A GRID WITH NO SPLIT PARALLELEPIPED

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Delaunay (2) has proved the result:

THEOREM. *Let Γ be a two-dimensional grid of mesh Δ. Then we can find four points P_1, P_2, P_3, P_4 of Γ such that $P_1P_2P_3P_4$ is a parallelogram of area Δ with its four vertices in different quadrants.*

Such 'split parallelograms' have been used extensively by Barnes and Swinnerton-Dyer (1) to investigate the inhomogeneous minima of quadratic forms. It is natural to try to generalize the result to three dimensions, and we may make the conjecture:

If Γ is any three-dimensional grid of mesh Δ, then there are eight points $P_1, \ldots, P_8$ of Γ such that $P_1 \ldots P_8$ is a parallelepiped of volume Δ with its eight vertices in different octants.

This conjecture is in fact false; I give the following rather trivial counter-example, in order to save other people the trouble of trying to prove it.

Let Γ_0 be the grid generated by the four points $(1, 1, 1)$, $(-1, -1, 1)$, $(-1, 1, -1)$, $(1, -1, -1)$; *then we cannot find a parallelepiped $P_1 \ldots P_8$ with its vertices at points of Γ_0 satisfying the conditions of the conjecture.* (Note that Γ_0 just consists of all points whose coordinates are odd integers with sum congruent to 3 modulo 4.)

Suppose, on the contrary, that $P_1, \ldots, P_8$ is a parallelepiped satisfying the conditions of the conjecture, and suppose that $P_1, \ldots, P_4$ have their z-coordinates positive. Let P_i^* be the projection of P_i on to the x, y-plane, and suppose that P_i^* and P_{i+4}^* lie in the ith quadrant for $i = 1, \ldots, 4$.

The mesh of Γ_0 is 16, and so the volume of $P_1 \ldots P_8$ is 16. $P_1^*P_2^*P_3^*P_4^*$ and $P_5^*P_6^*P_7^*P_8^*$ both contain the square

$$\mathbf{Q} : |x| \leqslant 1, \quad |y| \leqslant 1$$

of area 4; let the difference between the z-coordinates of P_i and P_{i+4} be ζ; then the volume of $P_1 \ldots P_8$ is at least 4ζ, and is greater than 4ζ unless both $P_1^*P_2^*P_3^*P_4^*$ and $P_5^*P_6^*P_7^*P_8^*$ are just $\mathbf{Q}$. Thus either $\zeta = 4$ and $P_1^*P_2^*P_3^*P_4^*$ and $P_5^*P_6^*P_7^*P_8^*$ are just $\mathbf{Q}$, or else $\zeta < 4$. The first case is easily seen to be impossible, and so we can only have the second case, in which $\zeta = 2$; this implies that none of $P_1, \ldots, P_8$ can have z-coordinate with modulus greater than 1. Similarly, none of $P_1, \ldots, P_8$ can have x- or y-coordinate absolutely greater than 1; but Γ_0 has only four points with $|x| \leqslant 1$, $|y| \leqslant 1$, $|z| \leqslant 1$; and so this last case is impossible too.

REFERENCES

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