Math. Proc. Camb. Phil. Soc. (1983), **93**, 25
Printed in Great Britain

Homothetic ellipsoids[1]

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(*Received* 3 *May* 1982)

Abstract. We let K_1 and K_2 be two convex bodies in E^d ($d \geqslant 3$) and consider translates K_2' of K_2 for which $\partial K_1 \cap \partial K_2' \neq \varnothing$ and $K_1 \neq K_2'$. We show that if, for all such translates, $\partial K_1 \cap \partial K_2'$ is contained in a hyperplane then K_1 and K_2 are homothetic ellipsoids.

If K_1 and K_2 are two planar non-coincident convex sets whose interiors intersect we define $\alpha(K_1, K_2)$ to be the number of connected components of $\partial K_1 \cap \partial K_2$. The starting point for our investigations is the result of Fujiwara (5) and Bol (2) to the effect that K is a circular disc if and only if $\alpha(K, K') = 2$ for all congruent copies K' of K. An easy extension of this result is the observation that K_1 and K_2 are congruent circular discs if and only if $\alpha(K_1, K_2') = 2$ for all congruent copies K_2' of K_2. When working with such results it is natural to look for analogues in the situation where only translates are allowed rather than all possible congruences. The appropriate analogue was obtained in (7) and we state it here for completeness.

Theorem 1. *Let K_1, K_2 be planar convex bodies. Then $\alpha(K_1, K_2') = 2$ for all translates K_2' of K_2 if and only if K_2 is a translate of K_1.*

As observed in (7), the result of Fujiwara and Bol and its extension follow from this theorem. It is worth noting that the hypothesis of this theorem ensures that there is no translate K_2' of K_2 such that either

$$K_2' \subset \operatorname{int} K_1 \quad \text{or} \quad K_1 \subset \operatorname{int} K_2'.$$

So that in some sense the hypothesis requires the bodies to be of similar size. Perhaps a more natural hypothesis would have been $\alpha(K_1, K_2') \leqslant 2$ for all translates K_2' of K_2. This condition has been studied in (6). To explain the result obtained we need to introduce the notion of freely sliding convex sets. If K_1 and K_2 are convex sets we say that K_2 *slides freely inside* K_1 if for any $\mathbf{x} \in \partial K_1$ there is a translation vector $\mathbf{t}$ such that

$$\mathbf{x} \in K_2 + \mathbf{t} \subset K_1.$$

For example, if K_2 is a summand of K_1 then K_2 slides freely inside K_1. One of the results obtained in (6) can now be stated in the following form.

Theorem 2. *Let K_1, K_2 be planar convex bodies. Then $\alpha(K_1, K_2') \leqslant 2$ and $K_1 \backslash K_2' \neq \varnothing$ for all translates K_2' of K_2 if and only if K_2 slides freely inside K_1.*

We note that this slackening of the hypothesis results in a much weaker relationship between K_1 and K_2.

The theorem of Fujiwara and Bol was used by Yanagihara in (8) to obtain a higher

[1] This work was carried out at Oklahoma University whilst the author was in receipt of a Fulbright–Hays grant.

https://doi.org/10.1017/S0305004100060291 Published online by Cambridge University Press

dimensional analogue. He considered those congruent copies K' of the 3-dimensional convex set K for which $K' \neq K$ and $\partial K' \cap \partial K \neq \varnothing$. He showed that if for all such K' we know that $\partial K' \cap \partial K$ is a planar curve then K is a ball. In (7) we obtained the analogous result in the case of translations. More precisely we obtained the case $d = 3$ of the following theorem.

THEOREM 3. *Let K_1 and K_2 be convex bodies in E^d $(d \geqslant 3)$ for which $K_1 = K_2$ whenever K_2' is a translate of K_2 with either $K_2' \subset K_1$ or $K_1 \subset K_2'$. Then $\partial K_2' \cap \partial K_1$ is contained in a hyperplane for all translates K_2' of K_2 with $K_2' \neq K_1$ if and only if K_1 and K_2 are translates of the same ellipsoid.*

It is interesting to note that although Theorem 1 gives a much weaker relationship between K_1 and K_2 than that obtained by Fujiwara and Bol, when we work in higher dimensions we obtain strong connections between K_1 and K_2 very much in the spirit of Yanagihara's result. We note once more that the hypothesis of Theorem 3 requires the bodies K_1 and K_2 to be of a similar size in some sense. With this in mind we turn our attention to the higher dimensional analogue of Theorem 2. Here again, despite the planar result, we shall obtain a result analogous to that of Yanagihara.

