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Spectra of derivations

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1. *Introduction.* This paper follows (4) in a quantitative study of derivations. The object is to investigate the properties of the spectrum of an inner derivation. In particular, we shall be interested in obtaining a general description of the spectrum of a derivation on a W*-algebra.

When A is a Banach algebra, and A belongs to A, the (inner) *derivation* implemented by A is the mapping Δ_A of A into itself which sends X to XA − AX.

The spectrum of an element A in the algebra A is denoted by Sp (A; A). For any Banach space X, $L(X)$ denotes the set of all bounded linear operators on X.

2. *Spectra of derivations on* C*-*algebra.* Before proceeding, we quote the following two results from (5).

2·1. Proposition. *Let A be any Banach algebra. Then for* A *in* A

$$\mathrm{Sp}\,(\Delta_A; L(A)) \subseteq \mathrm{Sp}\,(A; A) - \mathrm{Sp}\,(A; A).$$

2·2. Proposition. *Suppose $A = L(X)$ for some Banach space X. Then*

$$\mathrm{Sp}\,(\Delta_A; L(L(X))) = \mathrm{Sp}\,(A; L(X)) - \mathrm{Sp}\,(A; L(X)).$$

An earlier proof of Proposition 2·2 may be found in (6).

Throughout the rest of this section, A will always denote a C*-algebra. U will denote the universal representation of A and H will denote the Hilbert space on which U acts. It is known that $U(A)^{**}$ is a W*-algebra, and as such is isomorphic to the weak closure $\overline{U(A)}$ of $U(A)$. A proof of this may be found in (7). However, for our purposes it is enough to notice that the map Φ which implements this isomorphism is obtained as follows: for F in $U(A)^{**}$ and f in $U(A)^*$, we know that $f = \omega_{x,y}$ for some x, y in H ($\omega_{x,y}(A) = \langle Ax, y\rangle$). Hence the map $(x, y) \to F(f) = F(\omega_{x,y})$ defines a sesquilinear form on H and so by the Riesz representation of such forms, there is a unique bounded linear operator S_F on H, such that $\langle S_F x, y\rangle = F(\omega_{x,y})$. That S_F lies in $\overline{U(A)}$ is established via the double commutant theorem.

It follows from this that $\Phi(\widehat{U(A)}) = U(A)$ for all A in A. (Here $\widehat{\ }$ indicates the canonical map into the second dual).

Furthermore, routine calculations with the Arens multiplication show that

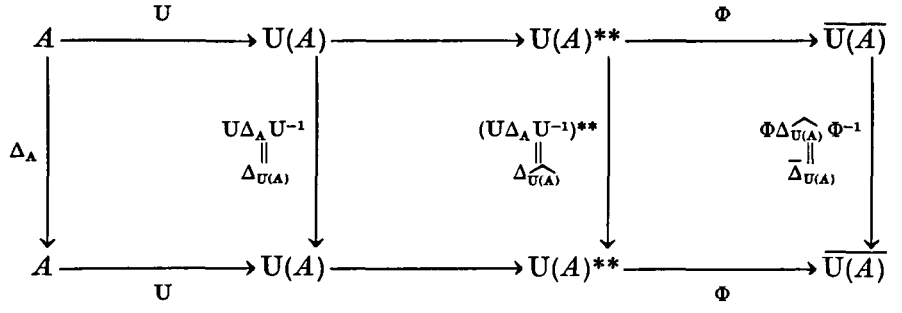
$$(U \circ \Delta_A \circ U^{-1})^{**} = \Delta_{\widehat{U(A)}}$$

and so

$$\Phi(U \circ \Delta_A \circ U^{-1})^{**} \Phi^{-1}(X) = XU(A) - U(A)X$$

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for all X in $\overline{\mathrm{U}(A)}$. The following diagram illustrates this:



Here $\bar{\Delta}_{\mathrm{U}(A)}$ denotes the unique extension of $\Delta_{\mathrm{U}(A)}$ to $\overline{\mathrm{U}(A)}$. Having established the above notation, we can now prove:

2·3. LEMMA. $\mathrm{Sp}\,(\Delta_{\mathrm{U}(A)}; L(\mathrm{U}(A))) = \mathrm{Sp}\,(\bar{\Delta}_{\mathrm{U}(A)}; L(\overline{\mathrm{U}(A)}))$.

