Mathematika 65 (2019) 1018–1032 © 2019 University College London
doi:10.1112/S0025579319000299

INTERVALS BETWEEN CONSECUTIVE NUMBERS WHICH ARE SUMS OF TWO SQUARES

ALEXANDER KALMYNIN

Abstract. In this paper, we improve the moment estimates for the gaps between numbers that can be represented as a sum of two squares of integers. We consider a certain sum of Bessel functions and prove the upper bound for its mean value. This bound provides estimates for the γth moments of gaps for all $\gamma \leqslant 2$.

§1. *Introduction.* Let $\mathcal{S} = \{s_0 < s_1 < s_2 < \cdots < s_n < \cdots\} = \{0, 1, 2, 4, 5, 8, \ldots\} \subset \mathbb{Z}$ be the set of all numbers that are expressible as the sum of two squares of integers. The classical area in the research of the properties of this set is the study of gaps between consecutive elements, i.e. the quantity $s_{n+1} - s_n$ or, equivalently, the value distribution of the distance function of $\mathcal{S}$

$$R(x) = \min_{n \in \mathcal{S}} |x - n|$$

for $x \to +\infty$.

The following fact is well known (see [1, 6]).

THEOREM 1. *The inequality*

$$R(x) \ll x^{1/4}$$

holds.

Proof. Let us note that if $f(x) = x - [\sqrt{x}]^2$, where $[y]$ is an integer part of the number y, then $0 \leqslant f(x) \ll \sqrt{x}$ for $x \gg 1$ and $x - f(x) = [\sqrt{x}]^2$ is a square of an integer. Therefore,

$$x^{1/4} \gg f(f(x)) = x - (x - f(x) + f(x) - f(f(x))) \geqslant 0.$$

But $x - f(x)$ and $f(x) - f(f(x))$ are squares of integers, so for some integers a and b we have

$$|x - a^2 - b^2| \ll x^{1/4},$$

which was to be proved. □

This estimate was probably known to L. Euler and, unfortunately, was not improved after that (it is still unknown if the identity $R(x) = o(x^{1/4})$ as $x \to +\infty$ is true). Conjecturally, the correct order of growth of $R(x)$ is much smaller.

Received 30 July 2018, published online 14 August 2019.
MSC (2010): 11E25 (primary), 11B05 (secondary).

https://doi.org/10.1112/S0025579319000299 Published online by Cambridge University Press

CONJECTURE 1. *For any $\varepsilon > 0$, the inequality*

$$R(x) \ll_\varepsilon x^\varepsilon$$

is true.

As for the large values of $R(x)$, by the work [10] of Richards, we have the following.

THEOREM 2. *For any $\varepsilon > 0$, there exist infinitely many positive integers x such that*

$$R(x) > \left(\tfrac{1}{4} - \varepsilon\right) \ln x.$$

In addition to the upper and lower bounds, one can also consider the mean values of our function. The best result of this type was proved by Hooley [5].

THEOREM 3. *For any $0 < \gamma < \frac{5}{3}$, we have*

$$\sum_{s_{n+1} \leqslant x} (s_{n+1} - s_n)^\gamma \ll x (\ln x)^{(\gamma-1)/2}.$$

Therefore, for almost all n the inequality $s_{n+1} - s_n \ll \sqrt{\ln s_n}$ holds. More precisely, we have the following.

COROLLARY 1. *Let $g(x)$ be any function that tends to infinity. Then the number of $s_n \leqslant x$ with $s_{n+1} - s_n \gg g(x)\sqrt{\ln x}$ is $O(x/g(x)^\gamma \sqrt{\ln x})$ for any $\gamma < 5/3$.*

The main goal of the present work is to improve Hooley's theorem, i.e. to prove an analogous estimate for a wider range of values of γ.

THEOREM 4. *For any $1 < \gamma \leqslant 2$, the inequality*

$$\sum_{s_{n+1} \leqslant x} (s_{n+1} - s_n)^\gamma \ll x (\ln x)^{(3/2)(\gamma-1)} \delta(x, \gamma)$$

is true, where

$$\delta(x, \gamma) = \begin{cases} 1 & \text{if } \gamma < 2, \\ \ln x & \text{if } \gamma = 2. \end{cases}$$

Remark 1. Theorem 4 is non-optimal, as for $\gamma < 5/3$ the moment of order γ is bounded from above by $c(\gamma) x (\ln x)^{(\gamma-1)/2}$ for some $c(\gamma) > 0$. Therefore, our estimate is worse than Hooley's theorem for $\gamma < 5/3$, but yields a new and non-trivial estimate for $5/3 \leqslant \gamma \leqslant 2$.

