Ergod. Th. & Dynam. Sys. (2002), **22**, 947–949
DOI: 10.1017/S0143385702000494
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Printed in the United Kingdom

A zero topological entropy map for which periodic points are not a G_δ set

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(*Received* 1 *September* 2000 *and accepted in revised form* 11 *January* 2001)

Abstract. We exhibit an example of a continuous map of the interval of type 2^∞, and hence of zero topological entropy, for which the set of periodic points is not a G_δ set. This disproves a conjecture by Sharkovsky from 1965. Unfortunately, this conjecture has been incorrectly quoted as a true statement by other authors in many papers and books.

1. *The Main Theorem and preliminaries*

In the following, $C(I, I)$ denotes the space of continuous maps of the unit compact interval I. By the period of a periodic point we mean its smallest period. The ω-limit set of the trajectory of $x \in I$ is denoted by $\omega_f(x)$, and $\omega(f) = \bigcup\{\omega_f(x);\ x \in I\}$ is the set of ω-limit points of f. The set of periodic points of f is denoted by $\mathrm{Per}(f)$ and the set of fixed points by $\mathrm{Fix}(f)$. Let $\overline{A}$ be the closure of a set A. If the periods of points in $\mathrm{Per}(f)$ are the numbers $1, 2, \ldots, 2^n$ then f is of *type* 2^n, and if the periods are all powers of 2 then f is of *type* 2^∞. A periodic point p is *repelling* if the derivative $|Df^n(p)| > 1$. A map $f \in C(I, I)$ is *unimodal* if there exists $c \in (0, 1)$ such that f is strictly increasing on $[0, c]$ and strictly decreasing on $[c, 1]$.

Let $f \in C(I, I)$. If, for some n, f is of type 2^n then clearly $\mathrm{Per}(f)$ is a closed set. On the other hand, if f has a periodic point whose period is not a power of 2, then $\mathrm{Per}(f)$ cannot be a G_δ set. This was proved in 1965 by Sharkovsky [**9**]. The argument is simple. For any such map f there is an n and a nowhere-dense perfect (i.e. compact, without isolated points) set F such that $f^n|F$ is 'almost' conjugate to the shift. Consequently, $\mathrm{Per}(f^n)$ is a dense countable subset of F and hence, by the Baire theorem on complete spaces, it cannot be a G_δ set.

Sharkovsky [**9**] also conjectured that, for any function of type 2^∞, $\mathrm{Per}(f)$ is a G_δ set. This conjecture was proved in 1968 by Sharkovsky himself [**8**] but his argument is false. It is based on the assumption that a continuous function of type 2^∞ can have only countably many infinite minimal sets, which is not true; a counterexample was given in 1989 by Sharkovsky [**7**].

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However, in a series of papers and books (cf. [**3**, **4, 10–12**], for example) we find that, for any $f \in C(I, I)$, the properties 'the topological entropy is zero' and 'Per(f) is a G_δ set' are equivalent. We show that this statement is false. In view of the above remarks it suffices to prove the following.

THEOREM. *There is a map $f \in C(I, I)$ with zero topological entropy such that* Per(f) *is not a G_δ set.*

2. *Proof of theorem*

Let g be the Feigenbaum's map (unimodal, of type 2^∞, see [**2**] for its description). Let $J \subset I$ be the minimal compact interval containing the critical point $c = \frac{1}{2}$ of g such that $g(J) = J$, and let $f_0 \in C(I, I)$ be the map linearly conjugate to the restriction $g|J$. Define by induction a sequence $\{f_n\}_{n=0}^\infty$ of maps in $C(I, I)$ such that for any n:

(i) f_n is of type 2^∞;
(ii) f_n is strictly monotone in a neighbourhood of any $x \in \mathrm{Per}(f_n)$;
(iii) $\mathrm{Per}(f_n)$ is isolated and any point in $\mathrm{Per}(f_n)$ is repelling;
(iv) for any $u < v$ in $\mathrm{Per}(f_n)$ there is a $z \in \omega(f_n)\backslash\mathrm{Per}(f_n)$ such that $u < z < v$;
(v) $\omega(f_n) = \overline{\mathrm{Per}(f_n)}$;
(vi) $\omega(f_n) \supset \omega(f_{n-1})$;
(vii) $f_n|\omega(f_n) = f_{n+1}|\omega(f_n)$;
(viii) $\|f_n - f_{n+1}\| < 1/2^n$;
(ix) for any $x \in \mathrm{Per}(f_n)$, $\delta_{n+1}(x) < \frac{1}{2}\delta_n(x)$;

where $\delta_n(x)$ is defined for any x as the distance of x from the set $\mathrm{Per}(f_n)\backslash\{x\}$.