Proof. Since a linear operator has a bounded inverse if, and only if, the adjoint of that operator has a bounded inverse ((1), VI, 2·7), we conclude that

$$\begin{aligned}\mathrm{Sp}\,(\Delta_{\mathrm{U}(A)}; L(\mathrm{U}(A))) &= \mathrm{Sp}\,((\mathrm{U}\circ\Delta_A\circ\mathrm{U}^{-1})^{**}; L(\mathrm{U}(A)^{**}))\\ &= \mathrm{Sp}\,(\Delta_{\widehat{\mathrm{U}(A)}}; L(\mathrm{U}(A)^{**}))\\ &= \mathrm{Sp}\,(\Phi\Delta_{\widehat{\mathrm{U}}(A)}\Phi^{-1}; L(\overline{\mathrm{U}(A)}))\\ &= \mathrm{Sp}\,(\bar{\Delta}_{\mathrm{U}(A)}; L(\overline{\mathrm{U}(A)}))\end{aligned}$$

as required.

2·4. LEMMA. *Let A be a C*-algebra and let π be any faithful representation of A. Then*

$$\mathrm{Sp}\,(\bar{\Delta}_{\pi(A)}; L(\overline{\pi(A)})) \subseteq \mathrm{Sp}\,(\bar{\Delta}_{\mathrm{U}(A)}; L(\overline{\mathrm{U}(A)})).$$

Proof. Since U is the universal representation, we can find a projection Q in the centre of $\overline{\mathrm{U}(A)}$ and an isomorphism $\alpha: \overline{\pi(A)} \to \overline{\mathrm{U}(A)}\,\mathrm{Q}$ such that

$$\alpha(\pi(\mathrm{X})) = \mathrm{U}(\mathrm{X})\,\mathrm{Q} \quad \text{for all X in } A.$$

Now
$$\begin{aligned}\mathrm{Sp}\,(\bar{\Delta}_{\pi(A)}; L(\overline{\pi(A)})) &= \mathrm{Sp}\,(\overline{\pi\Delta_A\pi^{-1}}; L(\overline{\pi(A)}))\\ &= \mathrm{Sp}\,(\overline{\alpha\pi\Delta_A(\alpha\pi)^{-1}}; L(\overline{\mathrm{U}(A)}\,\mathrm{Q})).\end{aligned}$$

Now $\overline{(\alpha\pi)\,\Delta_A(\alpha\pi)^{-1}} = \bar{\Delta}_{\mathrm{U}(A)}|_{\overline{\mathrm{U}(A)}\,\mathrm{Q}}$ and, since $\overline{\mathrm{U}(A)}\,\mathrm{Q}$ and $\overline{\mathrm{U}(A)}\,(\mathrm{I}-\mathrm{Q})$ are both invariant under $\bar{\Delta}_{\mathrm{U}(A)}$, we know that

$$\mathrm{Sp}\,(\bar{\Delta}_{\mathrm{U}(A)}; L(\overline{\mathrm{U}(A)})) = \mathrm{Sp}\,(\bar{\Delta}_{\mathrm{U}(A)}|_{\overline{\mathrm{U}(A)}\,\mathrm{Q}}; L(\overline{\mathrm{U}(A)}\,\mathrm{Q})) \cup \mathrm{Sp}\,(\bar{\Delta}_{\mathrm{U}(A)}|_{\overline{\mathrm{U}(A)}\,(\mathrm{I}-\mathrm{Q})}; L(\overline{\mathrm{U}(A)}\,(\mathrm{I}-\mathrm{Q}))),$$

which completes the proof of the lemma.

