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Continuous data dependence for the equations of classical elastodynamics†

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1. *Introduction.* Consider a linear elastic solid occupying a bounded regular three-dimensional region B with smooth surface ∂B. The components of displacement u_i referred to cartesian axes x_i are then well known to satisfy the system of governing equations

$$\rho(x)\frac{\partial^2 u_i}{\partial t^2}-\frac{\partial}{\partial x_j}\left\{c_{ijkl}(x)\frac{\partial u_k}{\partial x_l}\right\}=\rho(x)\mathscr{F}_i(x,t) \text{ in } B\times[0,T] \tag{1·1}$$

in which t denotes the time variable, $x=(x_1,x_2,x_3)$ denotes the position vector, $\rho(x)$ is the non-homogeneous density, assumed positive, $\mathscr{F}_i(x,t)$ are the Cartesian components of body force per unit mass, and $c_{ijkl}(x)$ are the non-homogeneous elasticities, which apart from certain smoothness conditions stated later, are assumed to possess the symmetry

$$c_{ijkl}=c_{klij}. \tag{1·2}$$

Throughout this paper, all suffixes range over the values 1, 2, 3 and the usual convention of summing over repeated indices is adopted. Except where it is in the interest of clarity we avoid explicit mention of the dependence of functions on their arguments.

A classical solution of (1·1) and (1·2) is assumed to exist subject to the prescribed initial conditions

$$u_i(x,0)=f_i(x), \quad (\partial u_i/\partial t)(x,0)=g_i(x) \quad \text{in} \quad B\times O \tag{1·3}$$

and the prescribed boundary conditions

$$u_i(x,t)=h_i(x,t) \quad \text{on} \quad \partial B_1\times[0,T],$$

$$n_j c_{ijkl}\frac{\partial u_k}{\partial x_l}(x,t)=l_i(x,t) \quad \text{on} \quad \partial B_2\times[0,T], \tag{1·4}$$

where B_1 and B_2 are disjoint sets, $\overline{\partial B_1}$ is the closure of ∂B_1, and $\overline{\partial B_1}\cup\partial B_2=\partial B$. Also, n_j are the cartesian components of the unit normal on B_2. In both (1·1) and (1·4) a finite time interval has been introduced to assist in the later definition of certain norms and classes of solutions.

The purpose of the present paper is to prove that the standard initial boundary-value problems for (1·1) and (1·2) have certain classes of solutions whose members depend continuously for finite times on the body force, elasticities and initial data. We indicate also how to obtain continuous dependence of the solution on the boundary

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data, and in a final section we give a simple example to illustrate continuous dependence on geometry.

In all the problems considered only weak restrictions are imposed on the elasticities; in particular, except in the traction boundary-value problem, no definiteness conditions are assumed. This remarkable fact accords with a previous result (8) of the authors, also established without any definiteness condition on the elasticities. It is shown in (8) that solutions of (1·1) and (1·2) lying in a certain class $\mathcal{N}$ depend continuously (in L_2 over B) on the initial data for all t in $0 \leqslant t < T$. This class $\mathcal{N}$ of solutions to the standard initial boundary value problems for (1·1) and (1·2), is defined as the class of solutions whose L_2 integrals over B at time T are bounded by a prescribed constant N. Continuous dependence of solutions on initial data is of course closely related to Lyapounov stability (a full discussion is contained in (8)), and previous stability proofs have relied heavily on a definiteness assumption (see, e.g. Movchan (10), Shield (13)). A slight weakness of the conclusions of (8) is that they involve the magnitude of the elasticities and in order to remove this difficulty we reformulate the problem in section 4.

In another recent paper (7) the authors treated the question of uniqueness of the solution to the standard initial boundary-value problems for (1·1) and (1·2). Uniqueness is proved for a certain class of weak solutions that includes the classical ones, and again it is found that the elasticities need satisfy no definiteness condition. This sharply contrasts with the uniqueness proofs of Neumann (11), Gurtin and Sternberg (3), Gurtin and Toupin (4) and Hayes and Knops (5) where such properties are always required. We note of course that the continuous dependence results of (8) imply uniqueness of the classical solution.

For convenience, the following theorems are established only for classical solutions, but it will be readily seen from the techniques of (7) that in each case the results can be generalized to certain classes of weak solutions. Also, we do not relate the present results with Lyapounov stability, although this may be easily done following the pattern of (8).

As in the previous papers (7), (8) (see also (12)) our approach is based on logarithmetic convexity arguments. However, we develop this argument only for the dependence of the solution on the body force since we show that dependence upon the other data considered can be reduced to this single case.