THEOREM 4. *Let K_1 and K_2 be convex bodies in E^d $(d \geqslant 3)$. Then $\partial K_2' \cap \partial K_1$ is contained in a hyperplane for all translates K_2' of K_2 with $K_2' \neq K_1$ if and only if K_1 and K_2 are homothetic ellipsoids.*

We recognize that Theorem 3 is a consequence of Theorem 4, but for the purposes of our proof it is convenient to have them stated as separate results.

If K_1 and K_2 are homothetic ellipsoids then there is a single affine transformation which transforms both K_1 and K_2 into balls. It follows that $\partial K_2' \cap \partial K_1$ is contained in a hyperplane whenever K_2' is a translate of K_2 with $K_2' \neq K_1$. So, in one direction, the proofs of Theorems 3 and 4 are complete. The remainder of this paper is devoted to establishing the reverse implications in both theorems.

PROPOSITION 1. *Let K_1 and K_2 be convex bodies in E^d $(d \geqslant 3)$ such that $\partial K_2' \cap \partial K_1$ is contained in a hyperplane for all translates K_2' of K_2 with $K_2' \neq K_1$. Then if H is a supporting hyperplane of K_1 we have* $\dim (H \cap K_1) < d-1$.

Proof. We assume the proposition is false and seek a contradiction.

So if $F = H \cap K_1$ we have

$$\dim F = d-1.$$

Let $\mathbf{n}$ be the outward unit normal vector to H and denote by J the supporting hyperplane of K_2 with outward normal vector $\mathbf{n}$. Then we may choose a translation vector $\mathbf{t}$ such that

$$J + \mathbf{t} = H,$$

$$\operatorname{relbd} F \cap (\operatorname{relint} (J \cap K_2) + \mathbf{t}) \neq \varnothing,$$

and

$$\operatorname{relbd} F \cap (H \backslash ((J \cap K_2) + \mathbf{t})) \neq \varnothing.$$

Here *relbd* and *relint* denote the relative boundaries and interiors, respectively. So we can find arbitrarily small translation vectors $\mathbf{v}$ such that

$$\operatorname{relbd} F \cap (\operatorname{int} K_2 + \mathbf{t} + \mathbf{v}) \neq \varnothing$$

and

$$\operatorname{relbd} F \cap (H \backslash (K_2 + \mathbf{t} + \mathbf{v})) \neq \varnothing.$$

https://doi.org/10.1017/S0305004100060291 Published online by Cambridge University Press

It follows that H is the hyperplane containing

$$\partial(K_2+\mathbf{t}+\mathbf{v}) \cap \partial K_1.$$

So if

$$\mathbf{x} \in \text{relbd}\, F \cap (\text{int}\, K_2+\mathbf{t}+\mathbf{v}),$$

and

$$\mathbf{y} \in \text{relbd}\, F \cap (H \backslash (K_2+\mathbf{t}+\mathbf{v})),$$

we can choose a path $P \subset \partial K_1$ joining $\mathbf{x}$ and $\mathbf{y}$ with

$$P \cap H = \{\mathbf{x}, \mathbf{y}\}.$$

Consequently

$$(P \cap (\partial K_2+\mathbf{t}+\mathbf{v})) \backslash H \neq \varnothing,$$

which yields a point of

$$\partial(K_2+\mathbf{t}+\mathbf{v}) \cap \partial K_1,$$

which is not in H. This contradiction shows, as required, that K_1 has no $(d-1)$-dimensional faces.

In the sequel, if S is a set contained in a hyperplane we shall denote the $(d-1)$-dimensional volume of S by $v(S)$.

