Probability in the Engineering and Informational Sciences, **15**, 2001, 273–293. Printed in the U.S.A.

ON PARTIAL ORDERINGS BETWEEN COHERENT SYSTEMS WITH DIFFERENT STRUCTURES

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In this article, we give some results on the preservation of orderings between the components under the formation of coherent systems with different structures. We consider the stochastic, failure rate, reversed failure rate, and likelihood ratio orderings to compare two coherent structures formed from a set of components and from two sets of components, in both cases, when components are independent and either identically distributed or not necessarily identically distributed. As a consequence, we get new stochastic comparisons of systems with different *k*-out-of-*n* structures, and some results of partial orderings given by other authors may be also obtained from ours.

1. INTRODUCTION

There are several ways in which one can assert that a random variable (r.v.) X is "greater" or "smaller" than another r.v. Y; for instance, stochastic, failure rate, reversed failure rate, and likelihood ratio orderings, which play an important role in reliability theory.

In the literature, most of authors have studied stochastic comparison between two coherent systems with the same structure and, in particular, *k*-out-of-*n* systems.

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https://doi.org/10.1017/S0269964801152095 Published online by Cambridge University Press

A k-out-of-n system is a system with n components which functions if and only if at least k of the components function (i.e., it works as long as at most $n - k$ components have failed). In particular, the parallel and series systems are 1-out-of-n and n-out-of-n systems, respectively.

If X_i is the lifetime to the ith component, $i = 1,2,\ldots,n$, the survival function of the k-out-of-n system is the same as that of the $(n - k + 1)$st order statistic $X_{n-k+1,n}$ from this set of n random variables (i.e., $\tau_{k,n} = X_{n-k+1,n}$), so the study of order statistics is also of interest for comparing the aging of these systems.

In this context, most of the results are devoted to the preservation of some aging ordering between independent and identically distributed (i.i.d.) components, under the formation of systems with the same k-out-of-n structure. For instance, convex transform, star, and superadditive orderings are studied in Barlow and Proschan [3], dispersive ordering in Bartoszewicz [4], Parzen ordering in Alzaid and Al-Osh [1], and likelihood ratio, failure rate, and stochastic orderings in Singh and Vijayasree [20].

Furthermore, Chan et al. [9] obtained a preservation of the likelihood ratio ordering between independent and not necessarily identically distributed components under the formation of k-out-of-n systems.

Also, as can be seen in Nanda et al. [16], the likelihood ratio ordering is preserved under the formation of certain coherent systems with i.i.d. components, and Lynch et al. [15] and Nanda et al. [16] give sufficient conditions to compare, in the failure rate and reversed failure rate orderings, systems formed from two sets of independent components and the same coherent structure.

Recently, the study of partial orderings between coherent systems with different structures has been of increasing interest, and, in particular, between k-out-of-n and l-out-of-m systems, which allows us to design more reliable systems.

For instance, Raqab and Amin [17] compare in the failure rate and likelihood ratio orderings, k-out-of-n and l-out-of-m systems formed from a set of i.i.d. components, and Khaledi and Kochar [13,14] show the preservation of the stochastic and dispersive orderings between k-out-of-n and l-out-of-m systems formed from a set of i.i.d. components.

Likewise, some authors have studied stochastic comparisons between the lifetimes of some kinds of k-out-of-n systems formed from a set of n independent components and not necessarily identically distributed. In this setting, Boland et al. [8] proved that given two systems of k-out-of-n type formed by the same number of components, it is greater in the stochastic, failure rate, and reversed failure rate orderings which support more failures. The last two results were proved by Boland et al. [7] and Block et al. [5], respectively, when components are absolutely continuous. Concerning the failure rate order, Boland et al. [7] proved that if one of the lifetimes which form a k-out-of-n system is greater (in this order) than the remaining, then the lifetime of the k-out-of-$(n-1)$ system that results from eliminating this component is smaller in the failure rate order than the lifetime of the initial k-out-of-n system. They also gave a similar result but assuming that one of the lifetimes is smaller than the remaining in the failure rate order. Block et al. [5] proved that if one

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of the lifetimes that form a k-out-of-n system is smaller than the remaining in the reversed failure rate order, then the lifetime of the $(k-1)$-out-of-$(n-1)$ system that results from eliminating this component is greater (in this order) than the lifetime of the initial k-out-of-n system. Hu and He [11] showed that a k-out-of-n system is smaller than a $(k-1)$-out-of-$(n-1)$ system in the failure rate order and greater than a k-out-of-$(n-1)$ system in the reversed failure rate order.

The main objective of this article is to study when one ordering between the components is preserved under the formation of coherent systems with different structures. In this way, we discuss the following two cases:

(i) Two coherent systems formed from a set of independent components with lifetimes $\mathbf{X} = (X_1, \ldots, X_n)$
(ii) Two coherent systems formed from two sets of independent components with lifetimes $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_m)$

when the components of each set are either identically distributed or not necessarily identically distributed.

In Section 2, we give the definitions and technical results which will be used in the following sections. In Sections 3–6, we show the comparison results in the stochastic, failure rate, reversed failure rate, and likelihood ratio orderings, respectively. These results allow us to establish new comparisons for k-out-of-n systems, and some former preservations of other authors mentioned for coherent and k-out-of-n systems, may be obtained from ours.

2. PREVIOUS RESULTS

Let us recall the definitions of some orderings which can be found in the above references and that will be used in the remainder of the article.

Let X and Y be the lifetime of two components (i.e., nonnegative absolutely continuous r.v.'s), with lifetime distributions F and G, respectively; f and g are their probability density functions, respectively. It is said that X is smaller than Y in the stochastic ordering, $X \leq_{\text{st}} Y$, if

$$\bar{F}(x) \leq \bar{G}(x) \quad \text{for all } x \geq 0, \tag{2.1}$$

where $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$ are the reliability functions of X and Y, respectively.

It is said that X is smaller than Y in the failure rate ordering, $X \leq_{\text{fr}} Y$, if

$$r(x) \geq q(x) \quad \text{for all } x \geq 0, \tag{2.2}$$

where $r(x) = f(x)/\bar{F}(x)$ and $q(x) = g(x)/\bar{G}(x)$ are the failure rate functions of X and Y, respectively, and it is said that X is smaller than Y in the reversed failure rate ordering, $X \leq_{\text{rf}} Y$, if

$$\tilde{r}(x) \leq \tilde{q}(x) \quad \text{for all } x \geq 0, \tag{2.3}$$

where $\tilde{r}(x) = f(x)/F(x)$ and $\tilde{q}(x) = g(x)/G(x)$ are the reversed failure rate functions of X and Y, respectively.

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Likewise, it is said that X is smaller than Y in the likelihood ratio ordering, $X \leq_{\mathrm{lr}} Y$, if

$$\frac{f(x)}{g(x)} \text{ is a nonincreasing function of } x, \quad \text{for all } x \geq 0. \tag{2.4}$$

The relationships among these orderings are shown in the following diagram (see, e.g., Shaked and Shanthikumar [19]):

$$\begin{array}{ccc} X \leq_{\mathrm{lr}} Y & \Rightarrow & X \leq_{\mathrm{fr}} Y \\ \Downarrow & & \Downarrow \\ X \leq_{\mathrm{rf}} Y & \Rightarrow & X \leq_{\mathrm{st}} Y, \end{array}$$

so all of these orderings allow one to determine that components with lifetime Y are more reliable than components with lifetime X.