Remark 2. In 1987 Plaksin published an article [8] in which he presented a proof of a result similar to Theorem 3, but for a larger range of γ, namely $0 < \gamma < 2$. However, later his paper was shown to contain some mistakes (see [4]) and attempts to fix the arguments were not successful (see [9]).

https://doi.org/10.1112/S0025579319000299 Published online by Cambridge University Press

§2. *Notation.* Throughout this paper, $f = O(g)$ means that there is an absolute constant $C > 0$ such that $|f| \leqslant Cg$; $f \ll g$ means that $f = O(g)$. The notation $f = o(g)$ as $x \to a$ indicates that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0$$

and $f \sim g$ as $x \to a$ is equivalent to $f - g = o(g)$ as $x \to a$.

For a positive real number M and a real number x, we define the function $\vartheta_M(x)$ by the formula

$$\vartheta_M(x) = \sum_{n \in \mathbb{Z}} \mathrm{e}^{-\pi M(x+n)^2}.$$

If x is a complex number, then the Bessel function of the first kind of order zero is equal to

$$J_0(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}x\cos\varphi}\, d\varphi.$$

For any non-negative integer n, the number of pairs (a, b) of integers with $a^2 + b^2 = n$ is denoted by $r_2(n)$.

For positive real numbers N and M, we define $I(N, M)$, $S(N, M)$ and $J(N, M)$ by the formulas

$$I(N, M) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \vartheta_M(\sqrt{N}\sin\varphi)\vartheta_M(\sqrt{N}\cos\varphi)\, d\varphi,$$
$$S(N, M) = \sum_{n \geqslant 0} r_2(n) J_0(2\pi\sqrt{Nn})\mathrm{e}^{-\pi n/M}$$

and

$$J(N, M) = \int_0^N (S(x, M) - 1)^2\, dx.$$

§3. *Proof of the main theorem.* In this section, we will prove Theorem 4 by reducing the original problem to a question about the distribution of values of a certain sum of Bessel functions.

To do this, we need some lemmas. Let us start with the transformation formula for the theta function.

LEMMA 1. *Let M be a positive real number. For any real x, the equality*

$$\vartheta_M(x) = \sum_{n \in \mathbb{Z}} \mathrm{e}^{-\pi M(x+n)^2} = \frac{1}{\sqrt{M}} \sum_{m \in \mathbb{Z}} \mathrm{e}^{2\pi \mathrm{i} m x} \mathrm{e}^{-\pi m^2/M}$$

is true.

https://doi.org/10.1112/S0025579319000299 Published online by Cambridge University Press

Proof. Consider the function $g(x) = \mathrm{e}^{-\pi M x^2}$ on the real line. It is easy to show that g is a Schwartz function and

$$\vartheta_M(x) = \sum_{n\in\mathbb{Z}} g(x+n).$$

By the Poisson summation formula, for any $x \in \mathbb{R}$, we get

$$\sum_{n\in\mathbb{Z}} g(x+n) = \sum_{m\in\mathbb{Z}} \mathrm{e}^{2\pi \mathrm{i} m x}\widehat{g}(m),$$

where $\widehat{g}(\xi) = \int_{\mathbb{R}} g(x)\mathrm{e}^{-2\pi \mathrm{i}\xi x}dx$ is a Fourier transform of our function.

On the other hand, it is well known that

$$\int_{\mathbb{R}} \mathrm{e}^{-\pi M x^2 - 2\pi \mathrm{i}\xi x}\,dx = \frac{1}{\sqrt{M}}\mathrm{e}^{-\pi\xi^2/M}.$$

Using this relation, we obtain the required result. □

With the help of Lemma 1, we will prove the following identity, which will be crucial for the subsequent considerations.

LEMMA 2. *Let M and N be some positive real numbers. Then*

$$\begin{aligned} I(N,M) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \vartheta_M(\sqrt{N}\sin\varphi)\vartheta_M(\sqrt{N}\cos\varphi)\,d\varphi \\ &= \frac{1}{M}\sum_{n\geqslant 0} r_2(n) J_0(2\pi\sqrt{Nn})\mathrm{e}^{-\pi n/M} = \frac{1}{M}S(N,M). \end{aligned}$$

Proof. Using Lemma 1, we find that

$$\begin{aligned} I(N,M) = \frac{1}{2\pi M}\int_{-\pi}^{\pi} &\left(\sum_{a\in\mathbb{Z}} \mathrm{e}^{2\pi \mathrm{i} a\sqrt{N}\sin\varphi}\mathrm{e}^{-\pi a^2/M}\right) \\ &\times \left(\sum_{b\in\mathbb{Z}} \mathrm{e}^{2\pi \mathrm{i} b\sqrt{N}\cos\varphi}\mathrm{e}^{-\pi b^2/M}\right) d\varphi. \end{aligned}$$

Both series are absolutely and uniformly convergent, so we can replace the product of their sums by the double sum and interchange summation and integration. Consequently,

$$I(N,M) = \frac{1}{2\pi M}\sum_{(a,b)\in\mathbb{Z}} \mathrm{e}^{-(a^2+b^2)\pi/M}\int_{-\pi}^{\pi} \mathrm{e}^{2\pi \mathrm{i}\sqrt{N}(a\sin\varphi + b\cos\varphi)}\,d\varphi.$$