PROPOSITION 2. *Let K_1 and K_2 satisfy the hypothesis of Proposition* 1. *Assume H and J are parallel hyperplanes with*

$$v(H \cap K_1) = v(J \cap K_2) > 0.$$

Then

(i) *if $d = 3$, $H \cap K_1$ is a translate of $J \cap K_2$;*
(ii) *if $d > 3$ and Theorem* 3 *holds in dimension $d-1$, $H \cap K_1$ is a translate of $J \cap K_2$.*

Proof. We put.

$$T = \{\mathbf{t} \in E^d : J+\mathbf{t} = H\}.$$

First we assume that there is a $\mathbf{t} \in T$ such that

$$\dim \text{aff}\,((\text{relbd}\,((J \cap K_2)+\mathbf{t})) \cap (\text{relbd}\,(H \cap K_1))) = d-1, \tag{1}$$

where *aff* denotes the affine hull. Then clearly, either $K_2+\mathbf{t} = K_1$ or H is the hyperplane containing

$$\partial(K_2+\mathbf{t}) \cap \partial K_1.$$

We aim to show that

$$H \cap (K_2+\mathbf{t}) = H \cap K_1. \tag{2}$$

If this were not the case we could use the fact that

$$v(H \cap (K_2+\mathbf{t})) = v(H \cap K_1) > 0, \tag{3}$$

to choose

$$\mathbf{x} \in \text{relbd}\,(H \cap (K_2+\mathbf{t})) \backslash K_1,$$

and

$$\mathbf{y} \in (\text{relbd}\,(H \cap (K_2+\mathbf{t}))) \cap \text{relint}\,(H \cap K_1).$$

But then we can choose a path $P \subset \partial K_2+\mathbf{t}$ joining $\mathbf{x}$ and $\mathbf{y}$ with

$$P \cap H = \{\mathbf{x}, \mathbf{y}\}.$$

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Now Proposition 1 shows that $\mathbf{y} \in \operatorname{int} K_1$ and so it follows that

$$(\partial(K_2+\mathbf{t}) \cap \partial K_1) \backslash H \neq \varnothing .$$

This contradiction establishes (2) and the proof of the proposition follows.

So now we must assume there is no $\mathbf{t} \in T$ satisfying (1). If $d = 3$ it follows from Theorem 2 and (3) that $H \cap K_1$ is a translate of $J \cap K_2$. If $d > 3$ and Theorem 3 holds in dimension $d-1$ then again $H \cap K_1$ is a translate of $J \cap K_2$, in fact both sets are ellipsoids. This completes the proof of the proposition.

PROPOSITION 3. *If K_1 and K_2 satisfy the hypothesis of Proposition 1 then there are no proper line segments in ∂K_1 or ∂K_2 when either of the following conditions is satisfied*:

(i) $d = 3$;

(ii) $d > 3$ *and Theorem 3 holds in dimension* $d-1$.

Proof. First we note that the proof of Proposition 1 could easily be adapted to show that if J is a supporting hyperplane of K_2 then

$$\dim (J \cap K_2) < d-1.$$

So it suffices to use Proposition 1 to show that there are no proper line segments in ∂K_1, the case of ∂K_2 then follows immediately.

We let $[\mathbf{a}, \mathbf{b}]$ be a maximal line segment in ∂K_1 with $\mathbf{a} \neq \mathbf{b}$ and seek a contradiction. Let $\mathscr{H}$ denote the set of all hyperplanes containing $[\mathbf{a}, \mathbf{b}]$ and intersecting $\operatorname{int} K_1$. Choose any point $\mathbf{x} \in \operatorname{int} K_2$ and let $\alpha > 0$ be the smallest $(d-1)$-dimensional volume of a section of K_2 through $\mathbf{x}$. We can use Proposition 1 together with a continuity argument to choose an $H \in \mathscr{H}$ such that

$$v(H \cap K_1) < \alpha.$$

Let J^* be the hyperplane containing $\mathbf{x}$ parallel to H. Using Proposition 1 applied to K_2 we can, by translating J^*, find a hyperplane J parallel to H with