Throughout this article, we will consider $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_m)$ the lifetimes of two sets of independent components, and F_i, G_i, f_i, g_i, r_i, q_i, $\tilde{r}_i$, and $\tilde{q}_i$ will denote the distribution, density, failure rate, and reversed failure rate functions of X_i and Y_i, respectively. Likewise, we will use $h_1(\mathbf{X})$, $h_2(\mathbf{Y})$, $\tau_{k,n}$, and $\tau'_{k,n}$ for denoting the lifetime of the coherent and k-out-of-n systems formed from $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_m)$, respectively. In addition, we will write $\mathbf{a}_{|s|}$ to represent the s-dimensional vector $(a_1, a_2, \ldots, a_s)$.

First, we give a lemma which will be useful in the remainder of this article.

LEMMA 2.1: *Let $I_i, i = 1,2,\ldots,n$, be independent indicator random variables (i.e., binary r.v.'s which only can take the values* 0 *or* 1*). Then,*

(i) $\dfrac{P\left(\sum_{i=1}^{n} I_i = k\right)}{P\left(\sum_{i=1}^{n} I_i = k-1\right)}$ *is decreasing in k,*

(ii) $\dfrac{P\left(\sum_{i=1}^{n} I_i = k-1\right)}{P\left(\sum_{i=1}^{n} I_i \geq k\right)}$ *and* $\dfrac{P\left(\sum_{j\neq i}^{n} I_j = k-1\right)}{P\left(\sum_{j=1}^{n} I_j \geq k\right)}$ *are increasing in k,*

(iii) $\dfrac{P\left(\sum_{i=1}^{n} I_i = k-1\right)}{P\left(\sum_{i=1}^{n} I_i < k\right)}$ *and* $\dfrac{P\left(\sum_{j\neq i}^{n} I_j = k-1\right)}{P\left(\sum_{j=1}^{n} I_j < k\right)}$ *are decreasing in k.*

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PROOF: Part (i) is Eq. (6) of Samuels [18] and is also a consequence of a result of Karlin [12].

On the other hand, note that first affirmation of part (ii) is equivalent to

$$\frac{P\left(\sum_{i=1}^{n} I_i \geq k\right)}{P\left(\sum_{i=1}^{n} I_i = k-1\right)} \geq \frac{P\left(\sum_{i=1}^{n} I_i \geq k+1\right)}{P\left(\sum_{i=1}^{n} I_i = k\right)}$$

so it is sufficient to prove that

$$\frac{P\left(\sum_{i=1}^{n} I_i = k+j\right)}{P\left(\sum_{i=1}^{n} I_i = k-1\right)} \geq \frac{P\left(\sum_{i=1}^{n} I_i = k+j+1\right)}{P\left(\sum_{i=1}^{n} I_i = k\right)} \quad \text{for all } j = 0,1,\ldots,n-k-1,$$

which follows from part (i) of the lemma. The second affirmation of part (ii) can be obtained from the first one pivoting on the ith component.

Analogously, we obtain part (iii) by taking into account that

$$\frac{P\left(\sum_{i=1}^{n} I_i = j\right)}{P\left(\sum_{i=1}^{n} I_i = k\right)} \geq \frac{P\left(\sum_{i=1}^{n} I_i = j-1\right)}{P\left(\sum_{i=1}^{n} I_i = k-1\right)} \quad \text{for all } j = 1,2,\ldots,k,$$

which holds due to part (i) of the lemma. ■

3. STOCHASTIC ORDERING

The following theorem shows the preservation of the stochastic order under the formation of coherent systems with different structures.

THEOREM 3.1: *Let* $\mathbf{X} = (X_1,\ldots,X_n)$ *and* $\mathbf{Y} = (Y_1,\ldots,Y_m)$ *be the lifetimes of two sets of independent components and* $h_1(\mathbf{p}_{|n|})$ *and* $h_2(\mathbf{p}_{|m|})$ *be the reliability functions of two coherent systems with n and m components, respectively, such that*

$$h_1(p_1,\ldots,p_n) \leq h_2(p_1,\ldots,p_m) \quad \textit{for all } (p_1,\ldots,p_{\max(n,m)}). \tag{3.1}$$

If

$$X_i \leq_{\mathrm{st}} Y_i \quad \textit{for all } i = 1,\ldots,\min(n,m), \tag{3.2}$$

then $h_1(\mathbf{X}) \leq_{\mathrm{st}} h_2(\mathbf{Y})$.

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PROOF: First, suppose that $n \leq m$. Using Eqs. (3.1) and (3.2) and the fact that the reliability function of a coherent system is increasing in each argument, we get

$$\bar{F}_{h_1(\mathbf{X})}(t) = h_1(\bar{F}_1(t),\ldots,\bar{F}_n(t)) \leq h_1(\bar{G}_1(t),\ldots,\bar{G}_n(t))$$
$$\leq h_2(\bar{G}_1(t),\ldots,\bar{G}_m(t)) = \bar{G}_{h_2(\mathbf{Y})}(t),$$

so $h_1(\mathbf{X}) \leq_{\mathrm{st}} h_2(\mathbf{Y})$.

Now suppose that $n > m$. Then, we have that

$$\bar{F}_{h_1(\mathbf{X})}(t) = h_1(\bar{F}_1(t),\ldots,\bar{F}_n(t)) \leq h_1(\bar{G}_1(t),\ldots,\bar{G}_m(t),\bar{F}_{m+1}(t),\ldots,\bar{F}_n(t))$$
$$\leq h_2(\bar{G}_1(t),\ldots,\bar{G}_m(t)) = \bar{G}_{h_2(\mathbf{Y})}(t). \quad \blacksquare$$

In the next corollary, we prove that a k-out-of-n system, which has more reliable components, supports more failures and needs less working components because the system performance is more reliable than other system, which has less reliable components, supports less failures, and needs more working components for the system performance.

COROLLARY 3.2: *Let $(X_1,\ldots,X_n)$ and $(Y_1,\ldots,Y_m)$ be the lifetimes of two sets of independent components such that $X_i \leq_{\mathrm{st}} Y_i$ for all $i = 1,\ldots,\min(n,m)$. Then, $\tau_{k,n} \leq_{\mathrm{st}} \tau'_{l,m}$ for all $l \leq k$, $n-k \leq m-l$.*

PROOF: From Theorem 3.1, it is sufficient to prove that

$$h_{k,n}(p_1,\ldots,p_n) \leq h_{l,m}(p_1,\ldots,p_m) \quad \text{for all } (p_1,\ldots,p_{\max(n,m)}) \tag{3.3}$$

where $h_{k,n}$ is the reliability function of a k-out-of-n system.

Let

$$X_i(t) = \begin{cases} 0 & \text{if } X_i < t, \\ 1 & \text{otherwise} \end{cases} \quad \text{for } i = 1,\ldots,n$$

be an independent indicator r.v. of the state of the ith component at any time t; then, $P(X_i(t)=1) = \bar{F}_i(t)$. From the definition of a k-out-of-n system, Eq. (3.3) is equivalent to

$$P\left(\sum_{i=1}^{n} X_i(t) \geq k\right) \leq P\left(\sum_{i=1}^{m} X_i(t) \geq l\right). \tag{3.4}$$

If $n \leq m$, we have that

$$P\left(\sum_{i=1}^{n} X_i(t) \geq k\right) \leq P\left(\sum_{i=1}^{m} X_i(t) \geq k\right) \leq P\left(\sum_{i=1}^{m} X_i(t) \geq l\right) \quad \text{for all } l \leq k.$$

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If $n > m$, we take $a = n - m > 0$ and the condition $n - k \le m - l$ may be rewritten as $l \le k - a$. On the other hand, it is easy to see that

$$P\left(\sum_{i=1}^{n} X_i(t) \ge k\right) \le P\left(\sum_{i=1}^{n-1} X_i(t) \ge k-1\right) \tag{3.5}$$

because it is the same as

$$\bar{F}_n(t)P\left(\sum_{i=1}^{n-1} X_i(t) \ge k-1\right) + F_n(t)P\left(\sum_{i=1}^{n-1} X_i(t) \ge k\right) \le P\left(\sum_{i=1}^{n-1} X_i(t) \ge k-1\right);$$

that is,

$$P\left(\sum_{i=1}^{n-1} X_i(t) \ge k\right) \le P\left(\sum_{i=1}^{n-1} X_i(t) \ge k-1\right).$$

Therefore, from Eq. (3.5) and $l \le k - a$, we obtain

$$P\left(\sum_{i=1}^{n} X_i(t) \ge k\right) \le P\left(\sum_{i=1}^{n-a} X_i(t) \ge k-a\right) \le P\left(\sum_{i=1}^{m} X_i(t) \ge l\right). \qquad \blacksquare$$

Remark 3.1: Khaledi and Kochar [13] showed the above corollary when components are independent and identically distributed. In particular, for $n = m$, $k = l$, and i.i.d. components, Theorem 3.3 of Singh and Vijayasree [20] is obtained.