Let us now compute the inner integral for all integers a and b. If $(a,b) = (0,0)$, then the integrand is equal to 1 and so the integral equals 2π. If, in contrast, $a^2 + b^2 \neq 0$, then there exists some $\theta \in [-\pi, \pi]$ such that

$$\sin\theta = \frac{a}{\sqrt{a^2+b^2}}, \qquad \cos\theta = \frac{b}{\sqrt{a^2+b^2}}.$$

https://doi.org/10.1112/S0025579319000299 Published online by Cambridge University Press

Therefore, $a\sin\varphi + b\cos\varphi = \sqrt{a^2+b^2}\cos(\varphi-\theta)$. Applying the change of variables $\varphi = \varphi_1 + \theta$, we obtain

$$\int_{-\pi}^{\pi} \mathrm{e}^{2\pi \mathrm{i}\sqrt{N}(a\sin\varphi + b\cos\varphi)}\,d\varphi = \int_{-\pi-\theta}^{\pi-\theta} \mathrm{e}^{2\pi\mathrm{i}\sqrt{N(a^2+b^2)}\cos\varphi_1}\,d\varphi_1.$$

Since the integrand is periodic with the period 2π, the last integral is equal to the integral of the same function over the interval $[-\pi, \pi]$. Thus, we finally find that

$$\begin{aligned}\int_{-\pi}^{\pi} \mathrm{e}^{2\pi \mathrm{i}\sqrt{N}(a\sin\varphi + b\cos\varphi)}\,d\varphi &= \int_{-\pi}^{\pi} \mathrm{e}^{2\pi\mathrm{i}\sqrt{N(a^2+b^2)}\cos\varphi_1}\,d\varphi_1 \\ &= 2\pi J_0(2\pi\sqrt{N(a^2+b^2)}).\end{aligned}$$

Substituting the obtained result into the formula for $I(N, M)$, we deduce the identity

$$I(N,M) = \frac{1}{M}\sum_{(a,b)\in\mathbb{Z}} J_0(2\pi\sqrt{N(a^2+b^2)})\mathrm{e}^{-\pi(a^2+b^2)/M} = \frac{1}{M}S(N,M),$$

which was to be proved. □

Remark 3. Computing the integral $I(N, M)$ without using Lemma 1 yields

$$I(N,M) = \mathrm{e}^{-\pi NM}\sum_{n\geqslant 0} r_2(n) I_0(2\pi M\sqrt{Nn})\mathrm{e}^{-\pi nM}.$$

Therefore, for all $N, M > 0$, we have

$$M\mathrm{e}^{-\pi NM}\sum_{n\geqslant 0} r_2(n) I_0(2\pi M\sqrt{Nn})\mathrm{e}^{-\pi nM} = \sum_{n\geqslant 0} r_2(n) J_0(2\pi\sqrt{Nn})\mathrm{e}^{-\pi n/M}.$$

Now, if for all $z, \tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$, we have

$$f(\tau, z) = \sum_{n\geqslant 0} r_2(n) J_0(4\pi\sqrt{n}z)\mathrm{e}^{\pi i n\tau},$$

then we get $S(N, M) = f(i/M, \sqrt{N}/2)$ and by the identity principle

$$f\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = -\mathrm{i}\tau\exp\left(\frac{4\pi\mathrm{i}z^2}{\tau}\right) f(\tau, z).$$

So, the function $f(\tau, z)$ behaves like a Jacobi form. This phenomenon can be generalized to the Cohen–Kuznetsov series and provides some results on Rankin–Cohen brackets of modular forms (see [7] and [3]).

Now we want to show that if $R(N)$ is large, then the quantity $S(N, M)$ is rather small. To prove this, we will need some more additional lemmas.

https://doi.org/10.1112/S0025579319000299 Published online by Cambridge University Press

LEMMA 3. *Let $N > 0$, $M > 1$ and denote by $C(N) \subset \mathbb{R}^2$ the circle of radius $\sqrt{N}$ centered in the point $(0, 0)$. Let $d(N)$ be the distance between the sets $C(N)$ and $\mathbb{Z}^2$, that is,*

$$d(N) = \inf_{\substack{x \in C(N) \\ y \in \mathbb{Z}^2}} \rho(x, y),$$

where ρ is a standard euclidean distance on the plane. Then the inequality

$$S(N, M) \leqslant M\mathrm{e}^{-\pi M d(N)^2} + O(M\mathrm{e}^{-\pi M/4})$$

holds.