$$v(H \cap K_1) = v(J \cap K_2) > 0. \tag{4}$$

We can now use Proposition 2 to find a $\mathbf{t}$ such that

$$(J \cap K_2) + \mathbf{t} = H \cap K_1.$$

Now we put

$$\mathbf{v} = \tfrac{1}{2}(\mathbf{a}-\mathbf{b}),$$

and let

$$K_2' = K_2 + \mathbf{t} + \mathbf{v}.$$

We can find linearly independent (2-dimensional) planes $P_1, P_2, \ldots, P_{d-2}$ each intersecting $\operatorname{int} K_2'$ and such that

$$[\mathbf{a}, \mathbf{b}] \subset P_i \subset H,$$

for $i = 1, 2, \ldots, d-2$. Then for each $1 \leqslant i \leqslant d-2$ we choose a point

$$\mathbf{x}_i \in (P_i \cap \partial K_1 \cap \partial K_2') \backslash [\mathbf{a}, \mathbf{b}].$$

It follows that

$$\mathbf{b}, \tfrac{1}{2}(\mathbf{a}+\mathbf{b}), \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{d-2} \in H \cap \partial K_1 \cap \partial K_2',$$

with

$$\dim \operatorname{aff} (\mathbf{b}, \tfrac{1}{2}(\mathbf{a}+\mathbf{b}), \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{d-2}) = d-1.$$

Now

$$\mathbf{a} \in (\partial K_1 \backslash K_2') \cap H, \tag{5}$$

https://doi.org/10.1017/S0305004100060291 Published online by Cambridge University Press

and so the hyperplane containing $\partial K_2' \cap \partial K_1$ must be H. It follows from (4) and (5) that we can find a point

$$\mathbf{y} \in H \cap \partial K_1 \cap \operatorname{int} K_2'.$$

Once again we can choose a path $P \subset \partial K_1$ joining $\mathbf{a}$ and $\mathbf{y}$ with

$$P \cap H = \{\mathbf{a}, \mathbf{y}\}.$$

It follows that

$$(\partial K_2' \cap \partial K_1) \backslash H \neq \varnothing,$$

and this contradiction completes our proof.

PROPOSITION 4. (i) *Theorem* 4 *is true in dimension* 3; (ii) *If* $d > 3$ *and Theorem* 3 *holds in dimensions* $d-1$ *and* d, *then Theorem* 4 *is true in dimension* d.

Proof. Since the 3-dimensional version of Theorem 3 was established in (7), we note that if there were no translate K_2' of K_2 with $K_2' \neq K_1$ and either $K_2' \subset K_1$ or $K_1 \subset K_2'$ then the proposition follows immediately from Theorem 3. So we may assume without loss of generality that there is a translate K_2^* of K_2 with

$$K_2^* \neq K_1 \quad \text{and} \quad K_2^* \subset K_1.$$

We shall first show that K_2 is an ellipsoid by proving that each of its shadow boundaries lies in a hyperplane (see (3), p. 142). For each direction $\mathbf{v}$ let $B(\mathbf{v})$ denote the shadow boundary of K_2 in direction $\mathbf{v}$. For any fixed $\mathbf{v}$ and each sufficiently small $\alpha > 0$ we have

$$\operatorname{int} K_2 \cap \operatorname{int} (K_2 + \alpha\mathbf{v}) \neq \varnothing,$$

$$B(\mathbf{v}) \subset \mathscr{C}(K_2 + \alpha\mathbf{v}),$$

and

$$B(\mathbf{v}) + \alpha\mathbf{v} \subset \mathscr{C}K_2.$$

Now Proposition 3 implies that $B(\mathbf{v})$ is a $(d-2)$-manifold separating ∂K_2 into two residual domains. Consequently

$$\dim \operatorname{aff} B(\mathbf{v}) \geqslant d-1. \tag{6}$$

If $\mathbf{x} \in B(\mathbf{v})$ and $\mathbf{y} \in \operatorname{int} K_2 \cap \operatorname{int} (K_2 + \alpha\mathbf{v})$ for all small $\alpha > 0$ we let P denote the (2-dimensional) plane through $\mathbf{x}$ and $\mathbf{y}$ parallel to $\mathbf{v}$. Then

$$P \cap K_2 \quad \text{and} \quad P \cap (K_2 + \alpha\mathbf{v}),$$

are planar convex sets with

$$\mathbf{x} \in \partial(P \cap K_2), \mathbf{x} + \alpha\mathbf{v} \notin P \cap K_2,$$

and

$$\mathbf{y} \in (\operatorname{relint} P \cap K_2) \cap \operatorname{relint} (P \cap (K_2 + \alpha\mathbf{v})).$$