2·5. PROPOSITION. *Let A be a C*-algebra which has a faithful irreducible representation. Then*

$$\begin{aligned}\mathrm{Sp}\,(\Delta_A; L(A)) &= \mathrm{Sp}\,(\mathrm{A}; A) - \mathrm{Sp}\,(\mathrm{A}; A)\\ &= \{x-y : x, y \in \mathrm{Sp}\,(\mathrm{A}; A)\}.\end{aligned}$$

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Proof. Let π be any faithful irreducible representation of A. Then

$$\begin{aligned}\mathrm{Sp}\,(\Delta_A; L(A)) &= \mathrm{Sp}\,(U\Delta_A U^{-1}; L(U(A))) \quad \text{(since U is faithful)}\\ &= \mathrm{Sp}\,(\tilde{\Delta}_{U(A)}; L(\overline{U(A)})) \quad \text{(by Lemma 2·3)}\\ &\supseteq \mathrm{Sp}\,(\tilde{\Delta}_{\pi(A)}; L(\overline{\pi(A)})) \quad \text{(by Lemma 2·4).}\end{aligned}$$

However, since π is irreducible, $\overline{\pi(A)} = L(H_\pi)$ (where H_π is the Hilbert space on which π acts).

Now we can use Proposition 2·2 to conclude that

$$\mathrm{Sp}\,(A; A) - \mathrm{Sp}\,(A; A) \subseteq \mathrm{Sp}\,(\Delta_A; L(A)).$$

Proposition 2·1 provides the reverse inclusion, and the proof is complete.

2·6. Corollary. *If A is a simple* C**-algebra,* then

$$\mathrm{Sp}\,(\Delta_A; L(A)) = \mathrm{Sp}\,(A; A) - \mathrm{Sp}\,(A; A).$$

2·7. *Remarks.* If we concentrate on *-derivations (i.e. derivations Δ such that $\Delta(X^*) = \Delta(X)^*$) then we can show that $\mathrm{Sp}\,(\Delta; L(\pi(A))) = \mathrm{Sp}\,(\tilde{\Delta}; L(\overline{\pi(A)}))$. (The proof of this may be found in (5).)

3. *The Glimm quotients.* In the next section we shall determine the spectrum of Δ_A on any W*-algebra. In the proofs we make considerable use of the so-called 'Glimm quotients', and in this section we list their main properties and derive some technical results, which will be needed later.

Let A be a W*-algebra with centre Z. Let Ω be the maximal ideal space of Z and let ω in Ω be any maximal ideal of Z. $J(\omega)$ will denote the smallest norm-closed two-sided ideal of A containing ω, i.e. $J(\omega)$ is the norm closure of $\{\Sigma Z_j X_j : Z_j \in \omega \text{ and } X_j \in A\}$. Let $A(\omega)$ denote the quotient algebra $A|_{J(\omega)}$ and let $A(\omega)$ denote the canonical image of A in $A(\omega)$.

Glimm in (2) has shown that the map $\omega \to \|A(\omega)\|$ is continuous on Ω and that

$$\|A\| = \sup\{\|A(\omega)\| : \omega \in \Omega\}.$$

Halpern (3) has established that each $J(\omega)$ is a primitive ideal. It follows that each $A(\omega)$ has a faithful irreducible representation π_ω on some Hilbert space H_ω.

3·1. Proposition. *Let* P *be any projection in* Z *with* $P = \chi(\Lambda)$ *for some open and closed set* Λ *in* Ω. ($\chi(S)$ *denotes the characteristic function of* S.) *Then, for all* X *in* A, *we have*

$$\mathrm{Sp}\,(XP; AP) = \cup\{\mathrm{Sp}\,(X(\omega); A(\omega)) : \omega \in \Lambda\}.$$

3·2. Proposition. *For any* X *in* A *and* $\lambda \notin \mathrm{Sp}\,(X(\omega); A(\omega))$, *there is an open and closed neighbourhood* V *of* ω, *such that* $\lambda \notin \mathrm{Sp}\,(X(\tau); A(\tau))$ *whenever* $\tau \in V$.

