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A Weakening of the Odd Hadwiger's Conjecture

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Gerards and Seymour (see [10], p. 115) conjectured that if a graph has no odd complete minor of order l, then it is $(l-1)$-colourable. This is an analogue of the well-known conjecture of Hadwiger, and in fact, this would immediately imply Hadwiger's conjecture. The current best-known bound for the chromatic number of graphs with no odd complete minor of order l is $O(l\sqrt{\log l})$ by the recent result by Geelen, Gerards, Reed, Seymour and Vetta [8], and by Kawarabayashi [12] later, independently. But it seems very hard to improve this bound since this would also improve the current best-known bound for the chromatic number of graphs with no complete minor of order l.

Motivated by this problem, in this note we show that there exists an absolute constant $f(k)$ such that any graph G with no odd complete minor of order k admits a vertex partition $V_1, \ldots, V_{496k}$ such that each component in the subgraph induced on V_i $(i \geqslant 1)$ has at most $f(k)$ vertices. When $f(k) = 1$, this is a colouring of G. Hence this is a relaxation of colouring in a sense, and this is the first result in this direction for the odd Hadwiger's conjecture.

Our proof is based on a recent decomposition theorem due to Geelen, Gerards, Reed, Seymour and Vetta [8], together with a connectivity result that forces a huge complete bipartite minor in large graphs by Böhme, Kawarabayashi, Maharry and Mohar [3].

1. Hadwiger's conjecture and the odd Hadwiger's conjecture

Hadwiger's conjecture from 1943 suggests a far-reaching generalization of the Four Colour Theorem [1, 2, 19] and is considered by many as one of the deepest open problems in graph theory. Hadwiger's conjecture states the following.

Conjecture 1.1. *For all $k \geqslant 1$, every k-chromatic graph has a K_k-minor.*

Conjecture 1.1 is trivially true for $k \leqslant 3$, and reasonably easy for $k = 4$, as shown by Dirac [6] and Hadwiger himself [7]. However, for $k \geqslant 5$, Conjecture 1.1 implies the Four Colour Theorem. In 1937, Wagner [22] proved that the case $k = 5$ of Conjecture 1.1 is, in fact, equivalent to the

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Four Colour Theorem. In 1993, Robertson, Seymour and Thomas [18] proved that a minimal counter-example to the case $k = 6$ is a graph G which has a vertex v such that $G - v$ is planar. By the Four Colour Theorem, this implies Conjecture 1.1 for $k = 6$. Hence the cases $k = 5, 6$ are each equivalent to the Four Colour Theorem [1, 2, 19]. Conjecture 1.1 is open for $k \geqslant 7$. For the case $k = 7$, Toft and the author [15] proved that any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor. Recently, the author [11] proved that any 7-chromatic graph has K_7 or $K_{3,5}$ as a minor.

It is not known if there exists an absolute constant c such that any ck-chromatic graph has a K_k-minor. So far, it is known that there exists a constant c such that any $ck\sqrt{\log k}$-chromatic graph has a K_k-minor. This follows from the results by Kostochka [16, 17] or Thomason [20, 21].

Motivated by these facts, the following result was proved in [13].

Theorem 1.2. *For every positive integer k, there exists a computable constant $f(k)$ such that every graph G with no K_k-minor admits a vertex partition $V_1, \ldots, V_{\lceil 15.5k \rceil}$, such that every component in the subgraph of G induced on V_i has at most $f(k)$ vertices.*

By saying that $f(k)$ is *computable*, we mean that $f(k)$ can be expressed as a specific value, depending on k. The reader interested in this expression should consult [3].

When $f(k) = 1$, we get a colouring of G. Hence, Theorem 1.2 gives a relaxation of colouring, and this was the first result in this direction. Actually, the main result in [13] is the list-colouring version of Theorem 1.2. So the result in [13] is strictly stronger than Theorem 1.2.

Recently, the concept 'odd minor' has been paid much attention by many researchers.

