + H \mapsto \varphi(\mathbb{S}^1) \subseteq X$ for $\varphi \in G$. It follows that K has cardinality ℓ and so the canonical short exact sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 0$$

satisfies all the requirements from (2).

We verify implication (2) $\Rightarrow$ (1). Let G be an abelian group, H be a dense subgroup of $\mathbb{S}^1$, K be an abelian group with cardinality ℓ and assume that these groups fit into a short exact sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 0.$$

Without loss of generality, we may assume that $X = K \times \mathbb{S}^1$ and $G = K \times_f H$, where $f\colon K \times K \to H$ is a symmetric H-valued 2-cocycle over K. We define an action of G on X as follows: given $g \in G$, $g = (k, h)$, we define a map

$$\varphi_g = \varphi_{(k,h)}\colon K \times \mathbb{S}^1 \ni (a, b) \mapsto (k + a, b + h + f(k, a)) \in K \times \mathbb{S}^1.$$

Clearly, all the maps φ_g are continuous. Moreover, since f is a symmetric 2-cocycle over K, it follows that

- φ_0 is the identity on $K \times \mathbb{S}^1$,
- $\varphi_{g'}\varphi_g = \varphi_{g'+g}$ for all $g, g' \in G$.

Thus, the maps φ_g ($g \in G$) constitute an action of G on X by homeomorphisms. Since φ_g possesses a fixed point only for $g = 0$, the action of G on X is effective and hence G may be identified with a subgroup of $\mathcal{H}(X)$.

To finish the proof, we need to show that the action of G on X is minimal. This follows immediately from the following observations.

- The maps $\varphi_{(k,0)}$ ($k \in K$) permute the components of X transitively. That is, given components C, C' of X, there is $k \in K$ with $\varphi_{(k,0)}(C) = C'$.
- The maps $\varphi_{(0,h)}$ ($h \in H$) form the stabilizer of $\mathbb{S}^1 = 0 \times \mathbb{S}^1$ and they act on $\mathbb{S}^1$ via rotations by elements of H. Since H is a dense subgroup of $\mathbb{S}^1$, the family of maps $\varphi_{(0,h)}$ ($h \in H$) acts on $\mathbb{S}^1$ in a minimal way. □

In Theorem C we gave a necessary and sufficient condition for a given abelian group G to act in a minimal and effective way on a disjoint union X of ℓ circles. The following corollary of Theorem C gives an analogous necessary and sufficient condition for the existence of a minimal (not necessarily effective) action of G on X. It is based on a standard procedure of turning an action into an effective one; we present a detailed proof for completeness.

COROLLARY 15. *Let ℓ be a positive integer and X be a disjoint union of ℓ circles. Given an abelian group G, the following conditions are equivalent:*

(1) *G acts in a minimal way on X;*

https://doi.org/10.1017/etds.2018.4 Published online by Cambridge University Press

(2) *G possesses a quotient group, which is an extension of a group with ℓ elements by a dense subgroup of $\mathbb{S}^1$.*

Proof. We begin by showing that (2) follows from (1). To this end, fix a minimal action Φ of G on X. Denote by $h\colon G \to \mathcal{H}(X)$ the morphism of groups induced by Φ and write $G' = \ker(h)$ for its kernel. Then h factors through the canonical quotient morphism $p\colon G \to G/G'$ to a monomorphism of groups $h'\colon G/G' \to \mathcal{H}(X)$, that is, $h = h'p$. The action Φ' of G/G' on X corresponding to h' is then effective. Moreover, Φ and Φ' have the same set of orbits and, since Φ is minimal, so is Φ'. Thus, the group G/G' acts in a minimal and effective way on X and hence, by virtue of Theorem C, G/G' is an extension of a group with ℓ elements by a dense subgroup of $\mathbb{S}^1$.