Proofs of these may be found in (5) *or in* (8), *where the Glimm quotients are used in another context.*

3·3. Proposition. *Let* A *be a fixed element in* A. *Then the map* $\omega \to \rho(A(\omega))$ *is upper semi-continuous.* (*Here* $\rho(.)$ *denotes the spectral radius.*)

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Proof. If f_n denotes the function $\omega \to \|A(\omega)^n\|^{1/n}$, then each f_n is continuous and, since the mapping $\omega \to \rho(A(\omega))$ is the *infimum* of $\{f_n\}$, upper semi-continuity follows.

3·4. PROPOSITION. *Suppose* A *is a fixed element in* A *and that* $\lambda \notin \mathrm{Sp}(A(\omega); A(\omega))$. *Then there is a neighbourhood* V *of* ω *in* Ω *for which the map*

$$\tau \to \rho([A(\tau) - \lambda I(\tau)]^{-1})$$

is defined and is upper semi-continuous.

Proof. By Proposition 3·2, we may choose an open and closed neighbourhood V of ω, such that $\lambda \notin \mathrm{Sp}(A(\tau): A(\tau))$ whenever $\tau \in V$. Proposition 3·1 then shows that

$$\lambda \notin \mathrm{Sp}(AP; AP) \quad (\text{with } P = \chi(V)).$$

Let B be some element in A for which $BP = (AP - \lambda I)^{-1}$ in AP. Then, by Proposition 3·3, the map $\tau \to \rho(B(\tau))$ is upper semi-continuous. But, when τ is restricted to V, $B(\tau) = [A(\tau) - \lambda I(\tau)]^{-1}$, proving the proposition.

3·5. PROPOSITION. *For every* A *in* A, *the map* $\omega \to \mathrm{Sp}(A(\omega); A(\omega))$ *is upper semi-continuous with respect to the Hausdorff metric, i.e. given a neighbourhood* U *of* 0 *in* $\mathbb{C}$, *there is a neighbourhood* V *of* ω *in* Ω *such that*

$$\mathrm{Sp}(A(\tau); A(\tau)) \subseteq \mathrm{Sp}(A(\omega); A(\omega)) + U,$$

whenever $\tau \in V$.

Proof. Choose an arbitrary $\epsilon > 0$, and let $B(0, \epsilon)$ denote the open ball of radius ϵ centred at 0 in $\mathbb{C}$. Fix ω in Ω and choose λ not in $\mathrm{Sp}(A(\omega); A(\omega)) + B(0, \epsilon)$. Then

$$\begin{aligned}
&\mathrm{dist}(\lambda; \mathrm{Sp}(A(\omega); A(\omega))) \geqslant \epsilon \\
&\quad \Leftrightarrow \inf\{|\lambda - \mu| : \mu \in \mathrm{Sp}(A(\omega); A(\omega))\} \geqslant \epsilon \\
&\quad \Leftrightarrow \inf\{|\mu| : \mu \in \mathrm{Sp}(A(\omega) - \lambda I(\omega); A(\omega))\} \geqslant \epsilon \\
&\quad \Leftrightarrow \sup\{|\mu|^{-1} : \mu \in \mathrm{Sp}([A(\omega) - \lambda I(\omega)]^{-1}; A(\omega))\} \leqslant \epsilon^{-1} \\
&\quad \Leftrightarrow \rho([A(\omega) - \lambda I(\omega)]^{-1}) \leqslant \epsilon^{-1}.
\end{aligned}$$

Now, by Proposition 3·4, there is an open set $V(\lambda)$ in Ω, containing ω, on which the map $\tau \to \rho([A(\tau) - \lambda I(\tau)]^{-1})$ is defined and is upper semicontinuous. Hence, there is an open set $W(\lambda) \subseteq V(\lambda)$, such that $\rho([A(\tau) - \lambda I(\tau)]^{-1}) < \epsilon^{-1} + 1$ whenever $\tau \in W(\lambda)$. Reversing the argument above shows that

$$\mathrm{dist}(\lambda; \mathrm{Sp}(A(\tau); A(\tau))) \geqslant \epsilon(\epsilon + 1)^{-1}$$

$\tau \in W(\lambda)$. Assuming, as we may, that $\epsilon < 1$, we may conclude that

$$\mathrm{dist}(\lambda; \mathrm{Sp}(A(\tau); A(\tau))) \geqslant \tfrac{1}{2}\epsilon \quad \text{for all } \tau \text{ in } W(\lambda).$$

Thus

$$\mathrm{dist}(B(\lambda; \epsilon/4); \mathrm{Sp}(A(\tau); A(\tau))) \geqslant \tfrac{1}{4}\epsilon,$$

whenever $\tau \in W(\lambda)$.