We say that H has an *odd complete minor* of order l if there are l vertex-disjoint trees in H such that every two of them are joined by an edge, and in addition, all the vertices of trees are two-coloured in such a way that the edges within the trees are bichromatic, but the edges between trees are monochromatic. We say that H has an *odd K_l-minor* if H has an odd complete minor of order l. It is easy to see that any graph which has an odd K_l-minor certainly contains a K_l-minor.

Gerards and Seymour (see [10], p. 115) conjectured the following.

Conjecture 1.3. *For all $l \geqslant 1$, every graph with no odd K_{l+1}-minor is l-colourable.*

This is an analogue of Conjecture 1.1. In fact, it is easy to see that Conjecture 1.3 is strictly stronger than Conjecture 1.1. Again, Conjecture 1.3 is trivially true for the cases $l = 1, 2$. In fact, when $l = 2$, this means that if a graph has no odd cycles, then it is 2-colourable. This is easy since such a graph must be bipartite. The case $l = 3$ was proved by Catlin [4]. Recently, Guenin [9] announced a solution for the case $l = 4$. This result would imply the Four Colour Theorem because a graph having an odd K_5-minor certainly contains a K_5-minor. Conjecture 1.3 is open for $l \geqslant 5$.

Recently, Geelen, Gerards, Reed, Seymour and Vetta [8] gave a structural theorem with graphs with no odd K_k-minor, but having a huge clique minor. This result is enough to prove that there exists a constant c such that any graph with no odd K_k-minor is $ck\sqrt{\log k}$-colourable. A shorter and simpler proof without using the structural theorem has been obtained by the author [12]. This is an analogue of the results by Kostochka [16, 17] or Thomason [20, 21]. In fact, this immediately implies Kostochka's or Thomason's results. But it seems that improving the

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chromatic number of graphs with no odd K_k-minor is very hard, since we do not even know if there exists a constant c such that any ck-chromatic graph contains a K_k-minor.

Motivated by this fact and Theorem 1.2, we shall prove the following result.

Theorem 1.4. *For every positive integer k, there exists a computable constant $f(k)$ such that every graph G with no odd K_k-minor admits a vertex partition $V_1, \ldots, V_{496k}$ such that every component in the subgraph of G induced on V_i has at most $f(k)$ vertices.*

When $f(k) = 1$, we get a colouring of G. Hence, Theorem 1.4 gives a relaxation of colouring, and this is the first result in this direction for the odd Hadwiger's conjecture. We remark that, unlike the result in [13], the list-colouring version of our result does not hold. This is simply because a bipartite graph has no odd K_3-minors but may have arbitrarily large list-chromatic number [5].

For graph-theoretic terminology not explained in this paper, we refer the reader to [5].

2. Weakened connectivity

In the proof of Theorem 1.4, we shall need a corollary of the main result in [3]. Actually, the following is the main result in [3].

Theorem 2.1. *For any integers k, s and t, there exists a computable constant $N_0(k, s, t)$ such that every $(3k + 2)$-connected graph of minimum degree at least $15.5k$ and with at least $N_0(k, s, t)$ vertices either contains $K_{k,st}$ as a topological minor or a minor isomorphic to s disjoint copies of $K_{k,t}$.*

Let A and B be induced subgraphs of G such that $G = A \cup B$. If $V(A) \setminus V(B) \neq \emptyset$ and $V(B) \setminus V(A) \neq \emptyset$, then we say that the pair (A, B) is a *separation* of G. The *order* of this separation is equal to $|V(A \cap B)|$. Let $Z \subseteq V(G)$ be a vertex set. We say that the separation (A, B) of G is *Z-essential* if $(A - Z, B - Z)$ is a separation of $G - Z$. If l is a positive integer, we say that G is *l-connected relative to Z* if it has no Z-essential separations of order less than l.

We will also need a strengthening of Theorem 2.1, which is a corollary of the main result in [3].