2. *Continuous dependence on body force.* In this section we examine the continuous dependence of the solution of (1·1)–(1·4) on the body force. Because of linearity, it suffices to consider the solution v_i of the system

$$\rho \frac{\partial^2 v_i}{\partial t^2} - \frac{\partial}{\partial x_j}\left\{c_{ijkl}\frac{\partial v_k}{\partial x_l}\right\} = \rho \mathscr{F}_i \quad \text{in} \quad B \times [0, T] \tag{2·1}$$

$$c_{ijkl} = c_{klij}, \tag{2·2}$$

with the initial conditions $v_i = 0, \quad \frac{\partial v_i}{\partial t} = 0 \quad \text{in} \quad B \times 0$

and the boundary conditions $v_i = 0 \quad \text{on} \quad \overline{\partial B}_1 \times [0, T]$

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$$n_j c_{ijkl} \frac{\partial v_k}{\partial x_l} = 0 \quad \text{on} \quad \partial B_2 \times [0, T]. \tag{2·3}$$

The elasticities are assumed twice continuously differentiable in $B \times [0, T]$.

A solution is said to belong to the class $\mathscr{M}$ if for some prescribed constant M

$$\int_0^T \int_{B(t)} \rho v_i v_i dx < \mathrm{M}^2,$$

where $B(t)$ denotes integration over B at time t. For this class of solutions we establish the following:

THEOREM 2·1. *There exist constants K, and M_1 and a function $\delta(t)\,(0 \leqslant \delta \leqslant 1)$ independent of v_i such that any solution v_i of (2·1)–(2·3) in the class $\mathscr{M}$ satisfies the inequality.*

$$\|v\|_t^2 \leqslant K M_1^{2\delta} \|\mathscr{F}\|_T^{2(1-\delta)} \tag{2·4}$$

for

$$0 \leqslant t \leqslant t_1 < T.$$

Here,

$$\|w\|_t^2 \equiv \int_0^t \int_{B(\eta)} \rho w_i(x, \eta)\, w_i(x, \eta)\, dx\, d\eta. \tag{2·5}$$

To prove this theorem, we introduce the function

$$F(t) = \|v\|_t^2 + T^4 \|\mathscr{F}\|_T^2 \tag{2·6}$$

and then establish the convexity inequality

$$(d^2/dt^2)\{\log [e^{t^2/T^2} F(t)]\} \geqslant 0 \tag{2·7}$$

from which (2·4) follows by Jensen's inequality. Actually, (2·7) leads to an inequality of the form

$$F(t) \leqslant e^{\delta(1-\delta)} [F(T)]^{\delta}\, [F(0)]^{1-\delta}, \tag{2·7a}$$

where $\delta = t/T$, but in the term $F(t)$ we may assume without loss that $\|\mathscr{F}\|_T^2 \leqslant k_3 \mathrm{M}^2$ for a suitable constant K_3, thus reducing (2·7a) to the inequality (2·4).

To establish (2·7) we differentiate (2·6) to obtain

$$\frac{dF}{dt} = 2 \int_0^t \int_{B(\eta)} \rho\, v_i \frac{\partial v_i}{\partial \eta}\, dx\, d\eta \tag{2·8}$$

and

$$\frac{d^2F}{dt^2} = 2 \int_0^t \int_{B(\eta)} \rho \left\{ \frac{\partial v_i}{\partial \eta} \frac{\partial v_i}{\partial \eta} + v_i \frac{\partial^2 v_i}{\partial \eta^2} \right\} dx\, d\eta.$$

Upon using (2·1) and integrating by parts, this last expression becomes

$$\frac{d^2F}{dt^2} = 2 \left\| \frac{\partial v}{\partial \eta} \right\|_t^2 - 2 \int_0^t \int_{B(\eta)} c_{ijkl} \frac{\partial v_i}{\partial x_j} \frac{\partial v_k}{dx_l}\, dx\, d\eta + 2 \int_0^t \int_{B(\eta)} \rho v_i \mathscr{F}_i\, dx\, d\eta, \tag{2·9}$$

and thus by means of Schwarz's inequality we have

$$F \frac{d^2F}{dt^2} - \left(\frac{dF}{dt} \right)^2 \geqslant -2F \left\{ \left\| \frac{\partial v}{d\eta} \right\|_t^2 + \int_0^t \int_{B(\eta)} c_{ijkl} \frac{\partial v_i}{\partial x_j} \frac{\partial v_k}{\partial x_l}\, dx\, d\eta \right\}$$
$$+ 2F \int_0^t \int_{B(\eta)} \rho v_i \mathscr{F}_i\, dx\, d\eta + 4T^4 \|\mathscr{F}\|_T^2 \left\| \frac{\partial v}{\partial \eta} \right\|_t^2. \tag{2·10}$$