Now by Proposition 3·1 (with $P = I$), we know that $\mathrm{Sp}(A(\tau); A(\tau)) \subseteq \mathrm{Sp}(A; A)$ for all τ in Ω and so

$$S = \{\mu \in \mathrm{Sp}(A; A) : \mathrm{dist}(\mu; \mathrm{Sp}(A(\omega); A(\omega))) \geqslant \epsilon\}$$

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is a compact subset of $\mathbb{C}$. By the preceding argument, we see that S may be covered by the open sets $B(\lambda; \frac{1}{4}\epsilon)$ associated with each λ in S. Extracting a finite subcover, we can then find $\lambda_1, \ldots, \lambda_N$ such that

$$S \subseteq \bigcup_{j=1}^{N} B(\lambda_j; \tfrac{1}{4}\epsilon).$$

Let $V_j = W(\lambda_j)$ be the open set in Ω associated with each λ_j. Then each V_j is a neighbourhood of ω and so

$$V = \bigcap_{j=1}^{N} V_j$$

is also a neighbourhood of ω in Ω.

Furthermore, for τ in V and λ in S we have

$$\operatorname{dist}(\lambda; \operatorname{Sp}(A(\tau); A(\tau))) \geqslant \tfrac{1}{4}\epsilon,$$

i.e. for all τ in V, we have

$$\operatorname{Sp}(A(\tau); A(\tau)) \subseteq \operatorname{Sp}(A(\omega); A(\omega)) + B(0; \epsilon)$$

providing the semi-continuity we require.

4. *The spectrum of Δ_A on a* W**-algebra.* Throughout this section, A will denote a W*-algebra with centre Z which has a maximal ideal space Ω. If A is commutative, then clearly $\operatorname{Sp}(\Delta_A; L(A)) = \{0\}$, while if A is a factor it has been shown (in (5)) that $\operatorname{Sp}(\Delta_A; L(A))$ is as large as it can be, namely $\operatorname{Sp}(A; A) - \operatorname{Sp}(A; A)$. These cases represent the two extremes for a W*-algebra with respect to commutativity, and suggest that $\operatorname{Sp}(\Delta_A; L(A))$ may depend on 'how commutative' A is. This suggestion turns out to be well founded, and, in what follows, we shall show that there is a direct dependence of $\operatorname{Sp}(\Delta_A; L(A))$ on Z.

4·1. Lemma. *For ω in Ω, suppose that λ does not belong to* $\operatorname{Sp}(A(\omega); A(\omega)) - \operatorname{Sp}(A(\omega); A(\omega))$. *Then there is an open and closed set* V *in* Ω, *containing* ω, *for which*

$$\lambda \notin \operatorname{Sp}(A(\sigma); A(\sigma)) - \operatorname{Sp}(A(\tau); A(\tau))$$

whenever σ and τ belong to V.

Proof. Let ϵ denote the distance of λ from $\operatorname{Sp}(A(\omega); A(\omega)) - \operatorname{Sp}(A(\omega); A(\omega))$ (by compactness $\epsilon > 0$). We can now apply Proposition 3·5 to obtain a neighbourhood V of ω in Ω such that

$$\operatorname{Sp}(A(\tau); A(\tau)) \subseteq \operatorname{Sp}(A(\omega); A(\omega)) + B(0; \tfrac{1}{2}\epsilon), \quad \text{whenever} \quad \tau \in V.$$