Proof. Let us notice first that for any $\varphi \in [-\pi, \pi]$, the inequality

$$\|\sqrt{N}\cos\varphi\|^2 + \|\sqrt{N}\sin\varphi\|^2 \geqslant d(N)^2$$

holds. Indeed, for some integers a and b, we have

$$\begin{aligned}\|\sqrt{N}\cos\varphi\|^2 + \|\sqrt{N}\sin\varphi\|^2 &= (\sqrt{N}\cos\varphi - a)^2 + (\sqrt{N}\sin\varphi - b)^2 \\ &= \rho^2((\sqrt{N}\cos\varphi, \sqrt{N}\sin\varphi), (a, b)) \geqslant d(N)^2,\end{aligned}$$

because $(\sqrt{N}\cos\varphi, \sqrt{N}\sin\varphi) \in C(N)$ and $(a, b) \in \mathbb{Z}^2$. Then from Lemma 2 we obtain

$$\begin{aligned}S(N, M) &= \frac{M}{2\pi}\int_{-\pi}^{\pi} \vartheta_M(\sqrt{N}\cos\varphi)\vartheta_M(\sqrt{N}\sin\varphi)\,d\varphi \\ &= \frac{M}{2\pi}\int_{-\pi}^{\pi} (\mathrm{e}^{-\pi M\|\sqrt{N}\cos\varphi\|^2} + O(\mathrm{e}^{-\pi M/4})) \\ &\quad \times (\mathrm{e}^{-\pi M\|\sqrt{N}\sin\varphi\|^2} + O(\mathrm{e}^{-\pi M/4}))\,d\varphi.\end{aligned}$$

Therefore, from the previous observation, we get the desired inequality. □

It turns out that the distance between $C(N)$ and $\mathbb{Z}^2$ is closely related to the distance from N to the nearest sum of two squares. More precisely, the following proposition holds.

LEMMA 4. *For any $N > 0$, we have*

$$d(N) \geqslant \frac{2R(N)}{5\sqrt{N}}.$$

Proof. Let $O = (0, 0)$ and $A \in C(N)$, $B \in \mathbb{Z}^2$ be the points such that

$$d(N) = \rho(A, B)$$

(points with this condition certainly exist, because $C(N)$ is compact and $\mathbb{Z}^2$ is discrete). Then the points A, B and O are collinear, because if this is not the case then the triangle ABO is non-degenerate and thus

$$\rho(A, B) > |\rho(A, O) - \rho(B, O)|.$$

https://doi.org/10.1112/S0025579319000299 Published online by Cambridge University Press

But then if we take D to be the intersection point of the line BO and the circle $C(N)$, which lies closer to B than the second point, we get

$$\rho(D, B) = |\rho(D, O) - \rho(B, O)| = |\rho(A, O) - \rho(B, O)| < \rho(A, B),$$

which is a contradiction. Therefore,

$$d(N) = \rho(A, B) = |\rho(A, O) - \rho(B, O)| = |\sqrt{N} - \sqrt{a^2 + b^2}|$$

for some integers a and b. So, we get

$$d(N) = \frac{|N - a^2 - b^2|}{\sqrt{N} + \sqrt{a^2 + b^2}}$$

and we clearly can assume that $a^2 + b^2 \leqslant 2N$, as otherwise we have

$$d(N) \geqslant (\sqrt{2} - 1)\sqrt{N} > \frac{2}{5}\sqrt{N} \geqslant \frac{2R(N)}{5\sqrt{N}},$$

because $R(N) \leqslant |N - 0| = N$. Now, if $a^2 + b^2 \leqslant 2N$, then $\sqrt{N} + \sqrt{a^2 + b^2} \leqslant (\sqrt{2} + 1)\sqrt{N} < 5\sqrt{N}/2$. By the definition of $R(N)$, we also have

$$|N - a^2 - b^2| \geqslant R(N)$$

and so

$$d(N) \geqslant \frac{R(N)}{5\sqrt{N}/2} = \frac{2R(N)}{5\sqrt{N}}.$$ □

A combination of Lemmas 3 and 4 gives us the following.

LEMMA 5. *Let $H > 1, N > 3$ and suppose that $R(N) \geqslant H$. If $M \geqslant (2N \ln N)/H^2$, then*

$$S(N, M) \ll N^{-1/200}.$$

Proof. From Lemma 3, we get

$$S(N, M) \ll M\mathrm{e}^{-\pi M d(N)^2} + M\mathrm{e}^{-\pi M/4}.$$

As $R(N) \ll N^{1/4}$, the inequality

$$M \gg \frac{N \ln N}{R(N)^2} \gg \sqrt{N} \ln N$$

holds and so

$$M\mathrm{e}^{-\pi M/4} \ll N^{-1/200}.$$

Now, Lemma 4 gives us

$$d(N) \geqslant \frac{2R(N)}{5\sqrt{N}} \geqslant \frac{2H}{5\sqrt{N}}.$$

https://doi.org/10.1112/S0025579319000299 Published online by Cambridge University Press

Consider the function

$$t(M) = M\mathrm{e}^{-\pi d(N)^2 M}.$$

We have

$$t'(M) = (1 - \pi d(N)^2 M)t(M)/M,$$

so the quantity $t(M)$ is decreasing when

$$M > \frac{1}{\pi d(N)^2}.$$

Furthermore, from Lemma 4, we get

$$M \geqslant \frac{2N \ln N}{H^2} \geqslant \frac{25N}{4\pi H^2} \geqslant \frac{1}{\pi d(N)^2}.$$

