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A note on integral integer-valued functions of several variables

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1. The consequences of assuming that an integral function or its derivatives take integer values for some or all integer values of the variable have been widely investigated. The best-known result of this kind was obtained by Pólya (9, 10) and Hardy (6) in 1920 and states that among all transcendental integral functions which assume integer values for all non-negative integer values of the variable that of least increase is the function 2^z (cf. also (7)). Analogous results relating to integers in the Gaussian field were proved by Fukasawa (3) in 1926 and Gelfond (4) in 1929, and theorems relating to the function and its derivatives are included in the works of Gelfond (5), Selberg (14) and Straus (15). (For further results and references see (1, 2, 11).) The earlier of these investigations may be regarded as the genesis of the celebrated Gelfond–Schneider solution to the seventh problem of Hilbert and the much important work arising therefrom.

The object of the present note is to establish results of Gelfond–Straus type for functions of several complex variables. The problem of obtaining generalizations of this kind was proposed by Straus at the end of (15). (The paper (13) of Schneider appeared at about the same time as (15) and established results of a similar character. Schneider remarks that there is no essential difficulty in generalizing his main theorem to functions of several complex variables; an extension of this type is indeed given in Lang (8). But Schneider's theorem, although more far-reaching with respect to the theory of transcendental numbers, does not include the precise results of Gelfond and Straus referred to above. The present note shows that these can in fact be obtained by simple modifications of Schneider's basic method, but the generalizations to functions of several variables require additional formulae which do not seem to have been stated explicitly hitherto.) Throughout the paper $f(z_1, \ldots, z_n)$ will denote an integral function of the n complex variables $z_1, \ldots, z_n$ and for each set of non-negative integers $m_1, \ldots, m_n$ we shall write for brevity

$$f_{m_1,\ldots,m_n}(z_1, \ldots, z_n) = \frac{\partial^{m_1+\ldots+m_n}}{\partial z_1^{m_1} \ldots \partial z_n^{m_n}} f(z_1, \ldots, z_n).$$

Further we shall denote by $M(r)$ (where r is written for the vector $(r_1, \ldots, r_n)$) the maximum of $|f(z_1, \ldots, z_n)|$ in the polycylinder $|z_1| \leqslant r_1, \ldots, |z_n| \leqslant r_n$. Our main results are then as follows.

THEOREM. (i) *If h denotes a positive integer for which all the partial derivatives $f_{m_1,\ldots,m_n}(z_1, \ldots, z_n)$ with $0 \leqslant m_k < h$ $(1 \leqslant k \leqslant n)$ assume integer values for all non-negative integers $z_1, \ldots, z_n$, and if further*

$$\limsup_{\|r\| \to \infty} \|r\|^{-1} \log M(r) < h \log (1 + e^{-(h-1)/h}), \tag{1}$$

where $\|r\| = r_1 + \ldots + r_n$, then $f(z_1, \ldots, z_n)$ is a polynomial.

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(ii) *If h denotes a positive integer for which all the partial derivatives $f_{m_1,\dots,m_n}(z_1,\dots,z_n)$ assume integer values for all integers $z_1,\dots,z_n$ with $0 \leqslant z_k < h$ $(1 \leqslant k \leqslant n)$ and if further*

$$\limsup_{\|r^{h'}\|\to\infty} \|r^{h'}\|^{-1} \log M(r) < \infty \tag{2}$$

for some positive $h' < h$, where $\|r^{h'}\| = r_1^{h'} + \dots + r_n^{h'}$, then $f(z_1,\dots,z_n)$ is a polynomial.

When $h = 1$ we see that the constant $\log 2$ on the right of (1) cannot be replaced by any larger number, that is $2^{z_1+\dots+z_n}$ may be viewed as that integral integer-valued function, transcendental with respect to each of the n variables, which possesses the least rate of increase. (In the Pólya–Hardy result, the inequality corresponding to (1) possesses a different form; our remark is in accord, for example, with the comments on page 175 of (1) but requires further justification if regarded in the light of the original papers.) When $h > 1$ there is no doubt that (1) could be slightly relaxed in the manner of Selberg (14), but the proof of an improvement of this kind would unduly complicate the present exposition. The inequality $h' < h$ in (ii) is best possible, as is shown by the example

$$f(z_1,\dots,z_n) = \exp\left(\sum_{k=1}^{n} \{z_k(z_k-1)\dots(z_k-h+1)\}\right).$$

However, the validity of (ii) with $h' = h$ would be retained if an explicit constant were introduced on the right of (2). A sharpening of this kind follows easily from the analysis and was in fact given in the case $n = 1$ by Straus.