4. FAILURE RATE ORDERING

First, we consider a set of independent components with lifetimes $(X_1, \ldots, X_n)$ and we establish comparisons in the failure rate ordering between two coherent systems with different structures formed from this set of components.

THEOREM 4.1: *Let $X_1, \ldots, X_n$ be the lifetimes of n independent components, and let $h_1(\mathbf{p}_{|n|})$ and $h_2(\mathbf{p}_{|m|})$ be the reliability functions of two coherent systems with $n \ge m$. If*

$$\frac{1}{h_1(\mathbf{p}_{|n|})}\frac{\partial h_1}{\partial p_i}(\mathbf{p}_{|n|}) \ge \frac{1}{h_2(\mathbf{p}_{|m|})}\frac{\partial h_2}{\partial p_i}(\mathbf{p}_{|m|}) \quad \textit{for all } i = 1, \ldots, m, \tag{4.1}$$

then $h_1(\mathbf{X}_{|n|}) = h_1(X_1, \ldots, X_n) \le_{\mathrm{fr}} h_2(X_1, \ldots, X_m) = h_2(\mathbf{X}_{|m|})$.

PROOF: The failure rate function $r_{h(\mathbf{X})}$ of a coherent system with n components and reliability function $h(p_1, \ldots, p_n)$ is given by

$$r_{h(\mathbf{X})}(t) = \sum_{i=1}^{n} r_i(t)\left[\frac{p_i}{h(\mathbf{p})}\frac{\partial h}{\partial p_i}(\mathbf{p})\right]_{\mathbf{p}=(\bar{F}_1(t), \ldots, \bar{F}_n(t))}, \tag{4.2}$$

where $\bar{F}_i$ and r_i are the survival and failure rate functions, respectively, of its components (see Boland et al. [7]).

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Note that each term in Eq. (4.2) is nonnegative, so from Eq. (4.1), there results

$$\begin{aligned} r_{h_1(\mathbf{X}_{|n|})}(t) &= \sum_{i=1}^{n} r_i(t) \left[\frac{p_i}{h_1(\mathbf{p}_{|n|})} \frac{\partial h_1}{\partial p_i}(\mathbf{p}_{|n|}) \right]_{\mathbf{p}_{|n|}=(\bar{F}_1(t),\ldots,\bar{F}_n(t))} \\ &\geq \sum_{i=1}^{m} r_i(t) \left[\frac{p_i}{h_1(\mathbf{p}_{|n|})} \frac{\partial h_1}{\partial p_i}(\mathbf{p}_{|n|}) \right]_{\mathbf{p}_{|n|}=(\bar{F}_1(t),\ldots,\bar{F}_n(t))} \\ &\geq \sum_{i=1}^{m} r_i(t) \left[\frac{p_i}{h_2(\mathbf{p}_{|m|})} \frac{\partial h_2}{\partial p_i}(\mathbf{p}_{|m|}) \right]_{\mathbf{p}_{|m|}=(\bar{F}_1(t),\ldots,\bar{F}_m(t))} \\ &= r_{h_2(\mathbf{X}_{|m|})}(t). \end{aligned}$$

■

In particular, we establish, in the next corollary, the comparison in this order between two systems with different structure k-out-of-n, given by Theorem 3.1 of [7], and we give an extension of their Theorem 3.4 [7] in Corollary 4.3, which coincides with Theorem 1(a) of [11].

COROLLARY 4.2: *Let $X_1, X_2, \ldots, X_n$ be the lifetimes of n independent components. Then, $\tau_{k,n} \leq_{\mathrm{fr}} \tau_{l,n}$ for all $l \leq k$.*

PROOF: Let us see that $\tau_{k+1,n} \leq_{\mathrm{fr}} \tau_{k,n}$. From Theorem 4.1, it is sufficient to prove that Eq. (4.1) holds with $h_1 = h_{k+1,n}$ and $h_2 = h_{k,n}$. Using the notation of the proof of Corollary 3.2 and taking into account that

$$\left[\frac{\partial h_{k,n}}{\partial p_i}(\mathbf{p}) \right]_{\mathbf{p}=(\bar{F}_1(t),\ldots,\bar{F}_n(t))} = P\left(\sum_{j\neq i}^{n} X_j(t) = k-1 \right),$$

Eq. (4.1) is rewritten as

$$\frac{P\left(\sum_{j\neq i}^{n} X_j(t) = k \right)}{P\left(\sum_{j=1}^{n} X_j(t) \geq k+1 \right)} \geq \frac{P\left(\sum_{j\neq i}^{n} X_j(t) = k-1 \right)}{P\left(\sum_{j=1}^{n} X_j(t) \geq k \right)}$$

which was proved in part (ii) of Lemma 2.1. ■

COROLLARY 4.3: *Let $X_1, X_2, \ldots, X_n$ be the lifetimes of n independent components. Then, $\tau_{k,n} \leq_{\mathrm{fr}} \tau_{k-1,n-1}$.*

PROOF: From Theorem 4.1, it is sufficient to see that Eq. (4.1) holds with $h_1 = h_{k,n}$ and $h_2 = h_{k-1,n-1}$; that is,

$$\frac{P\left(\sum_{j\neq i}^{n} X_j(t) = k-1 \right)}{P\left(\sum_{j=1}^{n} X_j(t) \geq k \right)} \geq \frac{P\left(\sum_{j\neq i}^{n-1} X_j(t) = k-2 \right)}{P\left(\sum_{j=1}^{n-1} X_j(t) \geq k-1 \right)}.$$

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Finally, the above expression can be obtained pivoting on the nth component and using part (ii) of Lemma 2.1. ■

In general, the next result shows the comparison in the failure rate order between systems with different k-out-of-n structures. Note that Raqab and Amin [17] obtained the next corollary for i.i.d. components.

COROLLARY 4.4: *Let $X_1, X_2, \ldots, X_n$ be the lifetimes of n independent components and $m \leq n$. Then, $\tau_{k,n} \leq_{\text{fr}} \tau_{l,m}$ for all $l \leq k$ such that $n - k \leq m - l$.*

PROOF: Taking $a = m - n \geq 0$, condition $n - k \leq m - l$ may be rewritten as $l \leq k - a$. Then, from Corollaries 4.2 and 4.3, we get

$$\tau_{k,n} \leq_{\text{fr}} \tau_{k-a,n-a} = \tau_{k-a,m} \leq_{\text{fr}} \tau_{l,m}$$

for all $l \leq k$ and $n - k \leq m - l$. ■

In the case of two sets of independent components with lifetimes $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_m)$, respectively, we give sufficient conditions under which the ordering between the components with respect to the failure rate is preserved under the formation of two coherent systems with different structures.