Therefore,

$$t(M) \leqslant t\left(\frac{2N \ln N}{H^2}\right) \ll N\mathrm{e}^{-2\pi N d(N)^2 \ln N/H^2}.$$

And now we have by the previous estimates

$$\frac{2\pi N d(N)^2 \ln N}{H^2} \geqslant \frac{8\pi}{25} \ln N > 1.005 \ln N,$$

so

$$t(M) \ll N^{-0.005} = N^{-1/200}.$$

Consequently,

$$S(N, M) \ll t(M) + M\mathrm{e}^{-\pi M/4} \ll N^{-1/200},$$

which was to be proved. □

So, if $R(N)$ is sufficiently large, then the quantity $S(N, M)$ is close to 0. But the function $J_0(x)$ is oscillating for $x \to \infty$ and so $S(x, M) - 1$ is an infinite sum of certain oscillating functions. Therefore, because of possible cancellation it is reasonable to expect that $S(x, M)$ is close to 1. More precisely, the following L^2-estimate holds.

LEMMA 6. *Let $N \geqslant 2$, $H \geqslant 40(\ln N)^{3/2}$ and $M = (2N \ln N)/H^2 \geqslant 1$. Then the inequality*

$$J(N, M) = \int_0^N (S(x, M) - 1)^2 \, dx \ll \sqrt{NM} \ln N$$

holds.

Remark 4. The constant 40 here is not optimal, but this has no effect on subsequent considerations.

https://doi.org/10.1112/S0025579319000299 Published online by Cambridge University Press

To prove this lemma, we need three more propositions. The first of them is very well known.

LEMMA 7. *For any $x \geqslant 1$, we have*

$$\sum_{0<n<x} r_2^2(n) \ll x \ln x.$$

Proof. See [2, Theorem 1]. □

We also need Weber's second exponential integral.

LEMMA 8. *For arbitrary $\alpha, \beta, \gamma > 0$, the formula*

$$\int_0^{+\infty} \mathrm{e}^{-\alpha x} J_0(2\beta\sqrt{x}) J_0(2\gamma\sqrt{x})\, dx = \frac{1}{\alpha} I_0\left(\frac{2\beta\gamma}{\alpha}\right) \exp\left(-\frac{\beta^2+\gamma^2}{\alpha}\right)$$

is true, where $I_0(x) = J_0(\mathrm{i}x)$ is the modified Bessel function.

Proof. See [11, p. 395, §13.31]. □

Furthermore, we will use the asymptotic formula for the modified Bessel function.

LEMMA 9. *For $x \to +\infty$, we have the following asymptotic formula:*

$$I_0(x) \sim \frac{\mathrm{e}^x}{\sqrt{2\pi x}}.$$

Proof. See [11, p. 203, §7.23]. □

Proof of Lemma 6. Let us define

$$J^*(N, M) = \int_0^{+\infty} (S(x, M) - 1)^2 \mathrm{e}^{-\pi x/N}\, dx.$$

For any $x \leqslant N$, we have $\mathrm{e}^{-\pi x/N} \geqslant \mathrm{e}^{-\pi}$, so for all N and M we get

$$J(N, M) \leqslant \mathrm{e}^{\pi} J^*(N, M)$$

and so it is enough to prove an analogous estimate for $J^*(N, M)$.

Now, by the definition of $S(x, M)$, we have

$$S(x, M) - 1 = \sum_{n \geqslant 1} r_2(n) J_0(2\pi\sqrt{nx}) \mathrm{e}^{-\pi n/M}$$

and the series converge absolutely and uniformly for $x \geqslant 0$. Therefore, we obtain

$$\begin{aligned} J^*(N, M) &= \int_0^{+\infty} (S(x, M) - 1)^2 \mathrm{e}^{-\pi x/N}\, dx \\ &= \sum_{n,m \geqslant 1} r_2(n) r_2(m) \mathrm{e}^{-\pi(n+m)/M} \\ &\quad \times \int_0^{+\infty} J_0(2\pi\sqrt{mx}) J_0(2\pi\sqrt{nx}) \mathrm{e}^{-\pi x/N}\, dx. \end{aligned}$$

https://doi.org/10.1112/S0025579319000299 Published online by Cambridge University Press

The integrals can be computed using the change of variables $x = (N/\pi)y$ and Lemma 8 as follows:

$$\begin{aligned}\int_0^{+\infty} & J_0(2\pi\sqrt{mx})J_0(2\pi\sqrt{nx})\mathrm{e}^{-\pi x/N}\,dx \\ &= \frac{N}{\pi}\int_0^{+\infty} J_0(2\sqrt{\pi nNy})J_0(2\sqrt{\pi mNy})\mathrm{e}^{-y}\,dy \\ &= \frac{N}{\pi}\mathrm{e}^{-\pi N(n+m)}I_0(2\pi N\sqrt{nm}).\end{aligned}$$

Therefore,

$$J^*(N, M) = \frac{N}{\pi}\sum_{n,m\geqslant 1} r_2(n)r_2(m)\mathrm{e}^{-\pi N(n+m)-\pi(n+m)/M}I_0(2\pi N\sqrt{nm}).$$