The method of proof of both (i) and (ii) depends on the employment of a generalized Newton interpolation formula. It may be regarded as the source of the multiple integrals which occur in Schneider's paper (12) and also in the related work of Lang (8).

2. Let η_{kl} $(1 \leqslant k \leqslant n,\ l \geqslant 0)$ be any complex numbers. Let $J_k = J_k(j_1,\dots,j_{k-1})$ be non-negative integers defined respectively, when $k > 1$, for all sets of integers $j_1,\dots,j_{k-1}$ satisfying $0 \leqslant j_l \leqslant J_l$ $(1 \leqslant l \leqslant k-1)$. Let $C_1,\dots,C_n$ denote circles in the complex plane with centres the origin, described in the positive sense. For each set of integers $j_1,\dots,j_n$ with $0 \leqslant j_l \leqslant J_l$ $(1 \leqslant l \leqslant n)$ we define $q_k = q_k(j_1,\dots,j_n)$ $(1 \leqslant k \leqslant n)$ as 1 if $j_k = J_k(j_1,\dots,j_{k-1})$ and as 0 otherwise. We write, for brevity

$$p_{j_k}(z) = \prod_{l=0}^{j_k} (z-\eta_{kl}) \quad (1 \leqslant k \leqslant n).$$

The identity

$$\frac{1}{(\zeta_1-z_1)\dots(\zeta_n-z_n)} = \sum_{j_1=0}^{J_1} \dots \sum_{j_n=0}^{J_n} \prod_{k=1}^{n} \left\{ \frac{1}{(z_k-\eta_{k,j_k})} \left(\frac{p_{j_k}(z_k)}{p_{j_k}(\zeta_k)}\right) \left(\frac{\zeta_k-\eta_{k,j_k}}{\zeta_k-z_k}\right)^{q_k} \right\},$$

valid for all $z_1,\dots,z_n$, $\zeta_1,\dots,\zeta_n$ for which the denominators do not vanish, is easily verified for $n = 1$ by induction on J_1, and in general by induction on n. If $z_1,\dots,z_n$ and those $\eta_{1l},\dots,\eta_{nl}$ which occur in the denominators are interior to $C_1,\dots,C_n$ respectively then, on multiplying the identity by $f(\zeta_1,\dots,\zeta_n)$ and integrating, we obtain

$$f(z_1,\dots,z_n) = \sum_{j_1=0}^{J_1} \dots \sum_{j_n=0}^{J_n} a_{j_1,\dots,j_n}(z_1,\dots,z_n) \prod_{k=1}^{n} \{(z_k-\eta_{k,j_k})^{-1} p_{j_k}(z_k)\}, \tag{3}$$

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where

$$a_{j_1,\dots,j_n}(z_1,\dots,z_n) = \frac{1}{(2\pi i)^n}\int_{C_1}\cdots\int_{C_n}\frac{f(\zeta_1,\dots,\zeta_n)}{p_{j_1}(\zeta_1)\dots p_{j_n}(\zeta_n)}\prod_{k=1}^{n}\left(\frac{\zeta_k-\eta_{k,j_k}}{\zeta_k-z_k}\right)^{q_k}d\zeta_1\dots d\zeta_n. \quad (4)$$

Note that $a_{j_1,\dots,j_n}(z_1,\dots,z_n)$ is independent of $z_1,\dots,z_n$ if $j_k < J_k$ for all k.

3. Let $h_1,\dots,h_n$ denote positive integers. Corresponding to any set of non-negative integers $j_1,\dots,j_n$ we shall write for brevity

$$u_k = [j_k/h_k], \quad v_k = j_k - h_k u_k \quad (1 \leqslant k \leqslant n).$$

We proceed to evaluate $a_{j_1,\dots,j_n} = a_{j_1,\dots,j_n}(z_1,\dots,z_n)$ for any $j_1,\dots,j_n$ satisfying $q_k(j_1,\dots,j_n) = 0$ for all k, first on the supposition that $\eta_{kl} = \eta_{kl'}$ if and only if $[l/h_k] = [l'/h_k]$ and secondly on the supposition that $\eta_{kl} = \eta_{kl'}$ if and only if $l \equiv l' \pmod{h_k}$.