THEOREM 4.5: *Let $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_m)$ be the lifetimes of two sets of independent components and let $h_1(\mathbf{p}_{|n|})$ and $h_2(\mathbf{p}_{|m|})$ be the reliability functions of two coherent systems with $n \geq m$. Suppose that*

$$\frac{1}{h_1(\mathbf{p}_{|n|})} \frac{\partial h_1}{\partial p_i}(\mathbf{p}_{|n|}) \geq \frac{1}{h_2(\mathbf{p}_{|m|})} \frac{\partial h_2}{\partial p_i}(\mathbf{p}_{|m|}) \quad \text{for all } i = 1, \ldots, m, \tag{4.3}$$

and

$$\sum_{i=1}^{n} \frac{p_i}{h_1(\mathbf{p}_{|n|})} \frac{\partial h_1}{\partial p_i}(\mathbf{p}_{|n|}) \text{ or } \sum_{i=1}^{m} \frac{p_i}{h_2(\mathbf{p}_{|m|})} \frac{\partial h_2}{\partial p_i}(\mathbf{p}_{|m|}) \text{ is decreasing in each } p_i. \tag{4.4}$$

If $X_i \leq_{\text{fr}} Y_j$ for all $i, j = 1, \ldots, m$, then $h_1(\mathbf{X}) \leq_{\text{fr}} h_2(\mathbf{Y})$.

PROOF: Using Eq. (4.2), we have that

$$\begin{aligned}
r_{h_1(\mathbf{X})}(t) &\geq \sum_{i=1}^{m} r_i(t) \left[\frac{p_i}{h_1(\mathbf{p}_{|n|})} \frac{\partial h_1}{\partial p_i}(\mathbf{p}_{|n|}) \right]_{\mathbf{p}_{|n|} = (\bar{F}_1(t), \ldots, \bar{F}_n(t))} \\
&\geq \min_{i=1,\ldots,m} r_i(t) \sum_{i=1}^{m} \left[\frac{p_i}{h_1(\mathbf{p}_{|n|})} \frac{\partial h_1}{\partial p_i}(\mathbf{p}_{|n|}) \right]_{\mathbf{p}_{|n|} = (\bar{F}_1(t), \ldots, \bar{F}_n(t))} \\
&\geq \max_{i=1,\ldots,m} q_i(t) \sum_{i=1}^{m} \left[\frac{p_i}{h_1(\mathbf{p}_{|n|})} \frac{\partial h_1}{\partial p_i}(\mathbf{p}_{|n|}) \right]_{\mathbf{p}_{|n|} = (\bar{F}_1(t), \ldots, \bar{F}_n(t))} \\
&\geq \sum_{i=1}^{m} q_i(t) \left[\frac{p_i}{h_2(\mathbf{p}_{|m|})} \frac{\partial h_2}{\partial p_i}(\mathbf{p}_{|m|}) \right]_{\mathbf{p}_{|m|} = (\bar{G}_1(t), \ldots, \bar{G}_m(t))} \\
&= q_{h_2(\mathbf{Y})}(t),
\end{aligned}$$

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where the third inequality follows from $X_i \leq_{\text{fr}} Y_j$ for all $i, j = 1, \ldots, m$ and the last inequality is deduced from this last condition, Eqs. (4.3) and (4.4), and the relationship between orderings $\leq_{\text{fr}} \Rightarrow \leq_{\text{st}}$. ■

In particular, we obtain, in the following corollary, Theorem 3.3 of [15] and Theorem 16.B.1 of [6] on the preservation of the failure rate order under the formation of *k*-out-of-*n* systems.

COROLLARY 4.6: *Let* $(X_1, \ldots, X_n)$ *and* $(Y_1, \ldots, Y_n)$ *be the lifetimes of two sets of n independent components such that* $X_i \leq_{\text{fr}} Y_j$ *for all* $i, j = 1, \ldots, n$. *Then,* $\tau_{k,n} \leq_{\text{fr}} \tau'_{k,n}$.

PROOF: Taking $n = m$ and $h_1 = h_2 = h_{k,n}$ in Theorem 4.5, it is sufficient to prove that

$$\sum_{i=1}^{n} \frac{p_i}{h_{k,n}(p_1, \ldots, p_n)} \frac{\partial h_{k,n}}{\partial p_i}(p_1, \ldots, p_n)$$

is decreasing in each p_i, which results from Theorem 4 of Esary and Proschan [10]. ■

Remark 4.1: From Corollaries 4.4 and 4.6, we get that if $X_i \leq_{\text{fr}} Y_j$ for all $i, j = 1, \ldots, m$, then

$$\tau_{k,n} \leq_{\text{fr}} \tau'_{l,m}$$

for all $m \leq n$, $l \leq k$, *and* $n - k \leq m - l$.

5. REVERSED FAILURE RATE ORDERING

Concerning the reversed failure rate ordering, we obtain the following results.

THEOREM 5.1: *Let* $X_1, X_2, \ldots, X_m$ *be the lifetimes of m independent components, and* $h_1(\mathbf{p}_{|n|}), h_2(\mathbf{p}_{|m|})$ *the reliability functions of two coherent systems with* $m \geq n$. *If*

$$\frac{1}{1 - h_1(\mathbf{p}_{|n|})} \frac{\partial h_1}{\partial p_i}(\mathbf{p}_{|n|}) \leq \frac{1}{1 - h_2(\mathbf{p}_{|m|})} \frac{\partial h_2}{\partial p_i}(\mathbf{p}_{|m|}) \quad \textit{for all } i = 1, \ldots, n \tag{5.1}$$

then $h_1(\mathbf{X}_{|n|}) \leq_{\text{rf}} h_2(\mathbf{X}_{|m|})$.

PROOF: It is easily seen that the reversed failure rate function $\tilde{r}_{h(\mathbf{X})}$ of a coherent system with reliability function $h(p_1, \ldots, p_n)$ is given by

$$\tilde{r}_{h(\mathbf{X})}(t) = \sum_{i=1}^{n} \tilde{r}_i(t) \left[\frac{1 - p_i}{1 - h(\mathbf{p})} \frac{\partial h}{\partial p_i}(\mathbf{p}) \right]_{\mathbf{p} = (\bar{F}_1(t), \ldots, \bar{F}_n(t))}, \tag{5.2}$$

where $\bar{F}_i$ and $\tilde{r}_i$ are the survival and reversed failure rate functions of its components.

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Using the above result, that $m \geq n$, and Eq. (5.1), we get

$$\begin{aligned}\tilde{r}_{h_1(\mathbf{x}_{|n|})}(t) &\leq \sum_{i=1}^{m} \tilde{r}_i(t) \left[\frac{1-p_i}{1-h_1(\mathbf{p}_{|n|})} \frac{\partial h_1}{\partial p_i}(\mathbf{p}_{|n|}) \right]_{\mathbf{p}_{|n|}=(\bar{F}_1(t),\ldots,\bar{F}_n(t))} \\ &\leq \sum_{i=1}^{m} \tilde{r}_i(t) \left[\frac{1-p_i}{1-h_2(\mathbf{p}_{|m|})} \frac{\partial h_2}{\partial p_i}(\mathbf{p}_{|m|}) \right]_{\mathbf{p}_{|m|}=(\bar{F}_1(t),\ldots,\bar{F}_m(t))} \\ &= r_{h_2(\mathbf{x}_{|m|})}(t). \end{aligned}$$

■

As a consequence of the last theorem, we obtain the following two corollaries, where the first one is Theorem 4.1 of [5] and the second is Theorem 1(b) of [11].