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The following identity can be easily deduced from (2·1) to (2·3):

$$\left\|\frac{\partial v}{\partial \eta}\right\|_t^2 + \int_0^t \int_{B(\eta)} c_{ijkl} \frac{\partial v_i}{\partial x_j} \frac{\partial v_k}{\partial x_l} dx d\eta = 2 \int_0^t \int_{B(\eta)} \rho (t-\eta) \frac{\partial v_i}{\partial \eta} \mathscr{F}_i dx d\eta. \tag{2·11}$$

Insertion of (2·11) into (2·10) then gives

$$F \frac{d^2F}{dt^2} - \left(\frac{dF}{dt}\right)^2 \geqslant 2F \left\{ \int_0^t \int_{B(\eta)} \rho \left[v_i - 2(t-\eta) \frac{\partial v_i}{\partial \eta} \right] \mathscr{F}_i dx d\eta \right\} + 4T^4 \|\mathscr{F}\|_T^2 \left\|\frac{\partial v}{\partial \eta}\right\|_t^2. \tag{2·12}$$

The arithmetic-geometric mean and Schwarz's inequalities show that

$$2F \int_0^t \int_{B(\eta)} \rho v_i \mathscr{F}_i dx d\eta \geqslant -T^{-2}F^2, \tag{2·13}$$

and

$$-4F \int_0^t \int_{B(\eta)} \rho (t-\eta) \frac{\partial v_i}{\partial \eta} \mathscr{F}_i dx d\eta \geqslant -T^{-2}F^2 - 4T^4 \|\mathscr{F}\|_T^2 \left\|\frac{\partial v}{\partial \eta}\right\|_t^2. \tag{2·14}$$

On combining (2·12) with (2·13) and (2·14) we are thus led to the inequality

$$F \frac{d^2F}{dt^2} - \left(\frac{dF}{dt}\right)^2 \geqslant -2T^{-2}F^2. \tag{2·15}$$

Since $F(t)$ is strictly positive unless $\mathscr{F}_i \equiv 0$ (except possibly on a set of measure zero), we may divide (2·15) by $F^2(t)$ to obtain

$$(d^2/dt^2)[\log F(t)] \geqslant -2T^{-2} \tag{2·16}$$

which yields the inequality (2·7a). This completes the proof of the theorem.

Let us finally observe that Theorem 2·1 may be stated in the alternative form:

THEOREM 2·1a. *Under perturbations v_i in the class $\mathscr{M}$ any solution u_i of* (1·1)–(1·4) *depends continuously in $\|(.)\|_{t_1}$ on the body force for any t_1 satisfying $0 \leqslant t_1 < T$.*

3. *Continuous dependence on the elasticities.* We assume the existence of solutions u_i^1 and u_i^2 of (1·1)–(1·4) corresponding to different elasticities c_{ijkl}^1 and c_{ijkl}^2. We assume further that in each case the elasticities have continuous derivatives in B. If we set

$$v_i = u_i^1 - u_i^2, \tag{3·1}$$

then v_i satisfies (2·1)–(2·4) with c_{ijkl} replaced by c_{ijkl}^1 and with

$$\rho \mathscr{F}_i = \frac{\partial}{\partial x_j} \left\{ (c_{ijkl}^1 - c_{ijkl}^2) \frac{\partial u_k^2}{\partial x_l} \right\}. \tag{3·2}$$

Thus, Theorem 2·1 yields the obvious:

COROLLARY 3·1. *Under perturbations v_i in the class $\mathscr{M}$ (with $\mathscr{F}_i$ given by* (3·2)) *any solution u_i of* (1·1)–(1·4) *satisfying*

$$\int_0^T \int_{B(\eta)} \rho \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} dx d\eta + \int_0^T \int_{B(\eta)} \rho \frac{\partial^2 u_i}{\partial x_j \partial x_k} \frac{\partial^2 u_i}{\partial x_j \partial x_k} dx d\eta < M_2^2$$

(*i.e. whose first and second spatial derivatives are in L_2*) *depends continuously in $\|(.)\|_{t_1}$ on the elasticities for any t_1 in $0 \leqslant t_1 < T$.*

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The specific stability inequality is given by (2·4) with $\mathscr{F}_i$ expressed by (3·2).

It is worth noting that in this problem only c^1_{ijkl} need satisfy the symmetry condition (1·2). Corollary (3·1) supplements a similar result in the static theory established by Bramble and Payne (2).