We can assume that V is open and closed (if necessary, take an open subset W of V, such that $\overline{W} \subseteq V$). Thus if x_1 and x_2 belong to

$$\cup \{\operatorname{Sp}(A(\tau); A(\tau)) : \tau \in V\},$$

we can find λ_j in $\operatorname{Sp}(A(\omega); A(\omega))$, such that $|x_j - \lambda_j| < \frac{1}{2}\epsilon$. Hence if $\lambda = (x_1 - x_2)$, we have

$$|\lambda - (\lambda_1 - \lambda_2)| \leqslant |(x_1 - \lambda_1) - (x_2 - \lambda_2)| < \epsilon,$$

which contradicts the definition of ϵ.

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4·2. PROPOSITION. *For a fixed* A *in* A, $(\Delta_A - \lambda I)$ *has an inverse in* $L(A)$ *whenever*

$$\lambda \notin \cup \{x-y : x, y \in \mathrm{Sp}(A(\omega); A(\omega))\},$$

where the union is taken over all ω *in* Ω.

Proof. By Lemma 4·1, to each ω in Ω we can associate an open and closed neighbourhood $U(\omega)$ of ω for which

$$\lambda \notin \mathrm{Sp}(A(\tau); A(\tau)) - \mathrm{Sp}(A(\sigma); A(\sigma)),$$

whenever τ and σ lie in $U(\omega)$. Thus, as we vary ω, we obtain an open cover

$$\{U(\omega) : \omega \in \Omega\} \text{ of } \Omega.$$

Compactness allows us to select a finite subcover $U_1, \ldots, U_N$ each of which is both open and closed. Taking intersections, if necessary, we may assume that $U_1, \ldots, U_N$ are pairwise disjoint. Let $P_j = \chi(U_j)$. Then $P_1, \ldots, P_N$ is a family of mutually orthogonal projections in Z with $\Sigma_{j=1}^{N} P_j = I$. Further, since

$$\Delta_A(AP_j) \subseteq AP_j \quad (j = 1, \ldots, N),$$

we know that
$$\mathrm{Sp}(\Delta_A; L(A)) = \bigcup_{j=1}^{N} \mathrm{Sp}(\Delta_A|_{AP_j}; L(AP_j)).$$

Now $\Delta_A|_{AP_j} = \Delta_{AP_j}$ on AP_j and so

$$\mathrm{Sp}(\Delta_A|_{AP_j}; L(AP_j)) \subseteq \{x-y : x, y \in \mathrm{Sp}(AP_j; AP_j)\}$$

and, since
$$\mathrm{Sp}(AP_j; AP_j) = \cup \{\mathrm{Sp}(A(\tau); A(\tau)) : \tau \in U_j\},$$
we conclude that

$$\mathrm{Sp}(\Delta_A|_{AP_j}; L(AP_j)) \subseteq \cup \{\mathrm{Sp}(A(\tau); A(\tau)) - \mathrm{Sp}(A(\sigma); A(\sigma))\},$$

where the union is taken over $\tau, \sigma \in U_j$. Since λ cannot belong to a set of this form, we see that $\lambda \notin (\mathrm{Sp}(\Delta_A|_{AP_j}; L(AP_j))$ and so $\lambda \notin \mathrm{Sp}(\Delta_A; L(A))$.

4·3. THEOREM. *Suppose* A *implements the derivation* Δ_A *on the* W*-*algebra* A. *Then*

$$\mathrm{Sp}(\Delta_A; L(A)) = \cup \{x-y : x, y \in \mathrm{Sp}(A(\omega); A(\omega))\},$$

where the union is taken over all ω *in* Ω.