COROLLARY 5.2: *Let $X_1, X_2, \ldots, X_n$ be the lifetimes of n independent components. Then, $\tau_{k,n} \leq_{\mathrm{rf}} \tau_{l,n}$ for all $l \leq k$.*

PROOF: From Theorem 5.1, we only need to see that Eq. (5.1) holds with $h_1 = h_{k+1,n}$ and $h_2 = h_{k,n}$; that is, that

$$\frac{P\left(\sum_{j\neq i}^{n} X_j(t) = k\right)}{P\left(\sum_{j=1}^{n} X_j(t) < k+1\right)} \leq \frac{P\left(\sum_{j\neq i}^{n} X_j(t) = k-1\right)}{P\left(\sum_{j=1}^{n} X_j(t) < k\right)}$$

which was proved in part (iii) of Lemma 2.1. ■

COROLLARY 5.3: *Let $X_1, X_2, \ldots, X_n$ be the lifetimes of n independent components. Then, $\tau_{k,n-1} \leq_{\mathrm{rf}} \tau_{k,n}$.*

PROOF: Using Theorem 5.1, it is sufficient to prove that Eq. (5.1) holds with $h_1 = h_{k,n-1}$ and $h_2 = h_{k,n}$; that is, that

$$\frac{P\left(\sum_{j\neq i}^{n-1} X_j(t) = k-1\right)}{P\left(\sum_{j=1}^{n-1} X_j(t) \leq k-1\right)} \leq \frac{P\left(\sum_{j\neq i}^{n} X_j(t) = k-1\right)}{P\left(\sum_{j=1}^{n} X_j(t) \leq k-1\right)}. \tag{5.3}$$

Note that Eq. (5.3) is equivalent to

$$\begin{aligned} P\left(\sum_{j\neq i}^{n-1} X_j(t) = k-1\right) &\left[\bar{F}_n(t) P\left(\sum_{j=1}^{n-1} X_j(t) \leq k-2\right)\right. \\ &\left. + F_n(t) P\left(\sum_{j=1}^{n-1} X_j(t) \leq k-1\right)\right] \\ \leq P\left(\sum_{j=1}^{n-1} X_j(t) \leq k-1\right) &\left[\bar{F}_n(t) P\left(\sum_{j\neq i}^{n-1} X_j(t) = k-2\right)\right. \\ &\left. + F_n(t) P\left(\sum_{j\neq i}^{n-1} X_j(t) = k-1\right)\right], \end{aligned}$$

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which holds if and only if

$$\frac{P\left(\sum_{j\neq i}^{n-1} X_j(t) = k-1\right)}{P\left(\sum_{j=1}^{n-1} X_j(t) \leq k-1\right)} \leq \frac{P\left(\sum_{j\neq i}^{n-1} X_j(t) = k-2\right)}{P\left(\sum_{j=1}^{n-1} X_j(t) \leq k-2\right)}.$$

Finally, the above inequality follows from part (iii) of Lemma 2.1. ■

In general, the next result shows the comparison in the reversed failure rate order between systems with different *k*-out-of-*n* structures, the proof of which is immediate from Corollaries 5.2 and 5.3.

COROLLARY 5.4: *Let $X_1, X_2, \ldots, X_m$ be the lifetimes of m independent components. Then, $\tau_{k,n} \leq_{\mathrm{rf}} \tau_{l,m}$ for all $n \leq m$ and $l \leq k$.*

In the case of two sets of components, we obtain the next results.

THEOREM 5.5: *Let $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_m)$ be the lifetimes of two sets of independent components and let $h_1(\mathbf{p}_{|n|})$ and $h_2(\mathbf{p}_{|m|})$ be the reliability functions of two coherent systems with $m \geq n$. Suppose that*

$$\frac{1}{1-h_1(\mathbf{p}_{|n|})}\frac{\partial h_1}{\partial p_i}(\mathbf{p}_{|n|}) \leq \frac{1}{1-h_2(\mathbf{p}_{|m|})}\frac{\partial h_2}{\partial p_i}(\mathbf{p}_{|m|}) \quad \textit{for all } i = 1, \ldots, n \qquad \textbf{(5.4)}$$

and

$$\sum_{i=1}^{n} \frac{1-p_i}{1-h_1(\mathbf{p}_{|n|})}\frac{\partial h_1}{\partial p_i}(\mathbf{p}_{|n|}) \text{ or } \sum_{i=1}^{m} \frac{1-p_i}{1-h_2(\mathbf{p}_{|m|})}\frac{\partial h_2}{\partial p_i}(\mathbf{p}_{|m|}) \qquad \textbf{(5.5)}$$

is increasing in each p_i.

If $X_i \leq_{\mathrm{rf}} Y_j$ for all $i, j = 1, \ldots, n$, then $h_1(\mathbf{X}) \leq_{\mathrm{rf}} h_2(\mathbf{Y})$.

Remark 5.1: Theorem 3.1 of [16] is a particular case of our Theorem 5.5 when $n = m$, $h_1 = h_2$, and $Y_i =_{\mathrm{st}} Y$ for all $i = 1, \ldots, n$.

Likewise, we have the next corollary on the preservation of the reversed failure rate ordering under the formation of *k*-out-of-*n* systems.

COROLLARY 5.6: *Let $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$ be the lifetimes of two sets of n independent components such that $X_i \leq_{\mathrm{rf}} Y_j$ for all $i, j = 1, \ldots, n$. Then, $\tau_{k,n} \leq_{\mathrm{rf}} \tau'_{k,n}$.*

PROOF: Taking $n = m$ and $h_1 = h_2 = h_{k,n}$ in Theorem 5.5, we only need to show that

$$\sum_{i=1}^{n} \frac{1-p_i}{1-h_{k,n}(p_1, \ldots, p_n)}\frac{\partial h_{k,n}}{\partial p_i}(p_1, \ldots, p_n)$$

is increasing in each p_i, which results from Theorem 3.2 of [16]. ■

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Remark 5.2: From Corollaries 5.4 and 5.6, if $X_i \leq_{\mathrm{rf}} Y_j$ for all $i, j = 1, \ldots, n$, then

$$\tau_{k,n} \leq_{\mathrm{rf}} \tau'_{l,m}$$

for all $n \leq m$ and $l \leq k$.

6. LIKELIHOOD RATIO ORDERING

Let us consider $X_1, X_2, \ldots, X_s$ to be the lifetimes of a set of s independent components ($s = n$ or $s = m$) and $h_1(p_1, p_2, \ldots, p_n)$ and $h_2(p_1, p_2, \ldots, p_m)$ to be the reliability functions of two coherent systems with n and m components, respectively. Now, we study sufficient conditions under which $h_1(X_1, \ldots, X_n) \leq_{\mathrm{lr}} h_2(X_1, \ldots, X_m)$.

THEOREM 6.1: *Let $X_1, \ldots, X_n$ be the lifetimes of n independent components and let $h_1(\mathbf{p}_{|n|})$ and $h_2(\mathbf{p}_{|m|})$ be the reliability functions of two coherent systems with $n \geq m$. Suppose that*

$$\frac{\dfrac{\partial h_2}{\partial p_i}(\mathbf{p}_{|m|})}{\dfrac{\partial h_1}{\partial p_i}(\mathbf{p}_{|n|})} \text{ is increasing in } t, \ \forall i = 1, \ldots, m, \tag{6.1}$$

$$\frac{\partial h_2}{\partial p_i}(\mathbf{p}_{|m|}) \frac{\partial h_1}{\partial p_j}(\mathbf{p}_{|n|}) \leq \frac{\partial h_2}{\partial p_j}(\mathbf{p}_{|m|}) \frac{\partial h_1}{\partial p_i}(\mathbf{p}_{|n|}), \quad \forall i < j, i, j \in \{1, \ldots, m\}, \tag{6.2}$$