Theorem 2.2. *For any integers k, x and z, there exists a constant $N_1(k, x, z)$ such that, for every graph G and every vertex set $Z \subseteq V(G)$ of cardinality at most z, if G is $(96k + 2)$-connected relative to Z, the degree of every vertex in $V(G) \setminus Z$ is at least $496k$, and G has at least $N_1(k, x, z)$ vertices, then G contains the complete bipartite graph $K_{32k,x}$ as a minor.*

Proof. Let G and Z be as assumed in the statement of the theorem. Let Z' be the set of all vertices in Z whose degree is at most $496k + z$. Let D be a set of vertices in $G - Z$ of cardinality $496k$ such that no vertex in Z' is adjacent to D. If $|V(G)| \geqslant (496k + z)z + 496k$ (which we may assume), then D exists. Let G' be the graph obtained from G by adding all edges between Z' and D.

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In G', every vertex in Z has at least $496k$ neighbours that are not in Z. Since G is $(96k+2)$-connected relative to Z and is a spanning subgraph of G', this implies that G' is $(96k+2)$-connected. Also, the minimum degree of G' is at least $496k$.

Suppose that $|V(G)| \geqslant N_0(32k, s, t)$, where $s = z+1$ and $t = x + 992k + 1$, and furthermore, $N_0(32k, s, t)$ comes from Theorem 2.1. By Theorem 2.1, G' either contains a subdivision of $K_{32k,st}$ or a minor isomorphic to s disjoint copies of $K_{32k,t}$. If G' has s copies of $K_{32k,t}$ as a minor, then G' contains one of the s copies of a $K_{32k,t}$-minor that is disjoint from Z. Hence G contains a $K_{32k,x}$-minor. Finally, suppose G' contains a subgraph K which is a subdivision of $K_{32k,st}$. Then, none of the vertices of degree st in K belongs to Z' since the vertices in Z' have degree less than $496k + z + 496k + 1 = 992k + z + 1 \leqslant st$. Therefore, $G' - Z'$ contains a subgraph which is a subdivision of $K_{32k,st-z}$. Since $st - z \geqslant x$, G has a $K_{32k,x}$-minor. So, the theorem holds for the value $N_1(k, x, z) = N_0(32k, z+1, x + 992k + 1)$. □

3. Odd minor

Recall that a complete minor of order l can be thought of l vertex-disjoint trees, every two of which are joined by an edge. We can similarly view a complete bipartite minor.

We need the following result.

Theorem 3.1. *Suppose G has a $K_{32k,\,(16k-1)\binom{32k}{16k}+1}$-minor. Then either G has an odd K_k-minor or G has a vertex set X of order at most $8k$ such that the (unique) block F that intersects all but at most $8k$ trees of the $K_{32k,\,(16k-1)\binom{32k}{16k}+1}$-minor is bipartite, and each odd cycle is contained in either components of $G - X$ that do not intersect F or blocks with a cut vertex to F.*

To prove this theorem, we need the following. We call a complete minor of order l *even* if the union of the trees together with the connecting edges is bipartite.

Geelen, Gerards, Reed, Seymour and Vetta [8] proved the following result.

Theorem 3.2. *If G has an even complete minor of order at least $16k$, then either G has an odd complete minor of order k or G has a vertex set X with $|X| < 8k$ such that the (unique) component F that intersects all but at most $8k$ trees of the even minor is bipartite, and each odd cycle is contained in either components of $G - X$ that do not intersect F or blocks with a cut vertex to F.*

Proof of Theorem 3.1. We follow the proof in [14]. We can think of a $K_{32k,\,(16k-1)\binom{32k}{16k}+1}$-minor as follows:

There are $32k + (16k-1)\binom{32k}{16k} + 1$ disjoint trees $T_1, \ldots, T_{32k}, T'_1, \ldots, T'_{(16k-1)\binom{32k}{16k}+1}$ such that there is an edge between T_i and T'_j for any i, j with $1 \leqslant i \leqslant 32k$ and $1 \leqslant j \leqslant (16k-1)\binom{32k}{16k} + 1$.