To see that (1) follows from (2), let G' be a subgroup of G such that the corresponding quotient group G/G' is an extension of a group with ℓ elements by a dense subgroup of $\mathbb{S}^1$. Then, by virtue of Theorem C, G/G' acts on X in a minimal and effective way. Fix such a minimal action Φ' of G/G' on X and denote by $h'\colon G/G' \to \mathcal{H}(X)$ the morphism of groups induced by Φ'. Consider the canonical quotient morphism $p\colon G \to G/G'$ and set $h = h'p$. Then $h\colon G \to \mathcal{H}(X)$ is a morphism of groups. Denote by Φ the action of G on X corresponding to h. Then, similarly to the above, the two actions Φ and Φ' have the same set of orbits and, since Φ' is minimal, so is Φ. This verifies condition (1) and finishes the proof. □

Let us now illustrate how Theorem C and Corollary 15 can be used to detect the (non-)existence of minimal actions in concrete situations.

Example 16. The group $G = \mathbb{Z}$ acts in a minimal way on a disjoint union of ℓ circles for every $\ell \in \mathbb{N}$. Indeed, given $\ell \in \mathbb{N}$, the subgroup $H = \ell\mathbb{Z}$ of $\mathbb{Z}$ is isomorphic to a dense subgroup of $\mathbb{S}^1$ and the corresponding quotient group $G/H = \mathbb{Z}_\ell$ has cardinality ℓ.

Example 17. We claim that the torsion subgroup $G = \mathrm{tor}(\mathbb{S}^1)$ of $\mathbb{S}^1$, consisting of the elements of $\mathbb{S}^1$ with a finite order, acts in a minimal way on a disjoint union of ℓ circles if and only if $\ell = 1$. The 'if' part is clear, since $\mathrm{tor}(\mathbb{S}^1)$ is dense in $\mathbb{S}^1$. To verify the 'only if' part, fix $\ell \in \mathbb{N}$ and assume that G acts in a minimal way on a disjoint union of ℓ circles. By virtue of Corollary 15, G factors onto a group K with cardinality ℓ. Since G is divisible, it follows that so is K. However, the only finite divisible abelian group is the trivial one, hence $\ell = 1$.

Example 18. Let $(p_n)_{n\in\mathbb{N}}$ be an increasing sequence of prime numbers and G be the direct sum $G = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}$. We claim that G acts in a minimal way on a disjoint union X of ℓ circles if and only if ℓ is expressible as a product $\ell = \prod_{n\in F} p_n$ for a finite subset F of $\mathbb{N}$ (the empty product being interpreted as 1).

To verify the 'if' part, let $F \subseteq \mathbb{N}$ be finite. Then G splits into a direct sum $G = K \oplus H$, where $K = \bigoplus_{n\in F} \mathbb{Z}_{p_n}$ and $H = \bigoplus_{n\in\mathbb{N}\setminus F} \mathbb{Z}_{p_n}$. The group H is infinite and is clearly isomorphic to a subgroup of $\mathrm{tor}(\mathbb{S}^1)$, hence also of $\mathbb{S}^1$. Moreover, the cardinality of K is $\prod_{n\in F} p_n = \ell$. Finally, since K is isomorphic to G/H, the group G acts in a minimal (and effective) way on X by Theorem C.

https://doi.org/10.1017/etds.2018.4 Published online by Cambridge University Press

To verify the 'only if' part, fix $\ell \in \mathbb{N}$ and let G act in a minimal way on X. Then Corollary 15 yields a quotient group K of G with cardinality ℓ. Let $q\colon G \to K$ be the underlying quotient morphism. Given $n \in \mathbb{N}$, we have either $\mathbb{Z}_{p_n} \subseteq \ker(q)$ or $\mathbb{Z}_{p_n} \cap \ker(q) = 0$. Set $F = \{n \in \mathbb{N} : \mathbb{Z}_{p_n} \cap \ker(q) = 0\}$. Since q is monic on $\mathbb{Z}_{p_n}$ for every $n \in F$, and the prime numbers p_n $(n \in F)$ are mutually distinct, it follows that q is also monic on $\bigoplus_{n\in F} \mathbb{Z}_{p_n}$. Also, q vanishes on $\bigoplus_{n\in\mathbb{N}\setminus F} \mathbb{Z}_{p_n}$. Consequently, $q\colon \bigoplus_{n\in F} \mathbb{Z}_{p_n} \to K$ is an isomorphism and the cardinality of K thus equals $\ell = \prod_{n\in F} p_n$.

5. *Minimal sets on compact spaces with a free interval*

Our aim in this section is to prove the following trichotomy for minimal sets of abelian semigroup actions, which intersect a free interval.