So for each α we can choose

$$\mathbf{z}_\alpha \in [\mathbf{y}, \mathbf{x} + \alpha\mathbf{v}] \cap \partial(P \cap K_2),$$

and we note that

$$\mathbf{z}_\alpha \to \mathbf{x} \quad \text{as} \quad \alpha \to 0.$$

On the arc of $\operatorname{relbd} (P \cap K_2)$ joining $\mathbf{z}_\alpha \in \operatorname{relint} (P \cap (K_2 + \alpha\mathbf{v}))$ and $\mathbf{x} \notin K_2 + \alpha\mathbf{v}$ there is a point

$$\mathbf{w}_\alpha \in (\operatorname{relbd} (P \cap K_2)) \cap \operatorname{relbd} (P \cap (K_2 + \alpha\mathbf{v})).$$

Clearly

$$\mathbf{w}_\alpha \to \mathbf{x} \quad \text{as} \quad \alpha \to 0.$$

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Now

$$\mathbf{w}_\alpha \in \partial K_2 \cap \partial(K_2+\alpha\mathbf{v}),$$

and so, by choosing α sufficiently small, we can find points of

$$\partial K_2 \cap \partial(K_2+\alpha\mathbf{v}),$$

arbitrarily close to $\mathbf{x}$. It follows from (6) that for sufficiently small α we have

$$\dim \operatorname{aff}\{\partial K_2 \cap \partial(K_2+\alpha\mathbf{v})\} \geqslant d-1.$$

So for some such α we may choose points

$$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d \in \partial K_2 \cap \partial(K_2+\alpha\mathbf{v}),$$

with

$$\dim \operatorname{aff}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d\} = d-1.$$

We let L denote the hyperplane containing $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d$ and put

$$J = L-\alpha\mathbf{v}.$$

Then $L \cap K_2$ and $J \cap K_2$ are $(d-1)$-dimensional sections of K_2 and we shall assume without loss of generality that

$$0 < v(L \cap K_2) \leqslant v(J \cap K_2).$$

Since $K_2^* \subset K_1$ we can use a continuity argument and Proposition 2 to choose a hyperplane H such that $H \cap K_1$ is a translate of $L \cap K_2$. Let $\mathbf{t}$ denote the translation vector and put

$$\mathbf{y}_i = \mathbf{x}_i+\mathbf{t} \in \partial K_1,$$

for $i = 1, 2, \ldots, d$. Then

$$\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_d\} \subset \partial K_1 \cap \partial(K_2+\mathbf{t}+\alpha\mathbf{v}),$$

and so

$$\partial K_1 \cap \partial(K_2+\mathbf{t}+\alpha\mathbf{v}) \subset H = J+\mathbf{t}+\alpha\mathbf{v}.$$

Since the boundaries of K_1 and $K_2+\mathbf{t}+\alpha\mathbf{v}$ intersect only in H it follows that either

$$(J \cap K_2)+\mathbf{t}+\alpha\mathbf{v} \subset H \cap K_1$$

or

$$H \cap K_1 \subset (J \cap K_2)+\mathbf{t}+\alpha\mathbf{v}.$$

But the former possibility implies that

$$(J \cap K_2)+\mathbf{t}+\alpha\mathbf{v} = H \cap K_1, \tag{7}$$

since

$$v((J \cap K_2)+\mathbf{t}+\alpha\mathbf{v}) = v(J \cap K_2) \geqslant v(L \cap K_2) = v(H \cap K_1).$$

In the latter case, since $V(K_2) < V(K_1)$ we know that

$$K_1 \backslash (K_2+\mathbf{t}+\alpha\mathbf{v}) \neq \varnothing,$$

and again it follows that (7) holds, since the boundaries of the sets intersect only in H. Consequently $J \cap K_2$ is a translate of $H \cap K_1$ and therefore of $L \cap K_2$.