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial h_2}{\partial p_i}(\mathbf{p}_{|m|})\right)\frac{\partial h_1}{\partial p_j}(\mathbf{p}_{|n|}) + \frac{d}{dt}\left(\frac{\partial h_2}{\partial p_j}(\mathbf{p}_{|m|})\right)\frac{\partial h_1}{\partial p_i}(\mathbf{p}_{|n|}) \\ &\quad \geq \frac{d}{dt}\left(\frac{\partial h_1}{\partial p_i}(\mathbf{p}_{|n|})\right)\frac{\partial h_2}{\partial p_j}(\mathbf{p}_{|m|}) + \frac{d}{dt}\left(\frac{\partial h_1}{\partial p_j}(\mathbf{p}_{|n|})\right)\frac{\partial h_2}{\partial p_i}(\mathbf{p}_{|m|}), \\ &\qquad \forall i < j, i, j \in \{1, \ldots, m\}, \end{aligned} \tag{6.3}$$

and

$$\frac{\dfrac{\partial h_2}{\partial p_j}(\mathbf{p}_{|m|})}{\dfrac{\partial h_1}{\partial p_i}(\mathbf{p}_{|n|})} \text{ is increasing in } t, \quad \forall j = 1, \ldots, m \text{ and } \forall i = m+1, \ldots, n, \tag{6.4}$$

with $\mathbf{p}_{|n|} = (\bar{F}_1(t), \ldots, \bar{F}_n(t))$ and $\mathbf{p}_{|m|} = (\bar{F}_1(t), \ldots, \bar{F}_m(t))$.

If

$$X_i \leq_{\mathrm{lr}} X_1 \leq_{\mathrm{lr}} X_2 \leq_{\mathrm{lr}} \cdots \leq_{\mathrm{lr}} X_m \quad \forall i = m+1, \ldots, n, \tag{6.5}$$

then $h_1(X_1, \ldots, X_n) \leq_{\mathrm{lr}} h_2(X_1, \ldots, X_m)$.

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PROOF: From the definition of the likelihood ratio order given by Eq. (2.4), $h_1(X_1,\ldots,X_n) \leq_{\mathrm{lr}} h_2(X_1,\ldots,X_m)$ if and only if

$$\frac{\sum_{i=1}^{m} f_i(t)\,\dfrac{\partial h_2}{\partial p_i}\,(\bar{F}_1(t),\ldots,\bar{F}_m(t))}{\sum_{i=1}^{n} f_i(t)\,\dfrac{\partial h_1}{\partial p_i}\,(\bar{F}_1(t),\ldots,\bar{F}_n(t))} \text{ is increasing in } t. \tag{6.6}$$

Now, we denote $f_i \equiv f_i(t)$, $a_i \equiv a_i(t) = (\partial h_2/\partial p_i)(\bar{F}_1(t),\ldots,\bar{F}_m(t))$, and $b_i \equiv b_i(t) = (\partial h_1/\partial p_i)(\bar{F}_1(t),\ldots,\bar{F}_n(t))$, so Eq. (6.6) is rewritten as

$$\frac{\sum_{i=1}^{m} a_i f_i}{\sum_{i=1}^{n} b_i f_i} \text{ is increasing in } t. \tag{6.7}$$

Assuming that second partial derivatives of h_1 and h_2 exist and components have derivable densities, Eq. (6.7) holds if and only if

$$\left[\sum_{i=1}^{m}(a_i' f_i + a_i f_i')\right]\left(\sum_{i=1}^{m} b_i f_i\right) + \left[\sum_{i=1}^{m}(a_i' f_i + a_i f_i')\right]\left(\sum_{i=m+1}^{n} b_i f_i\right)$$
$$\geq \left[\sum_{i=1}^{m}(b_i' f_i + b_i f_i')\right]\left(\sum_{i=1}^{m} a_i f_i\right) + \left[\sum_{i=m+1}^{n}(b_i' f_i + b_i f_i')\right]\left(\sum_{i=1}^{m} a_i f_i\right);$$

that is,

$$\sum_{i=1}^{m} f_i^2(a_i' b_i - a_i b_i') + \sum_{i<j}^{m}(a_i b_j - b_i a_j)(f_i' f_j - f_i f_j')$$
$$+ \sum_{i<j}^{m} f_i f_j(a_i' b_j - b_i' a_j + a_j' b_i - b_j' a_i) + \sum_{i=m+1}^{n}\sum_{j=1}^{m} f_i f_j(b_i a_j' - b_i' a_j)$$
$$+ \sum_{i=m+1}^{n}\sum_{j=1}^{m} b_i a_j(f_i f_j' - f_i' f_j)$$
$$\geq 0. \tag{6.8}$$

Let us see now that $T_i \geq 0$ for $i = 1,2,\ldots,5$, where T_i is the ith term of Eq. (6.8).

Note that T_1, T_3, and T_4 are nonnegative due to Eqs. (6.1), (6.3), and (6.4), respectively. On the other hand, from Eqs. (6.2) and (6.5) results, $T_2 \geq 0$, and using the fact that reliability functions are increasing in each argument, we get that $T_5 \geq 0$ from Eq. (6.5). ■

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Analogously, we obtain the following theorem when $n \leq m$.

THEOREM 6.2: *Let $X_1, X_2, \ldots, X_m$ be the lifetimes of m independent components and let $h_1(\mathbf{p}_{|n|})$ and $h_2(\mathbf{p}_{|m|})$ be the reliability functions of two coherent systems with $m \geq n$. Suppose that the following conditions hold:*

(i) *Equation (6.1) for all $i = 1,2,\ldots,n$.*
(ii) *Equation (6.2) for all $i < j$, $i,j \in \{1,2,\ldots,n\}$.*
(iii) *Equation (6.3) for all $i < j$, $i,j \in \{1,2,\ldots,n\}$.*
(iv) *Equation (6.4) for all $i = 1,\ldots,n$ and $j = n+1,\ldots,m$.*

If $X_1 \leq_{\text{lr}} X_2 \leq_{\text{lr}} \cdots \leq_{\text{lr}} X_n \leq_{\text{lr}} X_j$, $\forall j = n+1,\ldots,m$, then $h_1(X_1,\ldots,X_n) \leq_{\text{lr}} h_2(X_1,\ldots,X_m)$.

The next corollary shows that Theorem 4 of Bapat and Kochar [2] is a particular case of Theorem 6.1, as well as the likelihood ratio ordering given by Raqab and Amin [17] for i.i.d. components.

COROLLARY 6.3: *Let $X_1, X_2, \ldots, X_n$ be the lifetimes of n independent components. If $X_i \leq_{\text{lr}} X_j$ for all $i < j$, then $\tau_{k,n} \leq_{\text{lr}} \tau_{l,n}$ for all $l \leq k$.*

PROOF: From Theorem 6.1, we only need to show that Eqs. (6.1), (6.2), and (6.3) hold with $n = m$, $h_1 = h_{k+1,n}$, and $h_2 = h_{k,n}$.

First, note that Eq. (6.1) is equivalent to

$$\frac{P\left(\sum_{j\neq i}^{n} X_j(t) = k-1\right)}{P\left(\sum_{j\neq i}^{n} X_j(t) = k\right)} \text{ is increasing in } t,$$

which follows from Eq. (7) of Samuels [18].

On the other hand, Eq. (6.2) holds if and only if

$$\frac{P\left(\sum_{l\neq i}^{n} X_l(t) = k-1\right)}{P\left(\sum_{l\neq i}^{n} X_l(t) = k\right)} \leq \frac{P\left(\sum_{l\neq j}^{n} X_l(t) = k-1\right)}{P\left(\sum_{l\neq j}^{n} X_l(t) = k\right)} \tag{6.9}$$

for all $i < j$, $i,j \in \{1,2,\ldots,n\}$. Pivoting the above expression on the ith and jth components and simplifying, we have that Eq. (6.9) is equivalent to

$$\begin{aligned}(\bar{F}_j(t) - \bar{F}_i(t))P\left(\sum_{l\neq i,j}^{n} X_l(t) = k\right)P\left(\sum_{l\neq i,j}^{n} X_t(t) = k-2\right)\\ \leq (\bar{F}_j(t) - \bar{F}_i(t))\left(P\left(\sum_{l\neq i,j}^{n} X_l(t) = k-1\right)\right)^2,\end{aligned}$$

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which holds for all $i < j$ because of part (i) of Lemma 2.1 and the given condition $X_i \leq_{\mathrm{lr}} X_j$, $\forall i < j$, using the fact that likelihood ratio ordering is stronger than stochastic ordering.