We can split this sum into "diagonal" and "non-diagonal" summands as follows:

$$J^*(N, M) = \frac{N}{\pi}(S_0 + S_1),$$

where

$$S_0 = \sum_{n\geqslant 1} r_2(n)^2\mathrm{e}^{-2\pi n/M}I_0(2\pi nN)\mathrm{e}^{-2\pi nN}$$

and

$$S_1 = \sum_{\substack{n,m\geqslant 1\\ n\neq m}} r_2(n)r_2(m)\mathrm{e}^{-\pi N(n+m)-\pi(n+m)/M}I_0(2\pi N\sqrt{nm}).$$

Let us estimate S_1 first. Note that all the summands are positive and for any pair (n, m) at least one of three pairs of inequalities is true:

$$\begin{aligned} 1 \leqslant n, &\qquad m \leqslant 100M\ln N,\\ n \geqslant 1, &\qquad m > 100M\ln N\end{aligned}$$

or

$$m \geqslant 1, \qquad n > 100M\ln N.$$

Thus, we deduce that

$$\begin{aligned} S_1 \leqslant & \sum_{\substack{1\leqslant n,m\leqslant 100M\ln N\\ n\neq m}} r_2(n)r_2(m)\mathrm{e}^{-\pi N(n+m)-\pi(n+m)/M}I_0(2\pi N\sqrt{nm})\\ & + 2\sum_{n\geqslant 1, m>100M\ln N} r_2(n)r_2(m)\mathrm{e}^{-\pi N(n+m)-\pi(n+m)/M}I_0(2\pi N\sqrt{nm})\\ &= S_2 + 2S_3.\end{aligned}$$

To estimate the sum S_3, let us notice that Lemma 9 gives us

$$\mathrm{e}^{-\pi N(n+m)}I_0(2\pi N\sqrt{nm}) \ll \mathrm{e}^{-\pi N(n+m)+2\pi N\sqrt{nm}} = \mathrm{e}^{-\pi N(\sqrt{n}-\sqrt{m})^2} \leqslant 1.$$

https://doi.org/10.1112/S0025579319000299 Published online by Cambridge University Press

And so we obtain the following inequality:

$$S_3 \ll \sum_{n\geqslant 1, m>100M\ln N} r_2(n)r_2(m)\mathrm{e}^{-\pi(n+m)/M} = \Big(\sum_{n\geqslant 1} r_2(n)\mathrm{e}^{-\pi n/M}\Big)\Big(\sum_{n>100M\ln N} r_2(n)\mathrm{e}^{-\pi n/M}\Big).$$

Now, for any $x > 0$, we have

$$\sum_{n\leqslant x} r_2(n) \ll x;$$

therefore, dividing $\mathbb{R}_{>0}$ into intervals of the form $(mM, (m+1)M]$ and using the fact that for $n \in (mM, (m+1)M]$ we have $\mathrm{e}^{-\pi n/M} < \mathrm{e}^{-\pi m}$, we obtain

$$\sum_{n\geqslant 1} r_2(n)\mathrm{e}^{-\pi n/M} \leqslant \sum_{m=0}^{+\infty} \mathrm{e}^{-\pi m} \sum_{n\leqslant (m+1)M} r_2(n) \ll \sum_{m=0}^{+\infty} (m+1)\mathrm{e}^{-m}M \ll M$$

and

$$\begin{aligned}\sum_{n>100M\ln N} r_2(n)\mathrm{e}^{-\pi n/M} &\leqslant \sum_{m=100\ln N}^{+\infty} \mathrm{e}^{-\pi m} \sum_{n\leqslant (m+1)M} r_2(n)\\ &\ll \sum_{m=100\ln N}^{+\infty} \mathrm{e}^{-\pi m}(m+1)M\\ &\ll M\ln N\mathrm{e}^{-100\pi\ln N} \ll N^{-313},\end{aligned}$$

so we finally get

$$S_3 \ll MN^{-313} \ll N^{-312}.$$

Now we turn to S_2. For any integer pair n, m with the conditions $1 \leqslant n, m \leqslant 100M\ln N$, we get by Lemma 9 that

$$\mathrm{e}^{-\pi N(n+m)} I_0(2\pi N\sqrt{nm}) \ll \mathrm{e}^{-\pi N(n+m-2\sqrt{nm})}.$$

Let us notice that as $n \neq m$, we have

$$n+m-2\sqrt{nm} = (\sqrt{n}-\sqrt{m})^2 = \frac{(n-m)^2}{(\sqrt{n}+\sqrt{m})^2} \geqslant \frac{1}{400M\ln N}.$$