In the first case we have

$$p_{j_k}(z) = \prod_{l_k=0}^{u_k}(z-\eta_{k,l_k h_k})^{w_k(l_k)} \quad (1 \leqslant k \leqslant n),$$

where $w_k(l_k) = h_k$ if $0 \leqslant l_k < u_k$ and $w_k(u_k) = v_k + 1$. If $F(z)$ is any integral function of z and $F^{(m)}(z)$ denotes the mth derivative of F, we obtain from Cauchy's residue theorem

$$\frac{1}{2\pi i}\int_{C_k}\frac{F(z)}{p_{j_k}(z)}\,dz = \sum_{l_k=0}^{u_k}\frac{1}{s_k!}\left[\frac{d^{s_k}}{dz^{s_k}}\frac{F(z)\,(z-\eta_{k,l_k h_k})^{s_k+1}}{p_{j_k}(z)}\right]_{z=\eta_{k,l_k h_k}},$$

where $s_k = s_k(l_k) = w_k(l_k) - 1$. The number on the right is given by

$$\sum_{l_k=0}^{u_k}\sum_{m_k=0}^{s_k}\frac{1}{s_k!}\binom{s_k}{m_k}F^{(m_k)}(\eta_{k,l_k h_k})\left[\frac{d^{s_k-m_k}}{dz^{s_k-m_k}}\frac{(z-\eta_{k,l_k h_k})^{s_k+1}}{p_{j_k}(z)}\right]_{z=\eta_{k,l_k h_k}},$$

and thus we see that

$$\int_{C_k}\frac{F(z)}{p_{j_k}(z)}\,dz = \sum_{l_k=0}^{u_k}\sum_{m_k=0}^{s_k}\frac{1}{m_k!}F^{(m_k)}(\eta_{k,l_k h_k})\int_{\Gamma_{k,l_k}}\frac{(z-\eta_{k,l_k h_k})^{m_k}}{p_{j_k}(z)}\,dz,$$

where Γ_{k,l_k} denotes a circle in the complex plane, described in the positive sense, with centre $\eta_{k,l_k h_k}$ and radius at most $\frac{1}{2}\min|\eta_{kl}-\eta_{kl'}|$, where the minimum is taken over all l, l' not exceeding j_k for which $[l/h_k] \neq [l'/h_k]$. Now taking $F(z)$ to be a function of the type

$$\frac{1}{(2\pi i)^{n-k}}\int_{C_{k+1}}\cdots\int_{C_n}\frac{f(\zeta_1,\dots,\zeta_{k-1},z,\zeta_{k+1},\dots,\zeta_n)}{p_{j_{k+1}}(\zeta_{k+1})\dots p_{j_n}(\zeta_n)}\,d\zeta_{k+1}\dots d\zeta_n,$$

and proceeding inductively for $k = n, n-1,\dots,1$, it clearly follows that

$$a_{j_1,\dots,j_n} = \sum_{l_1=0}^{u_1}\cdots\sum_{l_n=0}^{u_n}\sum_{m_1=0}^{s_1}\cdots\sum_{m_n=0}^{s_n}\frac{1}{m_1!\dots m_n!}b(j,l,m)f_{m_1,\dots,m_n}(\eta_{1,l_1 h_1},\dots,\eta_{n,l_n h_n}), \quad (5)$$

where

$$b(j,l,m) = \frac{1}{(2\pi i)^n}\int_{\Gamma_1,l_1}\cdots\int_{\Gamma_n,l_n}\prod_{k=1}^{n}\left\{\frac{(\zeta_k-\eta_{k,l_k h_k})^{m_k}}{p_{j_k}(\zeta_k)}\right\}d\zeta_1\dots d\zeta_n. \quad (6)$$

In the second case, when $\eta_{kl} = \eta_{kl'}$ if and only if $l \equiv l' \pmod{h_k}$, we have

$$p_{j_k}(z) = \prod_{l_k=0}^{h_k-1}(z-\eta_{k,l_k})^{w_k(l_k)} \quad (1 \leqslant k \leqslant n),$$

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where $w_k(l_k) = u_k + 1$ if $0 \leqslant l_k \leqslant v_k$ and $w_k(l_k) = u_k$ if $v_k < l_k < h_k$. Proceeding as above we obtain the same formulae (5) and (6) but with $\eta_{k,l_k h_k}$ $(1 \leqslant k \leqslant n)$ replaced by η_{k,l_k}, u_k replaced by $h_k - 1$ and with the obvious modifications in the definition of Γ_{k,l_k}.

