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101.02 A (doubly) elementary formula for prime numbers

In [1, p. 5], the problem of giving a formula for the nth prime number was raised. Using a characterisation of prime numbers by Wilson's theorem, Willans gave such a formula [2]. In [3], new formulas were given. They were obtained using the definition of prime numbers, say that they possess only trivial factorisations. In [4], another such formula was obtained using a clever encoding of Eratosthenes's sieve. In [5], Regimbal gave an explicit formula for the nth prime number using only a primality definition. Recently Ruiz and Sondow [6] also exhibited such constructions.

The formulas in [2, 4, 6] make use of transcendental functions such as log or cos. The other works make use only of the four elementary operations $+, -, \times, /$, the floor function $\lfloor \cdot \rfloor$ and powers. With the exception of [4], formulas require an upper bound for the nth prime number. In most cases, this bound is obtained with the help of Bertrand's postulate. Alternatively, in [6], theorems of Rosser and Schoenfeld are used for this purpose. These results, although classical, are not accessible to audiences below first degree level. In this paper, we present two new formulas involving elementary operations. Moreover, only elementary results of number theory are required for the proofs so that the title of this note is justified.

Let $\pi(x)$ denote the prime counting function and let p_n denote the nth prime number. We will prove the two following formulas.

$$\pi(x) = \sum_{i=2}^{\lfloor x \rfloor} \left\lfloor \frac{1}{\sum_{j=1}^{i} \left\lfloor \frac{1}{i+1-j\left\lfloor \frac{i}{j} \right\rfloor} \right\rfloor - 1} \right\rfloor \tag{1}$$

$$p_n = \sum_{k=2}^{2^{2n}} k \left\lfloor \frac{1}{1 + (\pi(k) + \pi(k-1) - 2n + 1)^2} \right\rfloor \tag{2}$$

Let $r, s \in \mathbb{R}$. If we suppose that $r \geqslant 0$, we have

$$r = 0 \Rightarrow \left\lfloor \frac{1}{1+r} \right\rfloor = 1 \text{ and } r > 0 \Rightarrow \left\lfloor \frac{1}{1+r} \right\rfloor = 0. \tag{3}$$

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We suppose now that $r \geqslant s$. Using the previous equations we have

$$r = s \Rightarrow \left\lfloor \frac{1}{1 + r - s} \right\rfloor = 1 \text{ and } r > s \Rightarrow \left\lfloor \frac{1}{1 + r - s} \right\rfloor = 0. \tag{4}$$

In the general case, without hypothesis on r and s, we will also need the following result.

$$r = s \Rightarrow \left\lfloor \frac{1}{1 + (r - s)^2} \right\rfloor = 1 \text{ and } r \neq s \Rightarrow \left\lfloor \frac{1}{1 + (r - s)^2} \right\rfloor = 0. \tag{5}$$

Let $i, j \in \mathbb{N}$ with $j > 0$. Then, if $\operatorname{rem}(i, j)$ denotes the remainder of the division of i by j, we have

$$\operatorname{rem}(i, j) = i - j \left\lfloor \frac{i}{j} \right\rfloor.$$

Using (3), we have then

$$i \mid j \Rightarrow \left\lfloor \frac{1}{1 + \operatorname{rem}(i, j)} \right\rfloor = 1 \text{ and } i \nmid j \Rightarrow \left\lfloor \frac{1}{1 + \operatorname{rem}(i, j)} \right\rfloor = 0.$$

Let $n \in \mathbb{N}$ with $n \geqslant 2$. We define $d(n)$ as the number of divisors function for n:

$$d(n) = \sum_{d \mid n} 1.$$

From what precedes, we have

$$d(n) = \sum_{i=1}^{n} \left\lfloor \frac{1}{1 + \operatorname{rem}(n, i)} \right\rfloor.$$

Given $d(n)$, it is possible to characterise the primality of n, for $n \geqslant 2$:

$$n \text{ is prime} \Rightarrow d(n) = 2 \text{ and } n \text{ is not prime} \Rightarrow d(n) > 2. \tag{7}$$

Let us define $P(n)$ as the characteristic function of primality, so that $P(n) = 1$ if n is prime and $P(n) = 0$ if n is composite. Using(4) and (7), we obtain, for $n \geqslant 2$

$$P(n) = \left\lfloor \frac{1}{d(n) - 1} \right\rfloor.$$

Let x be a real positive number. We have then

$$\pi(x) = \sum_{i=2}^{\lfloor x \rfloor} P(i).$$

Putting all the preceding results together, we establish (1).

At this point we need an upper bound for prime numbers. Let $l \geqslant 1$ be a natural number. In [1, p. 12], it is shown that Euclid's proof on the infinitude of prime numbers can be used to show that

$$p_l \leqslant 2^{2^l}. \tag{8}$$

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This property is shown by recurrence using Euclid's argument that $p_l \leqslant p_1 p_2 \ldots p_{l-1} + 1$ for $l \geqslant 2$. This result can eventually be improved to $p_l \leqslant 2^{2^{l-1}} + 1$ by using Polya's proof [1, p. 14]. It is shown with elementary arguments that Fermat numbers are mutually relatively coprime, so that the preceding result is deduced.

For $n \geqslant 1$, p_n is the least number $k \geqslant 2$ such that $\pi(k) = n$. We have then $\pi(p_n) = n$ and $\pi(p_n - 1) = n - 1$, so that $\pi(p_n) + \pi(p_{n-1}) = 2n - 1$. Let $k \geqslant 2$ such that $\pi(k) + \pi(k-1) = 2n - 1$. The function π is increasing, thus we have $\pi(k-1) \leqslant n - \frac{1}{2} \leqslant \pi(k)$. Since $\pi(k) \leqslant \pi(k-1) + 1$, we necessarily have $\pi(k-1) = n - 1$ and then $\pi(k) = n$. The number k is thus the least one with $\pi(k) = n$. As a consequence $k = p_n$. From (5) and (8), and using the latter characterisation of p_n, (2) is established.

Finally, using results from [7, p. 26], we can also eventually reduce drastically the summation domain. For $n \geqslant 6$, we obtain then

$$p_n = \sum_{k=n(\log n + \log\log n - 1)}^{n(\log n + \log\log n)} k \left\lfloor \frac{1}{1 + (\pi(k) + \pi(k-1) - 2n + 1)^2} \right\rfloor.$$

The summation range is then linear and compares favorably to that of Ruiz and Sondow formulas which is of size $O(n \log n)$.

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