NON-MEASURABLE INTERPOLATION SETS

I. INTEGRAL FUNCTIONS

By M. E. NOBLE*

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1. *Introduction.* In two classical papers (1, 2) J. M. Whittaker introduced the study of integral functions bounded at the lattice points $m+in$ $(m, n = 0, \pm 1, \ldots,)$. He succeeded in showing (cf. also G. Polya (3)) that an integral function of at most the minimum type of order 2 uniformly bounded at the lattice points was necessarily constant. This result was improved almost simultaneously by A. Pflüger (5) and V. Ganapathy Iyer (11), who showed that the result was true also for functions of type $K < \frac{1}{2}\pi$ of order 2. The example of Weierstrass's $\sigma(z)$ function shows that theirs is a best possible result in this direction.

The object of the present paper is the proof of similar results involving sequences $\{z_n\}$ more general than the lattice points. A number of such generalizations are already in the literature.

B. J. Maitland (16) showed that the lattice points could be replaced by a set $z_{m,n}$ for which $m \leqslant \Re z_{m,n} \leqslant (m+1)$, $n \leqslant \Im z_{m,n} \leqslant (n+1)$ provided that $|z_{m,n} - z_{p,q}| \geqslant \delta > 0$ if $m \neq p$ or $n \neq q$. In a somewhat earlier paper (12, 13) (cf. also (15)†) V. Ganapathy Iyer generalized the problem in a different direction. He attempted to characterize sets $\{z_n\}$ 'effective for ρ, d' in the sense that

$$K(z_n) = \overline{\lim_{|z_n| \to \infty}} |z_n|^{-\rho} \log |f(z_n)| = \overline{\lim_{r \to \infty}} r^{-\rho} \log M(r) = K(f) \tag{1·1}$$

for all integral functions of type $K(f) < d$ of order ρ. His paper contains a number of conditions that must be satisfied by such a set, but the only sufficient set of conditions that he obtained depends on the canonical product of the z_n rather than on their geometrical distribution directly. Finally, he showed that under the same conditions the z_n form a set effective for the original Whittaker problem and functions of type $K(f) < d$ of order ρ.

Still more recently Pflüger (8, 9) (cf. also B. Lévine (10)) has shown, by a method based on his earlier work on Lindelöf's $h(\theta)$ function for integral functions with measurable zeros (5), that for the original Whittaker problem $\{z_n\}$ constituted a set effective ρ, d if

(i) for any θ_1, θ_2 with $|\theta_1 - \theta_2| \leqslant 2\pi$, as $r \to \infty$

$$r^{-\rho} n(r, \theta_1, \theta_2) \to \rho d(\theta_1 \sim \theta_2),$$

(ii) when ρ is an integer, $\sum_{|z_n|<r} z_n^{-\rho}$ is convergent as $r \to \infty$,

(iii) for some δ if $m \neq n$

$$|z_m - z_n| \geqslant \delta \min (|z_n|^{1-\frac{1}{2}\rho}, |z_m|^{1-\frac{1}{2}\rho}).$$

* Present address: The University, Nottingham.

† I am indebted to the referee for this reference.

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The condition (iii) is the analogue of Maitland's condition $|z_{m,n}-z_{p,q}|\geqslant\delta$ and was introduced into the general problem essentially by Lévine* (10). Pflüger's paper (8) contains a partial result under conditions which are much more restrictive than (i), (ii), (iii) if ρ is an integer. In both papers he uses an analogue of Whittaker's interpolation formula proved under very general conditions in which, for example,

$$\frac{1}{\theta_1\sim\theta_2}\lim_{r\to\infty}\frac{1}{r^\rho}n(r,\theta_1,\theta_2)$$

varies with θ_1, θ_2.

The related problem for functions regular in an angle has been studied by Whittaker (2), Pflüger (5, 6), Ganapathy Iyer (14) and M. L. Cartwright (17, 18).

2. The condition (i) which involves the existence of a sectorial density is essential to Pflüger's argument, and the essential novelty of the present paper consists in the partial elimination of this condition. Our method, however, will only deal with distributions of z_n with a substantial uniformity as to direction, and therefore our results are in part complementary to Pflüger's.

Throughout the paper we shall say that a sequence z_n satisfies conditions D_ρ if $0<\rho<\infty$ and

(i) for any two sectors $S_1(\theta_1\leqslant\arg z<\phi_1)$, $S_2(\theta_2\leqslant\arg z<\phi_2)$ as $r\to\infty$

$$\frac{n(r,S_1)}{n(r,S_2)}\to\frac{\phi_1-\theta_1}{\phi_2-\theta_2};$$

(ii) if ρ is an integer, for some α as $r\to\infty$

$$\sum_{|z_n|\leqslant r}z_n^{-\rho}\to\alpha;$$

(iii)
$$\lim_{\xi\to0}\overline{\lim_{r\to\infty}}\frac{n\{(1+\xi)r\}-n(r)}{\{(1+\xi)^\rho-1\}}=D<\infty;$$