Now if $\mathbf{x} \in L \cap K_2 \cap B(\mathbf{v})$ then we have $\mathbf{x}-\alpha\mathbf{v} \in B(\mathbf{v})$ and this would give a line segment in ∂K_2, contradicting Proposition 3. It follows that

$$L \cap B(\mathbf{v}) = J \cap B(\mathbf{v}) = \varnothing,$$

https://doi.org/10.1017/S0305004100060291 Published online by Cambridge University Press

and that $B(\mathbf{v})$ lies between the hyperplanes L and J. But since α could be arbitrarily small we know that $B(\mathbf{v})$ lies in a hyperplane and so K_2 is an ellipsoid as required.

Let $\alpha > 0$ be the smallest $(d-1)$-dimensional volume of a section of K_2 through its centre, and let ω_{d-1} denote the $(d-1)$-dimensional volume of the unit $(d-1)$-dimensional ball. We choose any hyperplane H such that the diameter of $H \cap K_1$ is less than $(\alpha/\omega_{d-1})^{1/d-1}$. Our aim is to show that $H \cap K_1$ is centrally symmetric. It will then follow from Theorem 2 of (4) that K_1 is an ellipsoid.

Now clearly

$$v(H \cap K_1) < \omega_{d-1}(\alpha/\omega_{d-1}) = \alpha.$$

So, if $H \cap \operatorname{int} K_1 \neq \varnothing$ we can use the definition of α and Proposition 2 to see that there is a hyperplane J such that $H \cap K_1$ is a translate of $J \cap K_2$. But, of course, all sections of K_2 are centrally symmetric and so $H \cap K_1$ is centrally symmetric, as required. If $H \cap \operatorname{int} K_1 = \varnothing$ then $H \cap K_1$ is either empty or a singleton and is therefore again centrally symmetric.

Finally, let θ be an affine transformation which transforms K_2 into a ball. Then θK_1 and θK_2 satisfy the hypothesis of Proposition 1 and so, by the arguments above, θK_1 is an ellipdoid whose sections of sufficiently small diameter are balls. Hence θK_1 is also a ball. So any transformation which transforms K_2 into a ball also transforms K_1 into a ball. Hence K_1 and K_2 are homothetic ellipsoids.

Proof of Theorem 3. This proof will be carried out by induction on the dimension d. The case $d = 3$ was established in (7). So we shall assume that $d \geqslant 4$ and that the theorem holds in all dimensions less than d and use this hypothesis to establish the d-dimensional case. Our proof will, in fact, closely follow that given in (7).

First we note that in order to show that K_1 is a translate of K_2 it suffices to prove that each projection of K_1 onto a $(d-1)$-dimensional subspace is a translate of the same projection of K_2, see (1). So we let π denote the orthogonal projection of E^d onto some $(d-1)$-dimensional subspace E say, and let $\mathbf{y}$ be a vector orthogonal to E. Our intention is to show that πK_1 is a translate of πK_2.

First we assume there is a translation vector $\mathbf{t}$ such that

$$\pi K_2 + \mathbf{t} \subset \operatorname{relint} \pi K_1. \tag{8}$$

Let the shadow boundary $B(\mathbf{v})$ of K_1 in direction $\mathbf{v}$ yield the two residual domains A and C of ∂K_1. It follows from (8) that by considering translations parallel to $\mathbf{v}$ we can find a translate K_2' of K_2 with

$$\partial K_2' \cap A \neq \varnothing \quad \text{and} \quad \partial K_2' \cap C \neq \varnothing, \tag{9}$$

since the hypothesis of the theorem shows that there is no translation $\mathbf{s}$ such that

$$K_2 + \mathbf{s} \subset \operatorname{int} K_1.$$

It also follows from (8) that

$$\pi K_2' \subset \operatorname{relint} \pi K_1, \tag{10}$$

and so $K_2' \neq K_1$. Consequently we may let H be a hyperplane containing $\partial K_2' \cap \partial K_1$. It follows from (9) and Proposition 3 that

$$H \cap \operatorname{int} K_1 \neq \varnothing \quad \text{and} \quad H \cap \operatorname{int} K_2' \neq \varnothing.$$