We first two-colour (using colours 1 and 2) the trees $T_1, \ldots, T_{32k}$ so that the colouring of each T_i is bichromatic. Then, for each j, we two-colour T'_j in such a way that the colouring of each T'_j is bichromatic and there are at least $16k$ bichromatic edges between T'_j and $\bigcup_{i=1}^{32k} T_i$, for $1 \leqslant j \leqslant (16k-1)\binom{32k}{16k} + 1$. This is possible since we have two choices for two-colouring of T'_j. Then, by the Pigeonhole Principle, there are $16k$ disjoint trees in $\{T_1, \ldots, T_{32k}\}$, say trees $T_1, \ldots, T_{16k}$, and there are $16k$ disjoint trees in $\{T'_1, \ldots, T'_{(16k-1)\binom{32k}{16k}+1}\}$, say trees $T'_1, \ldots, T'_{16k}$, in

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such a way that each edge between T_i and T'_j is bichromatic (with respect to the colouring of T_i and T'_j, respectively) for $i = 1, \ldots, 16k$ and $j = 1, \ldots, 16k$.

Now let $T^*_i = T_i \cup T'_i$, where $i = 1, \ldots, 16k$. $\bigcup_{i=1}^{16k} T^*_i$ is bipartite and forms an even complete minor of order $16k$ in G. By Theorem 3.2, either G has an odd complete minor of order k or G has a vertex set X of order at most $8k$ such that the (unique) component F that intersects all but at most $8k$ trees of the even minor is bipartite, and each odd cycle is contained in either components of $G - X$ that do not intersect F or blocks with a cut vertex to F. This clearly implies the second conclusion of Theorem 3.1, since X can intersect at most $8k$ trees of the complete bipartite minor.

This completes the proof of Theorem 3.1. □

4. Proof of Theorem 1.4

In this section we fix a positive integer k. We let $N(k) \geqslant N_1(k, (16k-1)\binom{32k}{16k} + 1, 192k + 2)$, where N_1 comes from Theorem 2.2.

The proof of Theorem 1.4 is by induction on $|V(G)|$. For induction purposes, we shall prove the following stronger statement. Let Z be a set of vertices of G. Let us recall that by 'precolouring of Z' we mean that we colour Z so that the graph induced by Z is properly coloured.

Theorem 4.1. *Let k and $N(k)$ be as above. Suppose that G is a graph with no odd K_k-minor, and $Z \subseteq V(G)$ is a vertex set with $|Z| \leqslant 192k + 2$. Suppose Z is precoloured. Then this colouring can be extended to a $496k$-colouring with the following properties.*

(a) *Each colour class has only components whose order is smaller than $N(k)$.*
(b) *If a vertex $v \in V(G) - Z$ is adjacent to a vertex $z \in Z$, then v receives a colour different from the colour of z.*

For the proof, let G be a minimal counter-example to Theorem 4.1. Before the proof, we shall use the following fact.

Fact 4.2. *For every fixed k, every $Z \subseteq V(G)$ with $|Z| \leqslant 192k + 2$, and every precolouring of Z, if G has a separation (A, B) of order at most $96k + 1$ such that both $B - A - Z$ and $A - B - Z$ are non-empty and $|(Z \cap B) \cup (A \cap B)| \leqslant 192k + 2$, and if the precolouring of Z can be extended to a desired colouring of $A \cup Z$ (in the sense of the colouring as described in Theorem 4.1), then the resulting colouring of $(Z \cap B) \cup (A \cap B)$ can be extended to a colouring B in the sense of the colouring as described in Theorem 4.1 (and hence the precolouring of Z can be extended to the whole graph G).*

The proof is clear, since if the precolouring of Z can be extended to a desired colouring of $A \cup Z$, then we can extend the precolouring of $(Z \cap B) \cup (A \cap B)$ to B since $|(Z \cap B) \cup (A \cap B)| \leqslant 192k + 2$, so the induction hypothesis of B with $Z = (Z \cap B) \cup (A \cap B)$ is satisfied. Note that every vertex in $A \cap B$ satisfies requirement (b) under the colouring of B. Therefore, every component of the colour class in the colouring of B is either contained in $A \cup Z$ (and is also a component of the colouring of $A \cup Z$), or is contained in $B - (A \cup Z)$. This shows that the colouring of G satisfies (a). Hence Fact 4.2 is true.