Proof. Suppose $\lambda \in \mathbb{C}$ and that $(\Delta_A - \lambda I)$ is non-singular in $L(A)$. Now,

$$(\Delta_A - \lambda I)(\sum_j X_j Z_j) = \sum_j (\Delta_A - \lambda I)(X_j) Z_j,$$

whenever $X_j \in A$ and $Z_j \in Z$. Thus $(\Delta_A - \lambda I)$ maps $J(\omega)$ into $J(\omega)$. Similarly, we can show that $(\Delta_A - \lambda I)^{-1}$ maps $J(\omega)$ into $J(\omega)$, and so $(\Delta_A - \lambda I)$ induces a non-singular map in $L(A(\omega))$. Routine calculations show that this induced map is given by

$$X \to XA(\omega) - A(\omega)X - \lambda I(\omega)$$

for all X in $A(\omega)$. Now, by Halpern's result (3), each $A(\omega)$ has a faithful irreducible representation and so, by Proposition 2·3, $\lambda \notin \mathrm{Sp}(A(\omega); A(\omega)) - \mathrm{Sp}(A(\omega); A(\omega))$ for all

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ω in Ω. This establishes one inclusion. Proposition 4·2 provides the other and completes the proof.

4·4. *Remark.* All the methods and results quoted so far apply equally well, with slight modification, to the more general operator Γ described by

$$\Gamma(X) = \sum_{j=1}^{N} f_j(A_1)\, X g_j(A_2),$$

where $A_1, A_2 \in A$ and f_j, g_j are rational functions defined on $\mathbb{C}$ such that the poles of f_j lie outside $\mathrm{Sp}(A_1; A)$ and the poles of g_j lie outside $\mathrm{Sp}(A_2; A)$. Details of these results may be found in (5).

5. *Another description of* $\mathrm{Sp}(\Delta_A; L(A))$ *when* A *is a* W**-algebra.* In this section, we give another description of $\mathrm{Sp}(\Delta_A; L(A))$ for A in a W*-algebra A. We use the results of section 4 to obtain a more elegant and accessible formula. As before, Z will denote the centre of A and Ω denotes the maximal ideal space of Z.

5·1. Definition. *Let* A *belong to* A. *Two members* x, y, *of* $\mathrm{Sp}(A; A)$ *are said to be* centrally separated *when we can find a central projection* P *in* A *for which*

$$x \in \mathrm{Sp}(AP; AP) \backslash \mathrm{Sp}(A(I-P); A(I-P))$$

and

$$y \in \mathrm{Sp}(A(I-P); A(I-P)) \backslash \mathrm{Sp}(AP; AP).$$

Using Definition 5·1, we may define a binary relation $\sim$ on $\mathrm{Sp}(A; A)$ as follows: $x \sim y$ if, and only if, x and y are not centrally separated. Unfortunately, $\sim$ is not an equivalence relation, as the following example shows.

Let $A = M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ and let $A = A_1 \oplus A_2$, where

$$A_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix}.$$

Then $\alpha \sim \beta$ and $\beta \sim \gamma$, but α and γ are centrally separated.

Despite this, $\sim$ is the relevant relation for our purposes, as the following result shows.

5·2. Theorem. *Let* A *be a* W**-algebra. Then for* A *in* A,

$$\mathrm{Sp}(\Delta_A; L(A)) = \{x - y : x, y \in \mathrm{Sp}(A; A) \quad \textit{and} \quad x \sim y\}$$

Proof. By virtue of Theorem 4·3 we need only show that

$$\bigcup_{\omega \in \Omega} \{(x-y) : x, y \in \mathrm{Sp}(A(\omega); A(\omega))\} = \{(x-y) : x, y \in \mathrm{Sp}(A; A) \quad \text{and} \quad x \sim y\}$$

and this will follow when we show that

$$\bigcup_{\omega \in \Omega} \mathrm{Sp}(A(\omega); A(\omega)) \times \mathrm{Sp}(A(\omega); A(\omega)) \quad \text{and} \quad \{(x, y) : x \sim y\}$$

are identical subsets of $\mathrm{Sp}(A; A) \times \mathrm{Sp}(A; A)$.

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Suppose, then, that $(x_0, y_0) \in \mathrm{Sp}(\mathrm{A}; A) \times \mathrm{Sp}(\mathrm{A}; A)$ and that, for all ω in Ω, (x_0, y_0) does not belong to $\mathrm{Sp}(\mathrm{A}(\omega); A(\omega)) \times \mathrm{Sp}(\mathrm{A}(\omega); A(\omega))$.