Note that Eq. (6.3) is rewritten as

$$\frac{\partial h_1}{\partial p_j}(\mathbf{p}_{|n|}) \sum_{l \neq i}^{n} f_l(t) \frac{\partial^2 h_2}{\partial p_i \partial p_l}(\mathbf{p}_{|m|}) + \frac{\partial h_1}{\partial p_i}(\mathbf{p}_{|n|}) \sum_{l \neq j}^{n} f_l(t) \frac{\partial^2 h_2}{\partial p_j \partial p_l}(\mathbf{p}_{|m|})$$
$$\leq \frac{\partial h_2}{\partial p_j}(\mathbf{p}_{|m|}) \sum_{l \neq i}^{n} f_l(t) \frac{\partial^2 h_1}{\partial p_i \partial p_l}(\mathbf{p}_{|n|}) + \frac{\partial h_2}{\partial p_i}(\mathbf{p}_{|m|}) \sum_{l \neq j}^{n} f_l(t) \frac{\partial^2 h_1}{\partial p_j \partial p_l}(\mathbf{p}_{|n|}),$$

so it is sufficient to see that

$$P\left(\sum_{s \neq j}^{n} X_s(t) = k\right)\left[P\left(\sum_{s \neq i,j}^{n} X_s(t) = k-2\right) - P\left(\sum_{s \neq i,j}^{n} X_s(t) = k-1\right)\right]$$
$$\leq P\left(\sum_{s \neq j}^{n} X_s(t) = k-1\right)\left[P\left(\sum_{s \neq i,j}^{n} X_s(t) = k-1\right) - P\left(\sum_{s \neq i,j}^{n} X_s(t) = k\right)\right] \tag{6.10}$$

and

$$P\left(\sum_{s \neq j}^{n} X_s(t) = k\right)\left[P\left(\sum_{s \neq i,l}^{n} X_s(t) = k-2\right) - P\left(\sum_{s \neq i,l}^{n} X_s(t) = k-1\right)\right]$$
$$\leq P\left(\sum_{s \neq i}^{n} X_s(t) = k-1\right)\left[P\left(\sum_{s \neq j,l}^{n} X_s(t) = k-1\right) - P\left(\sum_{s \neq j,l}^{n} X_s(t) = k\right)\right] \tag{6.11}$$

for all $i, j, l \in \{1, \ldots, n\}$ with $l \neq i, j$.

Finally, Eq. (6.10) is obtained pivoting on the ith component and using part (i) of Lemma 2.1. Likewise, Eq. (6.11) is proved pivoting on the ith or jth component and using part (i) of Lemma 2.1. ■

Let us see now some results for the case of two sets of i.i.d. components, and then we study comparisons when components are independent but not necessarily identically distributed.

THEOREM 6.4: *Let $h_1(p)$ and $h_2(p)$ be the reliability functions of two coherent systems with n and m components, respectively, such that*

$$\frac{h_1''(p)}{h_1'(p)} \geq \frac{h_2''(p)}{h_2'(p)} \quad \textit{for all } p \in [0,1] \tag{6.12}$$

and h_i with $i = 1$ or 2 verifies that there exists $a \in [0,1]$ such that

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(i) $\dfrac{ph_i''(p)}{h_i'(p)}$ *is decreasing and positive for all* $p \in [0,a)$,

(ii) $\dfrac{(1-p)h_i''(p)}{h_i'(p)}$ *is decreasing and negative for all* $p \in (a,1]$.

If $X \leq_{\mathrm{lr}} Y$, *then* $h_1(X) \leq_{\mathrm{lr}} h_2(Y)$.

PROOF: Using a similar development to the proof of Theorem 2.4 of [16], it is easy to get that $h_1(X) \leq_{\mathrm{lr}} h_2(Y)$ if and only if

$$(f'(t)g(t) - g'(t)f(t)) + f(t)g(t)\left[\frac{g(t)h_2''(\bar{G}(t))}{h_2'(\bar{G}(t))} - \frac{f(t)h_1''(\bar{F}(t))}{h_1'(\bar{F}(t))}\right] \leq 0.$$

The first term of the above expression is nonpositive due to the fact that $X \leq_{\mathrm{lr}} Y$. The nonpositivity of the second term is obtained using the inequalities

$$\frac{g(t)}{\bar{G}(t)}\,\frac{\bar{G}(t)h_2''(\bar{G}(t))}{h_2'(\bar{G}(t))} \leq \frac{f(t)}{\bar{F}(t)}\,\frac{\bar{F}(t)h_1''(\bar{F}(t))}{h_1'(\bar{F}(t))}$$

when $p < a$, and

$$\frac{g(t)}{G(t)}\,\frac{G(t)h_2''(\bar{G}(t))}{h_2'(\bar{G}(t))} \leq \frac{f(t)}{F(t)}\,\frac{F(t)h_1''(\bar{F}(t))}{h_1'(\bar{F}(t))}$$

when $p > a$, which follow from Eq. (6.12) and conditions (i) and (ii), respectively, using the fact that the likelihood ratio ordering is stronger than the failure rate, reversed failure rate, and stochastic orderings. ■

Theorem 2.3 of [16] is a particular case of our Theorem 6.4 for $n = m$ and $h_1 = h_2$. Note that Theorem 2.3 of [16] is valid only when the reliability function h is convex and therefore cannot be applied to general k-out-of-n systems (only to series systems). However, as a consequence of the last theorem, we will prove in the next corollary the preservation of the likelihood ordering under the formation of k-out-of-n systems with different number of components and failures supported. For that, we take into account that the density and survival functions of a k-out-of-n system with i.i.d. components are given by

$$\bar{F}_{\tau_{k,n}}(t) = B_{k,n-k+1}(\bar{F}(t)) \quad \text{and} \quad f_{\tau_{k,n}}(t) = b_{k,n-k+1}(\bar{F}(t))f(t), \tag{6.13}$$

where $B_{r,s}$ is the beta distribution with (r,s) parameters and $b_{r,s}(x) = [\Gamma(r+s)/\Gamma(r)\Gamma(s)]x^{r-1}(1-x)^{s-1}$ represents its density function (see Barlow and Proschan [3]), and the next lemma whose proof is obvious.

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LEMMA 6.5: *The following items hold:*

(i) *If $w_{r,s}$ is the function defined by*

$$w_{r,s}(u) = u^r(1-u)^{s-r} \quad \text{for all } u \in [0,1]$$

then $w_{r,s}$ is a decreasing (increasing) function of u on $(0,1)$ for all $r \leq (\geq) 0$ and $r \leq (\geq) s$.