4. *Continuous dependence on the initial data.* In (8) the authors established a stability inequality of the following type

$$G(t) \leqslant K[G(T)]^{t/T}[G(0)]^{1-t/T}, \tag{4·1}$$

where

$$G(t) = \int_{B(t)} \rho w_i(x,t)\, w_i(x,t)\, dx + \int_{B(0)} \left[\rho \frac{\partial w_i}{\partial t}(x,0) \frac{\partial w_i}{\partial t}(x,0) + \gamma \frac{\partial w_i}{\partial x_j}(x,0) \frac{\partial w_i}{\partial x_j}(x,0)\right] dx. \tag{4·2}$$

Here, w_i is a solution of (1·1)–(1·4) with $\mathscr{F}_i = h_i = l_i = 0$, and K is an explicit constant independent of w_i. Clearly, therefore, if w_i belongs to the class $\mathscr{N}$, defined in section 1, it follows that (4·1) yields continuous dependence in L_2 on the initial data, f_i and g_i. Note, however, that from (8) the 'magnitude' of the elasticities determines the constant γ and that the initial value of w_i must have square integrable derivatives over B. The purpose of this section is to show that continuous dependence of the solution on the initial data may be established under conditions less restrictive than those of (8). To this end, we use the norm $\|(.)\|^2_t$, given by (2·5), but with the function $v_i(x,t)$ defined by

$$v_i(x,t) = \int_0^t (t-\eta)\, w_i(x,\eta)\, d\eta. \tag{4·3}$$

Then $v_i(x,t)$ satisfies (1·1)–(1·4) with homogeneous initial and boundary conditions ($f_i = g_i = h_i = l_i = 0$) and

$$\mathscr{F}_i = t \frac{\partial w_i}{\partial t}(x,0) + w_i(x,0). \tag{4·4}$$

Thus Theorem 2·1 applies and we are led to the further:

COROLLARY 4·1. *Under perturbations v_i in the class $\mathscr{M}$ any solution u_i of* (1·1)–(1·4) *depends continuously in the appropriate norm on the initial data for any t_1 in $0 \leqslant t_1 < T$.*

It is understood that in this corollary $\mathscr{F}_i$ is given by (4·4) and that the appropriate norm is

$$\int_0^t \int_{B(\eta)} \rho \left\{\int_0^\eta (\eta-\sigma)\, w_i(x,\sigma)\, d\sigma\right\} \left\{\int_0^\eta (\eta-\tau)\, w_i(x,\tau)\, d\tau\right\} dx\, d\eta.$$

The data term in this case is given by

$$\|\mathscr{F}\|^2_T = T \int_{B(0)} \rho \left\{\frac{T^2}{3} \frac{\partial w_i}{\partial t} \frac{\partial w_i}{\partial t} + T w_i \frac{\partial w_i}{\partial t} + w_i w_i\right\} dx \tag{4·5}$$

and the 'magnitude' of the elasticities clearly does not appear in the stability inequality.

5. *Continuous dependence on the boundary data.* We consider in this section only a special case of the general problem in which the displacement is prescribed everywhere on the surface, i.e. $\partial B_1 = \partial B$. We shall only briefly mention the traction boundary-value problem (i.e. when $\partial B_2 = \partial B$) at the end of this section. Thus, let $u^1_i(x,t)$ be a solution of (1·1)–(1·4) corresponding to the boundary data $u^1_i(x,t) = h^1_i(x,t)$ on ∂B, and

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let $u_i^2(x,t)$ be a solution of the same system of equations, but with boundary data $u_i^2(x,t) = h_i^2(x,t)$ on ∂B. We put

$$w_i(x,t) = u_i^1(x,t) - u_i^2(x,t) \tag{5·1}$$

and then represent $w_i(x,t)$ as

$$w_i(x,t) = H_i(x,t) + v_i(x,t), \tag{5·2}$$

where H_i is a solution to the problem

$$\rho \frac{\partial^2 H_i}{\partial t^2} - \Delta H_i = 0 \quad \text{in} \quad B \times [0,T], \tag{5·3}$$

subject to homogeneous initial conditions,

$$H_i(x,0) = \frac{\partial H_i}{\partial t}(x,0) = 0 \quad \text{in} \quad B \times O, \tag{5·4}$$

and the boundary conditions

$$H_i(x,t) = w_i(x,t) \quad \text{on} \quad \partial B \times [0,T]. \tag{5·5}$$

In (5·3) the symbol Δ denotes the spatial Laplace operator. The existence of the function H_i under reasonable assumptions on the data (see, e.g. Wilcox(15)) is guaranteed. Then v_i satisfies