Consider the following two sets:

$$\mathrm{S}_1 = \{\omega \in \Omega : x_0 \in \mathrm{Sp}(\mathrm{A}(\omega); A(\omega))\},$$

$$\mathrm{S}_2 = \{\omega \in \Omega : y_0 \in \mathrm{Sp}(\mathrm{A}(\omega); A(\omega))\}.$$

Clearly S_1 and S_2 are non-empty, and by assumption $\mathrm{S}_1 \cap \mathrm{S}_2$ is empty. Furthermore, both S_1 and S_2 are closed. (Suppose that σ does not belong to S_1. Then

$$x_0 \notin \mathrm{Sp}(\mathrm{A}(\sigma); A(\sigma))$$

and so, by Proposition 3·2, there is a neighbourhood V of σ for which $x_0 \notin \mathrm{Sp}(\mathrm{A}(\tau); A(\tau))$, whenever $\tau \in \mathrm{V}$. Hence the complement of S_1 is open. A similar argument applies to S_2.)

Using the properties of extremally disconnected spaces, we can find sets U_1 and U_2 in Ω, which are both open and closed with $\mathrm{U}_1 \cap \mathrm{U}_2 = \varnothing$ and

$$\mathrm{S}_1 \subseteq \mathrm{U}_1; \quad \mathrm{S}_2 \subseteq \mathrm{U}_2.$$

Let $\mathrm{P} = \chi(\mathrm{U}_1)$. Then P is a projection in Z and, if $\mathrm{Q} = \chi(\mathrm{U}_2)$, we have $\mathrm{Q} \leqslant \mathrm{I} - \mathrm{P}$. Thus we have

$$x_0 \in \mathrm{Sp}(\mathrm{AP}; A\mathrm{P}) \backslash \mathrm{Sp}(\mathrm{A}(\mathrm{I}-\mathrm{P}); A(\mathrm{I}-\mathrm{P}))$$

and

$$y_0 \in \mathrm{Sp}(\mathrm{A}(\mathrm{I}-\mathrm{P}); A(\mathrm{I}-\mathrm{P})) \backslash \mathrm{Sp}(\mathrm{AP}; A\mathrm{P}),$$

i.e. x_0 and y_0 are centrally separated.

Conversely, if x_0 and y_0 belong to $\mathrm{Sp}(\mathrm{A}; A)$ and are centrally separated, we can find a projection P in Z with

$$x_0 \in \mathrm{Sp}(\mathrm{AP}; A\mathrm{P}) \backslash \mathrm{Sp}(\mathrm{A}(\mathrm{I}-\mathrm{P}); A(\mathrm{I}-\mathrm{P}))$$

and

$$y_0 \in \mathrm{Sp}(\mathrm{A}(\mathrm{I}-\mathrm{P}); A(\mathrm{I}-\mathrm{P})) \backslash \mathrm{Sp}(\mathrm{AP}; A\mathrm{P}).$$

Now suppose that $(x_0, y_0) \in \mathrm{Sp}(\mathrm{A}(\omega); A(\omega)) \times \mathrm{Sp}(\mathrm{A}(\omega); A(\omega))$ for some ω in Ω. Then if $\mathrm{P} = \chi(\Lambda)$, either $\omega \in \Lambda$ or $\omega \in \Omega \backslash \Lambda$. If $\omega \in \Lambda$, then both x_0 and y_0 lie in $\mathrm{Sp}(\mathrm{AP}; A\mathrm{P})$, while if $\omega \in \Omega \backslash \Lambda$, both belong to $\mathrm{Sp}(\mathrm{A}(\mathrm{I}-\mathrm{P}); A(\mathrm{I}-\mathrm{P}))$. In either case, we arrive at a contradiction. Thus

$$\bigcup_{\omega \in \Omega} \mathrm{Sp}(\mathrm{A}(\omega); A(\omega)) \times \mathrm{Sp}(\mathrm{A}(\omega); A(\omega)) = \{(x, y) : x \sim y\}$$

as required.

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