(ii) *$b_{r',s'}(x)/b_{r,s}(x)$ is increasing in x for all $r \leq r'$ and $s \geq s'$.*

COROLLARY 6.6: *If $X \leq_{\mathrm{lr}} Y$, then $\tau_{k,n} \leq_{\mathrm{lr}} \tau'_{l,m}$ for all $l \leq k$, $n-k \leq m-l$.*

PROOF: Using Theorem 6.4, it is sufficient to show that Eq. (6.12) and items (i) and (ii) of Theorem 6.4 hold with $h_1 = h_{k,n}$ and $h_2 = h_{l,m}$. First, it is easy to check items (i) and (ii) of Theorem 6.4. On the other hand, using Eq. (6.13), we have that Eq. (6.12) is rewritten as

$$\frac{b'_{k,n-k+1}(p)}{b_{k,n-k+1}(p)} \geq \frac{b'_{l,m-l+1}(p)}{b_{l,m-l+1}(p)};$$

that is,

$$(\ln b_{k,n-k+1}(p))' \geq (\ln b_{l,m-l+1}(p))'.$$

Thus, Eq. (6.12) is equivalent to

$$\frac{b_{k,n-k+1}(p)}{b_{l,m-l+1}(p)} \text{ is increasing in } p,$$

which holds for all $l \leq k$, $n-k \leq m-l$ using part (ii) of Lemma 6.5. ■

Remark 6.1: In particular, Theorem 3.1 of [20] is obtained from Corollary 6.6 taking $n = m$ and $l = k$.

In the rest of the article, we study comparisons with respect to the likelihood ratio ordering between two coherent systems formed from two sets of independent but not necessarily identically distributed components.

THEOREM 6.7: *Let $(X_1,\ldots,X_n)$ and $(Y_1,\ldots,Y_m)$ be the lifetimes of two sets of independent components and $h_1(\mathbf{p}_{|n|})$ and $h_2(\mathbf{q}_{|m|})$ be the reliability functions of two coherent systems. Suppose that*

$$\frac{\dfrac{\partial h_2}{\partial q_j}(\mathbf{q}_{|m|})}{\dfrac{\partial h_1}{\partial p_i}(\mathbf{p}_{|n|})} \text{ is increasing in } t \tag{6.14}$$

for all $i = 1,\ldots,n, j = 1,\ldots,m$, with $\mathbf{p}_{|n|} = (\bar{F}_1(t),\ldots,\bar{F}_n(t))$ and $\mathbf{q}_{|m|} = (\bar{G}_1(t),\ldots,\bar{G}_m(t))$.

If $X_i \leq_{\mathrm{lr}} Y_j$ for all $i = 1,\ldots,n, j = 1,\ldots,m$, then $h_1(\mathbf{X}) \leq_{\mathrm{lr}} h_2(\mathbf{Y})$.

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PROOF: By definition of the likelihood ratio ordering, $h_1(\mathbf{X}) \leq_{\text{lr}} h_2(\mathbf{Y})$ if and only if

$$\frac{\sum_{j=1}^{m} g_j(t)\, \frac{\partial h_2}{\partial q_j}\,(\bar{G}_1(t),\ldots,\bar{G}_m(t))}{\sum_{i=1}^{n} f_i(t)\, \frac{\partial h_1}{\partial p_i}\,(\bar{F}_1(t),\ldots,\bar{F}_n(t))} \text{ is increasing in } t,$$

so it is sufficient to see that

$$\frac{g_j(t)}{f_i(t)}\,\frac{\frac{\partial h_2}{\partial q_j}\,(\bar{G}_1(t),\ldots,\bar{G}_m(t))}{\frac{\partial h_1}{\partial p_i}\,(\bar{F}_1(t),\ldots,\bar{F}_n(t))} \text{ is increasing in } t \tag{6.15}$$

for all $i = 1,\ldots,n, j = 1,\ldots,m$.

Taking into account that each term of the product is nonnegative and increasing in t, due to the hypothesis $X_i \leq_{\text{lr}} Y_j, \forall i,j$, and Eq. (6.14), respectively, we have Eq. (6.15), and so the required result. ■

In the following corollary, we apply Theorem 6.7 to show the preservation of the likelihood ratio ordering under formation of two systems with different k-out-of-n structures, formed from two sets of independent and not necessarily identically distributed components.

COROLLARY 6.8: *Let $(X_1,\ldots,X_n)$ and $(Y_1,\ldots,Y_m)$ be the lifetimes of two sets of independent components such that $X_i \leq_{\text{lr}} Y_j$ for all $i = 1,\ldots,n, j = 1,\ldots,m$. Then, $\tau_{k,n} \leq_{\text{lr}} \tau'_{l,m}$ for all $l \leq k$, $n-k \leq m-l$.*

PROOF: From Theorem 6.7, we only need to prove that Eq. (6.14) holds with $h_1 = h_{k,n}$ and $h_2 = h_{l,m}$ for all $l \leq k$, $n-k \leq m-l$.

First, note that Eq. (6.14) is rewritten as

$$\frac{P\left(\sum_{s\neq j}^{m} Y_s(t) = l-1\right)}{P\left(\sum_{s\neq i}^{n} X_s(t) = k-1\right)} \text{ is increasing in } t;$$

that is,

$$\frac{\sum_{(\beta_1,\ldots,\beta_{m-1})\in S^{j}_{m,l-1}} \bar{G}_{\beta_1}(t)\cdots\bar{G}_{\beta_{l-1}}(t)G_{\beta_l}(t)\cdots G_{\beta_{m-1}}(t)}{\sum_{(\alpha_1,\ldots,\alpha_{n-1})\in S^{i}_{n,k-1}} \bar{F}_{\alpha_1}(t)\cdots\bar{F}_{\alpha_{k-1}}(t)F_{\alpha_k}(t)\cdots F_{\alpha_{n-1}}(t)} \text{ is increasing in } t$$

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with $S^j_{m,l-1} = \{(\beta_1,\ldots,\beta_{m-1}) \in \{1,\ldots,m\} - \{j\} : \beta_1 < \cdots < \beta_{l-1}$ and $\beta_l < \cdots < \beta_{m-1}\}$ and $S^i_{n,k-1} = \{(\alpha_1,\ldots,\alpha_{n-1}) \in \{1,\ldots,n\} - \{i\} : \alpha_1 < \cdots < \alpha_{k-1}$ and $\alpha_k < \cdots < \alpha_{n-1}\}$. So, it is sufficient to show that

$$\frac{\bar{G}_{\beta_1}(t)\cdots\bar{G}_{\beta_{l-1}}(t)G_{\beta_l}(t)\cdots G_{\beta_{m-1}}(t)}{\bar{F}_{\alpha_1}(t)\cdots\bar{F}_{\alpha_{k-1}}(t)F_{\alpha_k}(t)\cdots F_{\alpha_{n-1}}(t)} \text{ is increasing in } t \qquad \textbf{(6.16)}$$

for all $(\beta_1,\ldots,\beta_{m-1}) \in S^j_{m,l-1}$ and $(\alpha_1,\ldots,\alpha_{n-1}) \in S^i_{n,k-1}$, when $l \le k$ and $n-k \le m-l$.

In fact, Eq. (6.16) can be rewritten as

$$\frac{\bar{G}_{\beta_1}(t)\cdots\bar{G}_{\beta_{l-1}}(t)}{\bar{F}_{\alpha_1}(t)\cdots\bar{F}_{\alpha_{l-1}}(t)}\,\frac{G_{\beta_l}(t)\cdots G_{\beta_{n-k+l-1}}(t)}{F_{\alpha_k}(t)\cdots F_{\alpha_{n-1}}(t)}\,\frac{G_{\beta_{n-k+l}}(t)\cdots G_{\beta_{m-1}}(t)}{\bar{F}_{\alpha_l}(t)\cdots\bar{F}_{\alpha_{k-1}}(t)}$$

and using the fact that $X_{\alpha_i} \le_{\mathrm{lr}} Y_{\beta_j}$ is stronger than failure rate and reversed failure rate orderings, we have that the first and second terms of this product are increasing in t, and the third term is always increasing. ■

Finally, note that Theorem 3.5 of [9] is a particular case of Corollary 6.8 when $k = l$ and $n = m$.

Acknowledgments

The authors are grateful to the referee for his comments and careful reading of the manuscript. This work was partially supported by the Ministerio de Educación y Cultura DGES under Grant PB96-1105.

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