$$\rho \frac{\partial^2 v_i}{\partial t^2} - \frac{\partial}{\partial x_j}\left\{c_{ijkl}\frac{\partial v_k}{\partial x_l}\right\} = -\rho \frac{\partial^2 H_i}{\partial t^2} - \frac{\partial}{\partial x_j}\left\{c_{ijkl}\frac{\partial H_k}{\partial x_l}\right\} \quad \text{in} \quad B \times [0,T], \tag{5·6}$$

together with homogeneous initial and boundary conditions. In other words, $v_i(x,t)$ is a solution of (2·1)–(2·3) with the body force defined by

$$\rho \mathscr{F}_i = -\rho \frac{\partial^2 H_i}{\partial t^2} + \frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial H_k}{\partial x_l}\right). \tag{5·7}$$

For convenience of computation we shall assume that $w_i \in C^2$ in $\bar{B} \times [0,T]$. Since

$$\|w\|_t \leqslant \|H\|_t + \|v\|_t \tag{5·8}$$

and it is known that $\|H\|_t$ depends continuously on the boundary data,† it is sufficient to investigate the dependence of $\|v\|_t$. We first observe that for $v_i \in \mathscr{M}$, Theorem 2·1 states:

$$\begin{aligned}\|v\|_t^2 &\leqslant K M_1^{2\delta}\left\{\int_0^T\int_{B(\eta)} \rho\left[-\frac{\partial^2 H_i}{\partial \eta^2} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left\{c_{ijkl}\frac{\partial H_k}{\partial x_l}\right\}\right]\right. \\ &\qquad \left.\times\left[-\frac{\partial^2 H_i}{\partial \eta^2} + \frac{1}{\rho}\frac{\partial}{\partial x_m}\left\{c_{imns}\frac{\partial H_n}{\partial x_s}\right\}\right] dx\,d\eta\right\}^{1-\delta} \\ &\leqslant K M_1^{2\delta} 3^{1-\delta}\left[\left\|\frac{\partial^2 H}{\partial \eta^2}\right\|_T^2 + k_1\left\|\frac{1}{\rho} DH\right\|_T^2 + k_2\left\|\frac{1}{\rho} D^2 H\right\|_T^2\right]^{1-\delta},\end{aligned} \tag{5·9}$$

where

$$\left\|\frac{1}{\rho} DH\right\|_T^2 = \int_0^T\int_{B(\eta)} \rho^{-1}\frac{\partial H_i}{\partial x_j}\frac{\partial H_i}{\partial x_j}\,dx\,d\eta, \tag{5·10}$$

† For completeness we indicate in the Appendix a method for deriving an explicit continuous dependence inequality.

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and
$$\left\|\frac{1}{\rho}D^2H\right\|_T^2 = \int_0^T\int_{B(\eta)}\rho^{-1}\frac{\partial^2 H_i}{\partial x_j\partial x_k}\frac{\partial^2 H_i}{\partial x_j\partial x_k}\,dx\,d\eta. \tag{5·11}$$

In (5·9), k_2 and k_1 are uniform bounds, respectively, for the elasticities and their spatial derivatives.

Next, let us note that $\partial^2 H_i/\partial t^2$ is a solution of the initial boundary value problem (5·3), (5·4) with boundary data $\partial^2 h_i^1/\partial t^2 - \partial^2 h_i^2/\partial t^2$. Hence, it is possible (see Appendix) to obtain the explicit *a priori* inequality

$$\left\|\frac{\partial^2 H}{\partial\eta}\right\|_T^2 \leqslant k_3\beta^{-1}\int_0^T\oint_{\partial B(\eta)}\left(\frac{\partial^2 h_i^1}{\partial\eta^2}-\frac{\partial^2 h_i^2}{\partial\eta^2}\right)\left(\frac{\partial^2 h_i^1}{\partial\eta^2}-\frac{\partial^2 h_i^2}{\partial\eta^2}\right)ds\,d\eta \tag{5·12}$$

where k_3 and β are given in the Appendix.