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So the convex sets $H \cap K_1$ and $H \cap K_2'$ are $(d-1)$-dimensional. As previously, we can now use the definition of H to deduce that we must have either $H \cap K_1 \subset H \cap K_2'$ or $H \cap K_2' \subset H \cap K_1$. We assume without loss of generality that the former possibility holds. We know from the hypothesis of the theorem that

$$\partial K_1 \backslash K_2' \neq \varnothing.$$

If

$$(\text{relbd}\,(H \cap K_1)) \cap \text{relint}\,(H \cap K_2') \neq \varnothing,$$

we could again obtain a point in

$$(\partial K_1 \cap \partial K_2') \backslash H,$$

which would contradict the definition of H. It follows that

$$H \cap K_1 = H \cap K_2'.$$

But then by (9)

$$H \cap K_2' \cap B(\mathbf{v}) \neq \varnothing,$$

which contradicts (10). We deduce that there is no translation vector $\mathbf{t}$ satisfying (8). A similar argument shows that there is no $\mathbf{t}$ with

$$\pi K_1 \subset \text{relint}\,\pi K_2 + \mathbf{t}. \tag{11}$$

If

$$\dim \text{aff}\,((\text{relbd}\,\pi K_1) \cap (\text{relbd}\,\pi K_2 + \mathbf{t})) \leqslant d-2,$$

for all $\mathbf{t} \in E$ then Proposition 4 shows that πK_2 and πK_1 are homothetic ellipsoids. But since there is no $\mathbf{t}$ satisfying either (8) or (11) we deduce that πK_1 is in fact a translate of πK_2, as required.

We must therefore assume that for some $\mathbf{t} \in E$ we can choose points

$$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d \in (\text{relbd}\,\pi K_1) \cap (\text{relbd}\,\pi K_2 + \mathbf{t}),$$

with

$$\dim \text{aff}\,\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d\} = d-1. \tag{12}$$

We let Q be the hyperplane in E^d defined by

$$\pi Q = \text{aff}\,\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{d-1}\},$$

and consider the sections $Q \cap K_1$ and $Q \cap (K_2 + \mathbf{t})$. We know from Proposition 3 that ∂K_1 contains no proper line segments and so the same must be true of $\text{relbd}\,\pi K_1$. Hence

$$\pi Q \cap \text{relint}\,\pi K_1 \neq \varnothing$$

and so

$$Q \cap \text{int}\,K_1 \neq \varnothing.$$

A similar argument shows that

$$Q \cap (\text{int}\,K_2 + \mathbf{t}) \neq \varnothing.$$

So

$$\dim (Q \cap K_1) = \dim (Q \cap (K_2 + \mathbf{t})) = d-1.$$

If

$$\dim \text{aff}\,((\text{relbd}\,(Q \cap K_1)) \cap (\text{relbd}\,(Q \cap (K_2 + \mathbf{t}) + \mathbf{w})) \leqslant d-2,$$

for all translation vectors $\mathbf{w}$ parallel to Q, then Proposition 4 shows that $Q \cap K_1$ and $Q \cap (K_2 + \mathbf{t})$ are homothetic ellipsoids. Since $d \geqslant 4$ it follows that $\pi(Q \cap K_1)$ and $\pi(Q \cap (K_2 + \mathbf{t}))$ are homothetic $(d-2)$-dimensional ellipsoids. But

$$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{d-1} \in (\text{relbd}\,\pi(Q \cap K_1)) \cap \text{relbd}\,\pi(Q \cap (K_2 + \mathbf{t})),$$

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and

$$\dim \operatorname{aff}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{d-1}\} = d-2.$$