Consequently,

$$-\pi N(n+m) + 2\pi N\sqrt{nm} \leqslant -\frac{\pi N}{400M\ln N}.$$

From this inequality and the conditions $M = (2N\ln N)/H^2$ and $H \geqslant 40(\ln N)^{3/2}$, we obtain

$$-\pi N(n+m) + 2\pi N\sqrt{nm} \leqslant -\frac{\pi H^2}{800\ln^2 N} \leqslant -\frac{1600\pi\ln^3 N}{800\ln^2 N} \leqslant -2\pi\ln N.$$

https://doi.org/10.1112/S0025579319000299 Published online by Cambridge University Press

Therefore,

$$S_2 \ll \sum_{n,m \leqslant 100M \ln N} r_2(n) r_2(m) N^{-2\pi} = \left(\sum_{n \leqslant 100M \ln N} r_2(n) \right)^2 N^{-2\pi}$$
$$\ll N^{2-2\pi} \ll N^{-4}.$$

And we deduce for the "non-diagonal" summand the estimate

$$S_1 \ll N^{-4} + N^{-312} \ll N^{-4}.$$

It remains to prove that

$$N S_0 \ll \sqrt{NM} \ln N.$$

From Lemma 9, we deduce that

$$\mathrm{e}^{-2\pi Nn} I_0(2\pi Nn) \ll \frac{1}{\sqrt{Nn}}.$$

Consequently,

$$S_0 \ll \sum_{n \leqslant 100M \ln N} \frac{r_2(n)^2}{\sqrt{Nn}} \mathrm{e}^{-\pi n/M}.$$

Let K be the smallest integer with $2^{K+1} > 100M \ln N$. Then we obviously have

$$S_0 \ll \sum_{k=0}^{K} S_0(k),$$

where

$$S_0(k) = \sum_{2^k \leqslant n < 2^{k+1}} \frac{r_2(n)^2}{\sqrt{Nn}} \mathrm{e}^{-\pi n/M}.$$

From Lemma 7, we get

$$S_0(k) \leqslant \sum_{2^k \leqslant n < 2^{k+1}} \frac{r_2(n)^2}{\sqrt{N} 2^{k/2}} \mathrm{e}^{-2^k \pi/M}$$
$$\leqslant \frac{\mathrm{e}^{-2^k\pi/M}}{\sqrt{N} 2^{k/2}} \sum_{n < 2^{k+1}} r_2(n)^2 \ll \frac{(k+1) 2^{k/2} \mathrm{e}^{-2^k\pi/M}}{\sqrt{N}}.$$

Summing this over all $k \leqslant K$, we obtain

$$S_0 \ll \sqrt{\frac{M}{N}} \ln N.$$

Consequently,

$$J^*(N, M) = \frac{N}{\pi}(S_0 + S_1) \ll \sqrt{NM} \ln N + N^{-3} \ll \sqrt{NM} \ln N$$

and

$$J(N, M) \leqslant \mathrm{e}^{\pi} J^*(N, M) \ll \sqrt{NM} \ln N,$$

which concludes the proof. □

https://doi.org/10.1112/S0025579319000299 Published online by Cambridge University Press

LEMMA 10. *Let $x, H > 3$ and denote by $E(H, x)$ the set of all real $y \leqslant x$ with $R(y) \geqslant H$. Then the inequality*

$$\mu(E(H, x)) \ll \frac{x(\ln x)^{3/2}}{H}$$

holds, where μ is the Lebesgue measure.

Proof. As this bound is trivial for $H \leqslant 40(\ln x)^{3/2}$, we can assume that $H \geqslant 40(\ln x)^{3/2}$. Then from Lemma 5 we get

$$|S(y, M)| \ll y^{-1/200}$$

for any $y \in E(H, x)$, where $M = (2N \ln N)/H^2$. Therefore, for large enough $y \in E(H, x)$, we have

$$|S(y, M) - 1| \gg 1.$$

Thus,

$$J(x, M) \geqslant \int_{E(x,H)} (S(y, M) - 1)^2 dy \gg \mu(E(x, H)) - O(1).$$

Consequently, by Lemma 6,

$$\mu(E(x, H)) \ll \sqrt{xM} \ln x + O(1) = \frac{\sqrt{2}x(\ln x)^{3/2}}{H} + O(1) \ll \frac{x(\ln x)^{3/2}}{H},$$

which was to be proved. □

From this last lemma we deduce Theorem 4.

Proof of Theorem 4. Observe that $\sum_{s_{n+1} \leqslant x} (s_{n+1} - s_n)^\gamma \ll \int_0^x R(t)^{\gamma-1} dt$. Indeed, for any positive integer n, we have

$$\int_{s_n}^{s_{n+1}} R(t)^{\gamma-1}\, dt = \int_{s_n}^{s_{n+1}} \min(t - s_n, s_{n+1} - t)^{\gamma-1}\, dt = \frac{(s_{n+1} - s_n)^\gamma}{2^{\gamma-1}\gamma}.$$

Summation over all n with the condition $s_{n+1} \leqslant x$ gives the desired result.

Let k be some positive integer. Consider the set $B_k \subset [0, x]$ of all real y with $2^k \leqslant R(y) \leqslant 2^{k+1}$.