Similarly, with explicit constants $k_\alpha(\alpha = 4, \ldots, 10)$ it has been shown (e.g. (1), (14)) that the following *a priori* inequalities are valid:

$$\left\|\frac{1}{\rho}D^2H\right\|_T^2 \leqslant k_4\int_0^T\oint_{\partial B(\eta)}D_s^2H_iD_s^2H_i\,ds\,d\eta + k_5\int_0^T\oint_{\partial B(\eta)}D_sH_iD_sH_i\,ds\,d\eta$$
$$+ k_6\int_0^T\oint_{\partial B(\eta)}H_iH_i\,ds\,d\eta + k_7\|\Delta H\|_T^2, \tag{5·13}$$

and

$$\left\|\frac{1}{\rho}DH\right\|_T^2 \leqslant k_8\int_0^T\oint_{\partial B(\eta)}D_sH_iD_sH_i\,ds\,d\eta + k_9\int_0^t\oint_{\partial B(\eta)}H_iH_i\,ds\,d\eta + k_{10}\|\Delta H\|_T^2. \tag{5·14}$$

In (5·13) and (5·14) the expression $D_s^2H_iD_s^2H_i$ denotes the sums of squares of second tangential derivatives of H_i on the boundary ∂B evaluated at a given instant of time t, while $D_sH_iD_sH_i$ denotes the sums of squares of the first tangential derivatives of H_i on the boundary ∂B also evaluated at a given instant of time t.

Clearly, since $H_i = w_i$ on ∂B and

$$\|\Delta H\|_T \leqslant \rho_{\max}\|\partial^2 H/\partial t^2\|_T \tag{5·15}$$

the insertion of (5·12)–(5·14) into (5·9) establishes the existence of a constant K such that

$$\|v\|_t^2 \leqslant KM_1^{2\delta}\left[\int_0^T\oint_{\partial B(\eta)}\left\{D_s^2w_iD_s^2w_i + D_sw_iD_sw_i + w_iw_i + \rho\frac{\partial^2 w_i}{\partial\eta^2}\frac{\partial^2 w_i}{\partial\eta^2}\right\}ds\,d\eta\right]^{1-\delta}. \tag{5·16}$$

We have therefore proved the following:

THEOREM 5·1. *Let $u_i(x,t)$ be a solution of* (1·1)–(1·4) *with $\partial B_1 = \partial B$. Then, under perturbations w_i which are in the class $\mathscr{M}$ and which are also twice continuously differentiable in $\bar{B}\times[0,T]$, the solution $u_i(x,t)$ depends continuously in the norm $\|(.)\|_{t_1}$ on the boundary data for any t_1 in $0 \leqslant t_1 < T$.*

Note that no definiteness conditions are imposed on the elasticities. If in fact c_{ijkl} is a positive-definite form then stability in the ordinary sense (i.e. on $[0,T]$) could be established with less smoothness on the data.

It is possible by arguments similar to those of this section to obtain an analogous theorem for continuous dependence of the solution on the boundary data in the trac-

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tion boundary-value problem, i.e. when $\partial B_2 = \partial B$. Then, however, the elasticities must satisfy one or other of the following definiteness conditions: either

$$\int_{B(t)} c_{ijkl} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} dx \geqslant c_0 \int_{B(t)} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) dx, \tag{5·17}$$

or

$$\int_{B(t)} c_{ijkl} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} dx \leqslant -c_0 \int_{B(t)} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) dx, \tag{5·18}$$

for positive constant c_0 and for arbitrary vector gradients satisfying

$$\int_{B(t)} (u_{i,j} - u_{j,i})\, dx = 0. \tag{5·19}$$

In case (5·17) standard energy arguments may be used, but for case (5·18) these are inadequate and must be replaced by logarithmetic convexity arguments. Holden (6) has also investigated dependence of the solution on the tractions in the static boundary-value problem. He adopted a condition similar to (5·17) but his definitions are entirely different from those in this paper. In (9) the authors considered continuous dependence of the solution on the *initial* data in the traction boundary-value problem and used energy arguments based on (5·17).

6. *Continuous dependence on geometry.* In this final section we consider a simple problem which exhibits continuous dependence of the solution for certain geometric perturbations. Let us suppose that the initial data are not all taken at time $t = 0$ but that at each point x of B the assigned initial values of the displacement and velocity are actually measured at time $t = -f(x)$. For convenience, $f(x)$ is assumed to be a positive function satisfying

$$\sup_{x \in B} |f(x)| \leqslant \epsilon. \tag{6·1}$$

Let $\Omega(T)$ denote the space-time cylinder bounded by the surface $t = -f(x)$, the boundary ∂B and the plane $t = T$, and let $u_i^1(x,t)$ denote the true solution with initial data given on the surface $t = -f(x)$. We denote by $u_i^2(x,t)$ the solution to the problem for which the initial data are measured at $t = 0$. As in section 4, we set

$$w_i(x,t) = u_i^1(x,t) - u_i^2(x,t) \quad \text{in} \quad \bar{B} \times [0,T], \tag{6·2}$$

and write

$$v_i(x,t) = \int_0^t (t-\eta)\, w_i(x,\eta)\, d\eta. \tag{6·3}$$

The following theorem is then an immediate consequence of the results of section 4:

THEOREM 6·1. *If $u_i^1(x,t)$ satisfies the condition*

$$\iint_{\Omega(T) - B \times [0,T]} \frac{\partial^2 u_i}{\partial \eta^2} \frac{\partial^2 u_i}{\partial \eta^2} dx\, d\eta \leqslant M_2^2 \tag{6·4}$$

(i.e. second time derivatives in L_2) then for perturbations v_i in class $\mathscr{M}$ there exist constants K and M_1, and a variable $\delta(t) \in (0,1)$, all independent of v_i, such that

$$\|v\|_t^2 \leqslant K M_1^{2\delta} \epsilon^{1-\delta} \tag{6·5}$$

where ϵ is given by (6·1).

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To establish this result we note that

$$w_i(x,0) = u_i^1(x-t_1) - u_i^2(x,0) = \int_{-f(x)}^{0} \frac{\partial u_i^1}{\partial \eta}\, d\eta,$$

and so

$$w_i(x,0) = -f(x)\, g_i(x) - \int_{-f(x)}^{0} \eta \frac{\partial^2 u_i^1}{\partial \eta^2}\, d\eta, \tag{6·6}$$

where, as before, $g_i(x)$ is the initial value of the velocity. Likewise, we have

$$\frac{\partial w_i}{\partial t}(x,0) = \int_{-f(x)}^{0} \frac{\partial^2 u_i^1}{\partial \eta^2}\, d\eta. \tag{6·7}$$

Thus we clearly have,

$$\begin{aligned}\int_{B(0)} w_i(x,0)\, w_i(x,0)\, dx &\leqslant 2\epsilon^2 \left[\int_B g_i(x)\, g_i(x)\, dx + \int_B \left\{\int_{-f(x)}^{0} \left|\frac{\partial^2 u_i^1}{\partial \eta^2}\right| d\eta\right\}\left\{\int_{-f(x)}^{0} \left|\frac{\partial^2 u_i^1}{\partial \eta^2}\right| d\eta\right\} dx\right] \\ &\leqslant 2\epsilon^2 \left[\int_B g_i(x)\, g_i(x)\, dx + \epsilon \iint_{\Omega(T)-B\times[0,T]} \frac{\partial^2 u_i^1}{\partial \eta^2} \frac{\partial^2 u_i^1}{\partial \eta^2}\, dx\, d\eta\right] \\ &\leqslant 2\epsilon^2 \left[\epsilon M_2^2 + \int_B g_i(x)\, g_i(x)\, dx\right]. \end{aligned} \tag{6·8}$$

Similarly,

$$\int_{B(0)} \frac{\partial w_i}{\partial t}(x,0) \frac{\partial w_i}{\partial t}(x,0)\, dx \leqslant M_2 \epsilon. \tag{6·9}$$

The proof of the theorem now follows from the results of section 4 upon observing that from (4·5), (6·8) and (6·9) we have

$$\|\mathscr{F}\|_T^2 \leqslant \frac{T}{6} \int_{B(0)} \rho \left\{5T^2 \frac{\partial w_i}{\partial t} \frac{\partial w_i}{\partial t} + 9 w_i w_i\right\} dx \leqslant M_3 \epsilon$$

for some constant M_3.

APPENDIX

We indicate now a means for obtaining an explicit continuous dependence inequality for $\|H\|_t$ if H_i is defined by (5·3)–(5·5). We first introduce the auxiliary vector ψ_i defined by

$$\Delta \psi_i(x,\eta) - \rho \frac{\partial^2 \psi_i}{\partial \eta^2}(x,\eta) = \rho H_i(x,\eta) \quad \text{in} \quad B\times[0,t] \tag{A 1}$$

where

$$\psi_i(x,t) = \frac{\partial \psi_i}{\partial t}(x,0) = 0 \tag{A 2}$$

and

$$\psi_i(x,\eta) = 0 \quad \text{on} \quad \partial B \times [0,t]. \tag{A 3}$$

The existence of ψ_i is assured by known theorems (see e.g. (15)). We shall not require explicit knowledge of ψ_i.