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Proof of Theorem 4.1. We prove by induction on $|V(G)|$. If $V(G) = Z$, there is nothing to prove. In fact, if $|V(G)| < N(k)$, then the result is clearly true since $|Z| \leqslant 192k + 2$, so we can colour all the vertices of $G - Z$ using the colour that is not used in the precolouring of Z. This particularly implies that $G - Z$ has a vertex v. Suppose the degree of v is at most $496k - 1$. Then, by the induction hypothesis, $G - v$ has a desired colouring, and since v has degree at most $496k - 1$, we can colour v so that no neighbour of v receives the same colour of v. So, we may assume that every vertex $v \in V(G - Z)$ has degree at least $496k$.

Suppose that there is a Z-essential separation (A, B) of order at most $96k + 1$. We assume that (A, B) is a minimal Z-essential separation, and we let $S = A \cap B$. Note that the minimum degree of vertices in $V(G - Z)$ is at least $496k$. Since $|S| \leqslant 96k + 1$ and $|Z| \leqslant 192k + 2$, it follows that either $|S \cup (A \cap Z)| \leqslant 192k + 2$ or $|S \cup (B \cap Z)| \leqslant 192k + 2$, say $|S \cup (B \cap Z)| \leqslant 192k + 2$. We first apply induction to the subgraph of G induced on $A \cup Z$ with Z precoloured. Since (A, B) is a Z-essential separation, the subgraph on $A \cup Z$ is smaller than G, and hence the induction hypothesis can be applied.

Next, we apply induction to B with $Z' = S \cup (B \cap Z)$ precoloured, where the precolouring of vertices in S comes from the colouring of $A \cup Z$. Recall that $|Z'| \leqslant 192k + 2$ and B is smaller than G. Finally, the combination of the obtained colourings of B and A yields a desired colouring of G by Fact 4.2.

Therefore, we may now assume that G is $(96k + 2)$-connected relative to Z. Since every vertex in $V(G - Z)$ has degree at least $496k$ and since G has at least $N(k)$ vertices, G contains a $K_{32k,(16k-1)\binom{32k}{16k}+1}$-minor by Theorem 2.2. So, as in the proof of Theorem 3.1, G has an even K_{16k}-minor.

By Theorem 3.1, either G has an odd K_k-minor or G has a vertex set X of order at most $8k$ such that the (unique) block F that intersects all but at most $8k$ trees of the $K_{32k,(16k-1)\binom{32k}{16k}+1}$-minor is bipartite, and each odd cycle is contained in either components of $G - X$ that do not intersect F or blocks with a cut vertex to F. Since we are assuming that G has no odd K_k-minor, we may assume that only the second outcome happens.

Hence we now have one big component W in $G - X$ such that W contains the induced bipartite subgraph F and each odd cycle is contained in either components of $G - X$ that do not intersect F or blocks with a cut vertex to F. But since G is $(96k + 2)$-connected relative to Z, if there is a component B in $G - X$ that does not intersect W, then B consists of vertices in Z since $|X| < 8k$. Similarly, if there is a block B with a cut vertex v to F in $G - X$, then $B - v$ consists of vertices in Z. This observation implies that $F \cup Z \cup Z = G$.

Now, we can colour the vertices in the bipartite $F - Z$ with two colours that are not used in the precolouring of Z. We can also colour all the vertices in $X - Z$ with $|X| \leqslant 8k$ colours that are not used in the precolouring of Z and the two-colouring of $F - Z$. These colourings together with the precolouring of Z give rise to a desired colouring. This completes the proof. □

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