4. We suppose now that h is a positive integer and that $h_k = h$ for all k. Our object in this section is to prove that if in the first case specified above we take $\eta_{k,l_k h_k} = l_k$ $(0 \leqslant l_k \leqslant u_k)$ and if also we assume the hypotheses of part (i) of the theorem to hold then $a_{j_1,\ldots,j_n} = 0$ for all $j_1, \ldots, j_n$ with $j_1 + \ldots + j_n$ sufficiently large. Further we shall show that a similar result is valid if in the second case we take $\eta_{k,l_k} = l_k$ and assume the hypotheses of part (ii) of the theorem to hold.

We note first that $b(j,l,m)$, defined by (6), can be written as a product of factors

$$\frac{1}{2\pi i}\int_{\Gamma_k,l_k} \frac{(\zeta_k - l_k)^{m_k}}{p_{j_k}(\zeta_k)}\,d\zeta_k = \sum_{t(0)} \ldots \sum_{t(u_k)} \prod_{\substack{j=0 \\ j \neq l_k}}^{u_k} \binom{s_k(j)+t(j)}{t(j)} \frac{(-1)^{t(j)}}{(l_k - j)^{s_k(j)+t(j)+1}},$$

where the summation is over all sets of non-negative integers $t(0), \ldots, t(u_k)$ with $t(l_k) = 0$ and $t(0) + \ldots + t(u_k) = s_k(l_k) - m_k$. In the first case we have $s_k(j) + 1 = w_k(j) \leqslant h$ and thus the product of the factors $(l_k - j)^{s_k(j)+1}$ taken over all j with $0 \leqslant j \leqslant u_k$ $(j \neq l_k)$ divides $(u_k!)^h$. Further it is clear that the product of the factors $(l_k - j)^{t(j)}$, taken over the same j, divides $(\omega(u_k))^{h-1}$, where $\omega(u_k)$ denotes the least common multiple of $1, 2, \ldots, u_k$. Thus in the first case we see that $Xb(j,l,m)$ is an integer, where

$$X = (\omega(u_1) \ldots \omega(u_n))^{h-1} (u_1! \ldots u_n!)^h, \tag{7}$$

and it is now apparent from (5) and the hypotheses of (i) that $(h!)^n X a_{j_1,\ldots,j_n}$ is an integer. Similarly in the second case we have $s_k(j) + 1 = w_k(j) \leqslant u_k + 1$ and the product of the factors $(l_k - j)^{s_k(j)+t(j)+1}$, taken over all j with $0 \leqslant j < h$ $(j \neq l_k)$, divides $(h!)^{2(u_k+1)}$. Thus we see that $Yb(j,l,m)$ is an integer, where

$$Y = (h!)^{2(u_1+\ldots+u_n)+2n}. \tag{8}$$

Since $m_k!$ divides $u_k!$, it follows from the modified form of (5) that $u_1! \ldots u_n!\, Y a_{j_1,\ldots,j_n}$ is an integer.

We now observe the integral representation (4). In the first case we take the radius r_k of C_k $(1 \leqslant k \leqslant n)$ as $Eu_k + 1$, where $E = 1 + e^{(h-1)/h}$; in the second case we take the radius as $j_k^{1/h'} + h$. Then $|p_{j_k}(\zeta_k)|$ with ζ_k on C_k exceeds

$$U_k = \{r_k(r_k - 1) \ldots (r_k - u_k + 1)\}^h$$

in the first case, and

$$V_k = \{r_k(r_k - 1) \ldots (r_k - h + 1)\}^{u_k}$$

in the second case. Further, by (1) or (2) we see that $|f(\zeta_1 \ldots \zeta_n)|$ with ζ_k on C_k does not exceed $\{E/(E-1)\}^{(h-\delta)\|r\|}$ or $c\,e^{\|r^{h'}\|}$ respectively, where $\delta > 0$, $c > 0$ do not depend on r. Thus in the first case we have

$$|a_{j_1,\ldots,j_n}| \leqslant \prod_{k=1}^{n} \{r_k U_k^{-1} (E/(E-1))^{(h-\delta)r_k}\}, \tag{9}$$

and in the second case we have

$$|a_{j_1,\ldots,j_n}| \leqslant \prod_{k=1}^{n} \{c r_k V_k^{-1} \exp(r_k^{h'})\}. \tag{10}$$