Let f_0 be as above. It is well known that it satisfies (i)–(v). (For (i), (ii), (iv) and (v) see [**2**] or [**6**], for (iii) see [**13**]; cf. also [**5**, p. 155].) Let α be the unique fixed point of f_0. Assume by induction that f_n is given. Now the idea is to let $\mathrm{Per}(f_{n+1})$ be the disjoint union of small affine copies of $\mathrm{Per}(f_0)$. Any such copy K intersects $\mathrm{Per}(f_n)$ exactly at one point p, whose relative position in K is the same as the position of α in $\mathrm{Per}(f_0)$.

By (iii), $\mathrm{Per}(f_n)$ is countable so there is a sequence $\{p_i\}_{i=1}^\infty$ in $\mathrm{Per}(f_n)$ that contains exactly one point from any periodic orbit. By (i), the period of p_i is $2^{n(i)}$. By (iii) there is a neighbourhood U_i of p_i such that $f_n^l(U_i) \cap f_n^m(U_j) \neq \emptyset$, $0 \le l < 2^{n(i)}$, and $0 \le m < 2^{n(j)}$ imply $i = j$ and $l = m$. By (ii), for any $i > 0$ and any $l = 0, \ldots, 2^{n(i)} - 1$, there is a compact interval V_i^l contained in the interior of $f_n^l(U_i)$ and containing $f_n^l(p_i)$, with the following two properties: all the intervals V_i^l, for a fixed i, have the same length, and the point $f_n^l(p_i)$ divides V_n^l in the ratio $\alpha/(1-\alpha)$.

Let $\xi_{A,B} : A \to B$ denote the increasing linear bijection of closed intervals A, B. Define the map $f_{n+1} : I \to I$ as follows:

$$f_{n+1}|I\Big\backslash\bigcup_{i=0}^{\infty}\bigcup_{l=0}^{2^{n(i)}-1} f_n^l(U_i) = f_n|\Big\backslash\bigcup_{i=0}^{\infty}\bigcup_{l=0}^{2^{n(i)}-1} f_n^l(U_i),$$

and

$$f_{n+1}|V_i^0 = \xi_{I,V_i^1} \circ f_0 \circ \xi_{V_i^0,I},$$
$$f_{n+1}|V_i^l = \xi_{V_i^l,V_i^{l+1}},$$

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for any i and any $l = 1, \dots, 2^{n(i)} - 1$, where $V_i^{2^{n(i)}} = V_i^0$, and extend f_{n+1} linearly onto the whole of I. Then clearly f_{n+1} satisfies the conditions (i)–(vii), with n replaced by $n+1$. The conditions (viii) and (ix) will be satisfied if the intervals U_i are taken sufficiently small, which is possible.

Put $f = \lim_{n\to\infty} f_n$. By (i) and (viii), the Block's stability theorem [**1**] implies that f is of type 2^∞. Let $P = \bigcup_{n=0}^\infty \mathrm{Per}(f_n)$, $X = \overline{P}$, and $h = f|X$. Assume that $\mathrm{Per}(f)$ is a G_δ set. Then $\mathrm{Per}(h)$ is a G_δ set and, consequently, both $\mathrm{Per}(h) = \bigcup_{n=0}^\infty \mathrm{Fix}(h^n)$ and its complement in X is an F_σ set.

By (vi) and (vii), $P \subset \mathrm{Per}(h)$. Hence $\mathrm{Per}(h)$ is dense in X, and to finish the proof it suffices to show that $X\backslash\mathrm{Per}(h)$ is dense in X; this would imply that X is a first category set contrary to the fact that X is complete. So let G be an open interval with $G \cap X \neq \emptyset$. Then there is a u in $P \cap G$. By (ix), P has no isolated points, hence $P \cap G$ contains a point v different from u. We may assume $u < v$. By (v), (iv) and (vii) there is $z \in \omega(h)\backslash\mathrm{Per}(h)$ such that $u < z < v$. This proves that $G \cap X\backslash\mathrm{Per}(h) \neq \emptyset$. □

Acknowledgements. This research was supported, in part, by contracts 201/00/0859 from the Grant Agency of Czech Republic, and CEZ:J10/98:192400002 from the Czech Ministry of Education. The support of these institutions is gratefully acknowledged.

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