By introducing (A 1) into the expression for $\|H\|_t$ and making use of the conditions on H_i and ψ_i it is easily seen that†

$$\begin{aligned}\|H\|_t^2 &= \int_0^t \oint_{\partial B(\eta)} H_i \frac{\partial \psi_i}{\partial n}\, ds\, d\eta \\ &\leqslant \left[\int_0^t \oint_{\partial B(\eta)} H_i H_i\, ds\, d\eta \int_0^t \oint_{\partial B(\eta)} \frac{\partial \psi_i}{\partial n} \frac{\partial \psi_i}{\partial n}\, ds\, d\eta\right]^{\frac{1}{2}}. \end{aligned} \tag{A 4}$$

† We use the symbol $\partial/\partial n$ to denote the outward normal derivative on ∂B and let n_k denote the kth component of the unit normal.

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We must now obtain a bound for the second integral on the right in the last expression. To this end we introduce a vector field $f_i(x) \in C^1$ in $\bar{B} \times [0, t]$, which has the property that $f_k n_k \geqslant \beta > 0$ on ∂B and form, for constants K_1 and C, the expression

$$\int_0^t \int_{B(\eta)} \left[f_k \frac{\partial \psi_i}{\partial x_k} + K_1(\eta + C) \frac{\partial \psi_i}{\partial \eta} \right] \left[\Delta \psi_i - \rho \frac{\partial^2 \psi_i}{\partial \eta^2} \right] dx\, d\eta = \frac{1}{2} \int_0^t \oint_{\partial B(\eta)} f_k n_k \frac{\partial \psi_i}{\partial n} \frac{\partial \psi_i}{\partial n} ds\, d\eta$$
$$- \frac{1}{2} \int_0^t \int_{B(\eta)} \left[2 \frac{\partial f_k}{\partial x_j} - \frac{\partial f_l}{\partial x_l} \delta_{jk} \right] \left[\frac{\partial \psi_i}{\partial x_k} \frac{\partial \psi_i}{\partial x_j} \right] dx\, d\eta + \int_{B(0)} \rho f_k \frac{\partial \psi_i}{\partial x_k} \frac{\partial \psi_i}{\partial \eta} dx$$
$$+ \frac{K_1 C}{2} \int_{B(0)} \left[\frac{\partial \psi_i}{\partial x_j} \frac{\partial \psi_i}{\partial x_j} + \rho \frac{\partial \psi_i}{\partial \eta} \frac{\partial \psi_i}{\partial \eta} \right] dx + \frac{K_1}{2} \int_0^t \int_{B(\eta)} \left[\frac{\partial \psi_i}{\partial x_j} \frac{\partial \psi_i}{\partial x_j} + \rho \frac{\partial \psi_i}{\partial \eta} \frac{\partial \psi_i}{\partial \eta} \right] dx\, d\eta$$
$$- \frac{1}{2} \int_0^t \int_{B(\eta)} \frac{\partial}{\partial x_k} (\rho f_k) \frac{\partial \psi_i}{\partial \eta} \frac{\partial \psi_i}{\partial \eta} dx\, d\eta. \tag{A 5}$$

The right-hand side has been obtained by repeated use of the divergence theorem and (A 2), (A 3). Clearly we may apply the arithmetic–geometric mean inequality to the term on the left of (A 5) and upon using (A 1) obtain

$$\int_0^t \int_{B(\eta)} \left(f_k \frac{\partial \psi_i}{\partial x_k} + K_1(\eta + C) \frac{\partial \psi_i}{\partial \eta} \right) \rho H_i\, dx\, d\eta \leqslant \frac{1}{2\alpha} \|H\|_t$$
$$+ \frac{\alpha K_2}{2} \int_0^t \int_{B(\eta)} \left[\frac{\partial \psi_i}{\partial x_j} \frac{\partial \psi_i}{\partial x_j} + \rho \frac{\partial \psi_i}{\partial \eta} \frac{\partial \psi_i}{\partial \eta} \right] dx\, d\eta \tag{A 6}$$

where the constant K_2 is explicit and α is an arbitrary positive constant. It is clear that with K_1 and C chosen sufficiently large and α sufficiently small (note that K_2 depends on K_1 and C), (A 5) and (A 6) will lead to

$$\beta \int_0^t \oint_{\partial B(\eta)} \frac{\partial \psi_i}{\partial \eta} \frac{\partial \psi_i}{\partial \eta} ds\, d\eta \leqslant K_3 \|H\|_t^2 \tag{A 7}$$

with a computable constant K_3. The insertion of (A 7) into (A 4) then yields

$$\|H\|_t^2 \leqslant K_3 \beta^{-1} \left\{ \int_0^t \oint_{\partial B} H_i H_i ds \right\} \tag{A 8}$$

which is the desired continuous dependence inequality.

If B is starshaped with respect to the origin we may take $f_i = x_i$. If ∂B is smooth, f_i may be chosen as indicated in (1).

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