THEOREM D. *Let X be a compact Hausdorff space with a free interval J and let Φ be an action of an abelian semigroup S on X. Assume that $M \subseteq X$ is a minimal set for Φ, which intersects J. Then M is contained in a closed metrizable locally connected subspace of X and exactly one of the following conditions holds:*

(1) *M is finite;*
(2) *M is a disjoint union of finitely many circles;*
(3) *M is nowhere dense in X, $A = \overline{\operatorname{conv}(M \cap J)}$ is an arc or a circle and $M \cap A$ is a Cantor set.*

Proof. Fix an arc $B \subseteq J$, whose interior in X intersects M. By Lemma 4 applied to the restricted action of S on M, there is a finite set $F \subseteq S$ with $M \subseteq \bigcup_{r\in F} \varphi_r(B)$. Being a finite union of Peano continua, $Y = \bigcup_{r\in F} \varphi_r(B)$ is a desired closed metrizable locally connected subspace of X containing M.

Assume that M has a non-empty interior in X. As $M \subseteq Y$ and Y is locally connected, M contains a non-empty connected open subset V of X. Further, since M is a minimal set for Φ and it intersects J, there is $t \in S$ with $\varphi_t(V) \cap J \neq \emptyset$. By connectedness of V, there are two possibilities.

(a) The set $\varphi_t(V) \cap J$ contains an arc. Then the set $M \supseteq \varphi_t(V) \cap J$ contains a free interval and so it is a disjoint union of finitely many circles by Theorem A. Thus, in this case, condition (2) holds.
(b) The set $\varphi_t(V)$ is a singleton. Then the set M is finite by virtue of Lemma 5. Thus, in this case, condition (1) holds.

Now assume that the set M is infinite and nowhere dense in X and write $J = (0, 1)$. By the first step of the proof, M is contained in the closed metrizable locally connected subspace Y of X. Since the set M has no isolated point by Lemma 5, $\operatorname{conv}(M \cap J)$ is a non-degenerate interval with end points $0 \leq a < b \leq 1$; put $L = (a, b)$. Denote by M^+ (respectively, M^-) the set of all $x \in X \setminus L$ such that there is an increasing (respectively, decreasing) sequence $(x_n)_{n\in\mathbb{N}}$ in $M \cap L$ with $x_n \to x$. Since M is compact metrizable, both M^+ and M^- are non-empty. We show, in fact, that each of them is a singleton. We verify this for M^+, the argument for M^- being similar. So let $x, y \in M^+$. Fix an increasing sequence $(x_n)_{n\in\mathbb{N}}$ in $M \cap L$ with $x_n \to x$. Given a connected neighbourhood W of x in Y, we have $x_n \in W$ for all but finitely many n and so $W \supseteq (b - \varepsilon, b)$ for some $\varepsilon > 0$ by connectedness of W. Since Y is locally connected, each neighbourhood of x in

https://doi.org/10.1017/etds.2018.4 Published online by Cambridge University Press

Y contains a subset of L of the form $(b-\varepsilon, b)$ with $\varepsilon > 0$. However, the same argument applies to y and so x, y cannot be separated by disjoint open sets in Y. In view of the Hausdorff property of Y, this means that $x = y$.

Write $A = \overline{\mathrm{conv}(M \cap J)}$; obviously, $A = \overline{L}$. We claim that

$$A = L \cup M^+ \cup M^-.$$

The inclusion '$\supseteq$' is clear. To verify the converse inclusion, fix $x \in A \setminus L$. Then each neighbourhood of x in X intersects $L \setminus [a+\varepsilon, b-\varepsilon] \subseteq (a, a+\varepsilon) \cup (b-\varepsilon, b)$ for every sufficiently small $\varepsilon > 0$. Moreover, as observed above, each neighbourhood of $M^+ \cup M^-$ in X contains $(a, a+\varepsilon) \cup (b-\varepsilon, b)$ for some $\varepsilon > 0$. Due to the finiteness of $M^+ \cup M^-$ and the Hausdorff property of X, this means that $x \in M^+ \cup M^-$.