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Now proceeding as in (5), noting that, by Stirling's formula

$$u_k!\, U_k^{-1/h} < c' u_k^{\frac{1}{2}} (E-1)^{-u_k} (1-1/E)^{Eu_k},$$

where $c' > 0$ does not depend on j_k, and using also the well-known estimate

$$\omega(u_k) = e^{u_k + o(u_k)},$$

we obtain from (7) and (9)

$$(h!)^n X a_{j_1, \ldots, j_n} \to 0 \quad \text{as} \quad j_1 + \ldots + j_n \to \infty.$$

This implies that $a_{j_1, \ldots, j_n} = 0$ for all $j_1, \ldots, j_n$ with $j_1 + \ldots + j_n$ sufficiently large, as asserted. In the second case we note that $V_k > j_k^{(j_k - h)/h'}$, $u_k! < j_k^{j_k/h}$ and $r_k^{h'} = (j_k^{1/h'} + h)^{h'}$. Further, by hypothesis, we have $h' < h$. It now follows easily from (8) and (10) that

$$u_1! \ldots u_n!\, Y a_{j_1, \ldots, j_n} \to 0 \quad \text{as} \quad j_1 + \ldots + j_n \to \infty$$

and hence again $a_{j_1, \ldots, j_n} = 0$ if $j_1 + \ldots + j_n$ is sufficiently large.

5. Let N be any positive integer and let $J_1, \ldots, J_n$ be defined by the equations

$$J_1 = N, \quad j_1 + \ldots + j_{k-1} + J_k(j_1, \ldots, j_{k-1}) = N \quad (2 \leqslant k \leqslant n).$$

Further, let $R_N(z_1, \ldots, z_n)$ denote the sum of the products

$$a_{j_1, \ldots, j_n}(z_1, \ldots, z_n) \prod_{k=1}^{n} \{(z_k - \eta_{k, j_k})^{-1} p_{j_k}(z_k)\}$$

taken over all $j_1, \ldots, j_n$ for which $q_k(j_1, \ldots, j_n) = 1$ for some k, that is for which $j_1 + \ldots + j_n = N$. The number of these $j_1, \ldots, j_n$ is plainly at most N^n. Proceeding as in section 4 we see that, for given $z_1, \ldots, z_n$ and given $K > 0$,

$$X N^K a_{j_1, \ldots, j_n}(z_1, \ldots, z_n) \to 0 \quad \text{as} \quad j_1 + \ldots + j_n = N \to \infty$$

in the first case, and the same holds in the second case with X replaced by $u_1! \ldots u_n!$. Now in the first case we have

$$|(z_k - \eta_{k, j_k})^{-1} p_{j_k}(z_k)| \leqslant \prod_{l=0}^{u_k} (|z_k| + l)^h \leqslant (u_k!)^h (|z_k| + u_k)^{h(|z_k|+1)},$$

and in the second case it is clear that

$$|(z_k - \eta_{k, j_k})^{-1} p_{j_k}(z_k)| \leqslant (|z_k| + h)^{h(u_k+1)}.$$

Thus, denoting the maximum of the $|z_k|$ by Z, we deduce that in the first case $R_N(z_1, \ldots, z_n)$ has absolute value at most

$$(u_1! \ldots u_n!)^h N^n A_N(z_1, \ldots, z_n) (Z + N)^{nh(Z+1)},$$

where $A_N(z_1, \ldots, z_n)$ denotes the maximum of the $|a_{j_1, \ldots, j_n}(z_1, \ldots, z_n)|$ taken over all $j_1, \ldots, j_n$ with $j_1 + \ldots + j_n = N$, and in the second case we deduce that $R_N(z_1, \ldots, z_n)$ has absolute value at most

$$N^n A_N(z_1, \ldots, z_n) (Z + h)^{h(u_1 + \ldots + u_n + n)}.$$

Hence, in either case, we have $R_N(z_1, \ldots, z_n) \to 0$ as $N \to \infty$. It follows from (3) and the results of the preceding section that $f(z_1, \ldots, z_n)$ is a polynomial, and this completes the proof of the theorem.

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