Hence

$$\pi(Q \cap K_1) = \pi(Q \cap (K_2+\mathbf{t})). \tag{14}$$

It follows that $Q \cap K_1$ is in fact a translate of $Q \cap (K_2+\mathbf{t})$. So we can choose a vector $\mathbf{w}$ parallel to Q such that

$$Q \cap K_1 = Q \cap (K_2+\mathbf{t})+\mathbf{w} = Q \cap (K_2+\mathbf{t}+\mathbf{w}). \tag{15}$$

We note from (14) that $\pi\mathbf{w} = \mathbf{0}$. We let Q^+, Q^- be the open half-spaces of E^d separated by Q. It follows from (15) that Q is the hyperplane containing

$$\partial K_1 \cap \partial(K_2+\mathbf{t}+\mathbf{w}),$$

and so we may assume without loss of generality that

$$(K_2+\mathbf{t}+\mathbf{w}) \cap Q^+ \subset \operatorname{int} K_1,$$

and

$$K_1 \cap Q^- \subset \operatorname{int} K_2+\mathbf{t}+\mathbf{w}.$$

But since $\pi\mathbf{w} = \mathbf{0}$, this implies that

$$\pi(K_2+\mathbf{t}) \cap \pi Q^+ \subset \operatorname{relint} \pi K_1,$$

and

$$\pi K_1 \cap \pi Q^- \subset \operatorname{relint} \pi(K_2+\mathbf{t}).$$

On the other hand, (12) and (13) show that

$$((\operatorname{relbd} \pi K_1) \cap (\operatorname{relbd} \pi K_2+\mathbf{t}))\backslash \pi Q \neq \varnothing.$$

This contradiction shows that we must assume that there is a vector $\mathbf{w}$ parallel to Q with

$$\dim \operatorname{aff}((\operatorname{relbd}(Q \cap K_1)) \cap (\operatorname{relbd}(Q \cap (K_2+\mathbf{t})+\mathbf{w})) = d-1.$$

In this case either $K_1 = K_2+\mathbf{t}+\mathbf{w}$ or Q is the hyperplane containing

$$\partial K_1 \cap \partial(K_2+\mathbf{t}+\mathbf{w}).$$

In the latter case it follows that either

$$Q \cap K_1 \subset Q \cap (K_2+\mathbf{t}+\mathbf{w}),$$

or

$$Q \cap (K_2+\mathbf{t}+\mathbf{w}) \subset Q \cap K_1.$$

Since the hypothesis of the theorem guarantees that

$$\partial K_1\backslash(K_2+\mathbf{t}+\mathbf{w}) \neq \varnothing,$$

and

$$\partial(K_2+\mathbf{t}+\mathbf{w})\backslash K_1 \neq \varnothing,$$

we see that neither of these possibilities can occur unless

$$Q \cap K_1 = Q \cap (K_2+\mathbf{t}+\mathbf{w}).$$

But then

$$\pi(Q \cap K_1) = \pi(Q \cap (K_2+\mathbf{t}))+\pi\mathbf{w}.$$

If $1 \leqslant i < j \leqslant d-1$ we let p denote the orthogonal projection of Q onto the line through $\mathbf{x}_i$ and $\mathbf{x}_j$. Then

$$p(Q \cap K_1) = p(Q \cap (K_2+\mathbf{t}))+p\mathbf{w}.$$

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But

$$p(Q \cap K_1) = p(Q \cap (K_2 + \mathbf{t})) = [\mathbf{x}_i, \mathbf{x}_j],$$

and so $p\mathbf{w} = 0$. Since this is true for all such pairs i, j we see from (13) that $\pi\mathbf{w} = \mathbf{0}$. By considering the half-spaces Q^+, Q^- we can again obtain the desired contradiction.

It follows that πK_1 is a translate of πK_2 and thus that K_1 is a translate of K_2. We can now see that K_2 is an ellipsoid by proving that each of its shadow boundaries lies in a hyperplane exactly as in the proof of Proposition 4. This completes the proof of Theorem 3.

We note finally that Proposition 4 now establishes the truth of Theorem 4 also.

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