Thus, A is a (Hausdorff) compactification of an interval by at most two points, whence it follows that A is either an arc or a circle. Further, since L is a free interval in X and M is nowhere dense in X by the assumption, the set $M \cap L$ is totally disconnected and hence so is $M \cap A = (M \cap L) \cup M^+ \cup M^-$. Moreover, the compact metrizable set $M \cap A$ has no isolated points by Lemma 5. Thus, $M \cap A$ is a Cantor set.

Finally, since all the conditions (1)–(3) are clearly mutually exclusive, the proof of the theorem is complete. □

In the special situation where X is a compact metric space and $S = \mathbb{N}$, we know from [**2**, Theorem B] that in case (3) the set M is a cantoroid (that is, a compact metric space without isolated points, whose degenerate components form a dense set). For a general abelian semigroup S we do not know whether this is also the case. However, if all the acting maps are homeomorphisms, even the following stronger result is true.

COROLLARY 19. *Let X be a compact Hausdorff space with a free interval J and let Φ be an action of an abelian semigroup S on X. Assume that $M \subseteq X$ is a minimal set for Φ, which intersects J. If all the acting maps φ_s of Φ are homeomorphisms of X then exactly one of the following conditions holds:*

(1) *M is finite;*
(2) *M is open in X and it is a disjoint union of finitely many circles;*
(3) *M is a nowhere dense Cantor set in X.*

Proof. First we show that, in case (3) of Theorem D, M is a Cantor set. To this end, fix a Cantor set $C \subseteq M \cap J$ open in M. By Lemma 4 applied to the restricted action of S on M, there is a finite set $F \subseteq S$ with $M = \bigcup_{r \in F} \varphi_r(C)$. Since all φ_r are homeomorphisms of X, it follows that M is a union of finitely many Cantor sets and hence M itself is a Cantor set.

Second we show that, in case (2) of Theorem D, M is open in X. To this end, notice that $J \subseteq M$. By Lemma 4, there is a finite set $F \subseteq S$ with $M = \bigcup_{r \in F} \varphi_r(J)$. Since all φ_r are homeomorphisms of X, it follows that M is a union of open subsets of X and hence M itself is open in X. This finishes the proof. □

Acknowledgement. This work was supported by the Slovak Research and Development Agency under the contract No. APVV-15-0439 and by VEGA grant 1/0786/15.

https://doi.org/10.1017/etds.2018.4 Published online by Cambridge University Press

REFERENCES

[**1**] L. A. Beklaryan. Groups of homeomorphisms of the line and the circle, topological characteristics and metric invariants. *Russian Math. Surveys* **59**(4) (2004), 599–660.
[**2**] M. Dirbák, L'. Snoha and V. Špitalský. Minimality, transitivity, mixing and topological entropy on spaces with a free interval. *Ergod. Th. & Dynam. Sys.* **33**(6) (2013), 1786–1812.
[**3**] L. Fuchs. *Infinite Abelian Groups. Vol. I (Pure and Applied Mathematics, 36)*. Academic Press, Burlington, MA, 1970.
[**4**] É. Ghys. Groups acting on the circle. *Enseign. Math.* **47**(3/4) (2001), 329–407.
[**5**] E. Glasner. Short proofs of theorems of Malyutin and Margulis. *Proc. Amer. Math. Soc.* **145**(12) (2017), 5463–5467.
[**6**] S. Kolyada, L'. Snoha and S. Trofimchuk. Noninvertible minimal maps. *Fund. Math.* **168**(2) (2001), 141–163.
[**7**] A. Malyutin. Classification of the group actions on the real line and circle. *St. Petersburg Math. J.* **19**(2) (2008), 279–296.
[**8**] G. A. Margulis. Free subgroups of the homeomorphism group of the circle. *C. R. Acad. Sci. Paris Sér. I Math.* **331**(9) (2000), 669–674.
[**9**] A. Navas. *Groups of Circle Diffeomorphisms (Chicago Lectures in Mathematics)*. University of Chicago Press, Chicago, 2011.
[**10**] K. Shinohara. On the minimality of semigroup actions on the interval which are C^1-close to the identity. *Proc. Lond. Math. Soc. (3)* **109**(5) (2014), 1175–1202.

https://doi.org/10.1017/etds.2018.4 Published online by Cambridge University Press