Due to Lemma 10, $\mu(B_k) \ll x(\ln x)^{3/2} 2^{-k}$. Therefore,

$$\int_{B_k} R(t)^{\gamma-1}\, dt \leqslant 2^{(k+1)(\gamma-1)} \mu(B_k) \ll x(\ln x)^{3/2} 2^{(\gamma-2)k}.$$

Let B be the set of $y \in [0, x]$ such that $R(y) \leqslant c(\ln x)^{3/2}$ with some sufficiently large constant c. Trivially, we have $B \subset [0, x]$ and so $\mu(B) \leqslant x$. Consequently,

$$\int_B R(t)^{\gamma-1} dt \ll \mu(B)(\ln x)^{(3/2)(\gamma-1)} \leqslant x(\ln x)^{(3/2)(\gamma-1)}.$$

https://doi.org/10.1112/S0025579319000299 Published online by Cambridge University Press

Due to the fact that on the interval $[0, x]$ the inequality $R(x) \leqslant x$ holds, we have

$$[0, x] = B \cup \bigcup_{x \geqslant 2^k \geqslant c(\ln x)^{3/2}} B_k.$$

Therefore,

$$\int_0^x R(t)^{\gamma-1}\, dt = \int_B R(t)^{\gamma-1}\, dt + \sum_{x \geqslant 2^k \geqslant c(\ln x)^{3/2}} \int_{B_k} R(t)^{\gamma-1}\, dt.$$

Furthermore, we have

$$\begin{aligned}\int_0^x R(t)^{\gamma-1}\, dt &\ll x(\ln x)^{(3/2)(\gamma-1)} + \sum_{x \geqslant 2^k \geqslant c(\ln x)^{3/2}} x(\ln x)^{3/2} 2^{(\gamma-2)k} \\ &\ll x(\ln x)^{(3/2)(\gamma-1)} \delta(x, \gamma).\end{aligned}$$

This concludes the proof of Theorem 4. □

§4. *Conclusion.* In this work, we constructed a certain sum of Bessel functions that is unusually small in the points that are far from numbers that are sums of two squares. The estimate for some mean value of this sum allowed us to prove the upper bound for the measure of the set of points with this property. However, we were not able to prove a sharp enough bound for the function $S(N, M) - 1$. Non-trivial estimates for this quantity would allow us to improve the exponent in the inequality $R(x) \ll x^{1/4}$. One can show that our construction works not only for the sums of two squares, but also for the set of values of arbitrary positive-definite quadratic form with integer coefficients. It would also be interesting to generalize this construction to the case of indefinite forms.

Acknowledgements. The author thanks Maxim Aleksandrovich Korolev for useful comments and discussion. The author is partially supported by the Laboratory of Mirror Symmetry NRU HSE, RF Government grant, agreement no. 14.641.31.0001, the Simons Foundation, the Moebius Contest Foundation for Young Scientists, the "Young Russian Mathematics" contest and the Foundation for the Advancement of Theoretical Physics and Mathematics "BASIS".

References

1. R. P. Bambah and S. Chowla, On numbers which can be expressed as a sum of two squares. *Proc. Nat. Acad. Sci. India* **13** (1947), 101–103.
2. V. Blomer and A. Granville, Estimates for representation numbers of quadratic forms. *Duke Math. J.* **135**(2) (2006), 261–302.
3. H. Cohen, Sums involving the values at negative integers of L-functions of quadratic characters. *Math. Ann.* **217** (1975), 271–285.
4. G. Harman, Sums of two squares in short intervals. *Proc. Lond. Math. Soc.* **62** (1991), 225–241.
5. C. Hooley, On the intervals between numbers that are sums of two squares I. *Acta Math.* **127** (1971), 279–297.
6. A. A. Karatsuba, Euler and number theory. *Proc. Steklov Inst. Math.* **274**(1) (2011), 169–179.

https://doi.org/10.1112/S0025579319000299 Published online by Cambridge University Press

7. N. V. Kuznetsov, A new class of identities for the Fourier coefficients of modular forms. *Acta Arith.* **27** (1975), 505–519 (in Russian).
8. V. A. Plaksin, The distribution of numbers representable as a sum of two squares. *Izv. Akad. Nauk SSSR Ser. Mat.* **51**(4) (1987), 860–877. Engl. transl. *Math. USSR-Izv.* **31**(1) (1988), 171–191.
9. V. A. Plaksin, Letter to the editor: correction to the paper "The distribution of numbers representable as a sum of two squares". *Izv. Math.* **41**(1) (1993), 187–188.
10. I. Richards, On the gaps between numbers which are sums of two squares. *Adv. Math.* **46** (1982), 1–2.
11. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd edn., Cambridge University Press (1966).

Alexander Kalmynin,
Mathematics Department,
International Laboratory of Mirror Symmetry
and Automorphic Forms,
National Research University Higher School of
Economics,
Usacheva 6,
Moscow 119048,
Russia
E-mail: alkalb1995cd@mail.ru

https://doi.org/10.1112/S0025579319000299 Published online by Cambridge University Press