(iv) there exist Θ, d, $\Theta>0$, $d>0$, such that for every sufficiently small positive ξ and any $\epsilon>0$ there exists an $R_0(\xi,\epsilon)$ such that if $R>R_0$ and $E_{\xi,\epsilon}(R)$ is the set of $r\leqslant R$ for which

$$\frac{n\{(1+\xi)r\}-n(r)}{\{(1+\xi)^\rho-1\}r^\rho}\geqslant d(1-\epsilon),$$

then
$$\text{meas}\,E(R)\geqslant\Theta(1-\epsilon)R;$$

(v) for some $h>0$ if $m\neq n$

$$|z_m-z_n|\geqslant h\min(|z_n|^{1-\frac{1}{2}\rho},|z_m|^{1-\frac{1}{2}\rho}).$$

We remark concerning D_ρ that (i) is essential to our line of approach and may be compared to Pflüger's density condition § 1 (i). The conditions (iii) and (iv) roughly represent the generalization achieved in this paper; (iii) implies but is not implied by

(iii)′
$$\overline{\lim}\,r^{-\rho}n(r)<\infty.$$

The status of (iv) is more complicated. Unless $\Theta=1$ it is much weaker than

(iv)′
$$\lim_{\xi\to0}\overline{\lim_{r\to\infty}}\frac{n(1+\xi)r-n(r)}{\{(1+\xi)^\rho-1\}r^\rho}=d>0,$$

but both (iv) and (iv)′ imply $\lim r^{-\rho}n(r)>0$.

* My knowledge of Lévine's paper is derived from a review (*Math. Rev.* 2 (1941)) and Pflüger's comments ((7), Appendix).

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As permanent notation we shall write $p = [\rho]$ and $q = \rho - 1$ or $[\rho]$, according to whether ρ is or is not an integer. We also require the function $\tau_\rho(\theta)$ defined by

$$\tau_\rho(\theta) = \rho^2 \sup_E \int_E x^{\rho-1} \log \frac{1}{x} dx,$$

where E ranges through all subsets of $[0, 1]$ for which meas $E = \theta$. If $\rho \leqslant 1$,

$$\tau_\rho(\theta) = \rho^2 \int_0^\theta x^{\rho-1} \log \frac{1}{x} dx.$$

When $\{z_n\}$ satisfies D_ρ or a similar set of conditions which give meaning to the definition, we write

$$\sigma_g(z) = F(z) \prod_{\nu=1}^{\infty} E\left(\frac{z}{z_\nu}, p\right), \tag{2·1}$$

where $E(u, p)$ is Weierstrass's primary factor of genus p and $F(z) = \exp(-\alpha z^p/p)$ or 1 according to whether ρ is or is not an integer. Convergence of the product in (2·1) is assured by (iii)′ which implies the convergence of $\Sigma |z_n|^{-\rho-1}$.

Many of our arguments involve an exceptional set $\mathscr{E}$ which can be enclosed in a set S of circles the sum of whose radii does not exceed $M(S)$. We define the radial measure* of $\mathscr{E}$ by $m_r(\mathscr{E}) = \inf_S M(S)$.

The principal results of the paper are then the following:

THEOREM 1. *Suppose that a sequence $\{z_n\}$ satisfies D_ρ and that $\sigma_g(z)$ is defined by* (2·1). *Then to any $\epsilon > 0$ corresponds $R_1(\epsilon)$ such that*

(i) *if* $|z| = r \geqslant R_1(\epsilon)$
$$\log |\sigma_g(z)| \leqslant \left(\frac{D}{\rho} + \epsilon\right) r^\rho;$$

(ii) *if* $R \geqslant R_1$ *and* $|z| \leqslant R$
$$\log |\sigma_g(z)| \geqslant \left\{\frac{d}{\rho}\{1 - \tau(1-\Theta)\} - \epsilon\right\} r^\rho,$$

except for a set $\mathscr{E}(R)$ of z for which $m_r(\mathscr{E}) \leqslant \epsilon R$;

(iii) *if* $|z_n| \geqslant R_1$
$$\log |\sigma_g'(z_n)| \geqslant \left\{\frac{d}{\rho}\{1 - \tau(1-\Theta)\} - \epsilon\right\} |z_n|^\rho.$$

Theorem 1 yields several types of interpolation result.

THEOREM 2. *If $\{z_n\}$ satisfies D_ρ and $f(z)$ is an integral function of type $K(f)$ and order ρ for which*

$$\underline{\lim}\, r^{-\rho} \log M(r, f) < \frac{d}{\rho}\{1 - \tau(1-\Theta)\},$$

$$K(z_n) = \overline{\lim}\, |z_n|^{-\rho} \log |f(z_n)| < \frac{d}{\rho}\{1 - \tau(1-\Theta)\},$$

then
$$K(f) < K(z_n)\frac{d}{D}\{1 - \tau(1-\Theta)\} + \frac{1}{\rho}\left(D - \frac{d^2}{D}\{1 - \tau(1-\Theta)\}^2\right).$$

THEOREM 3. *Suppose that $\{z_n\}$ satisfies D_ρ and that $f(z)$ is an integral function of type $K < \frac{d}{\rho}\{1 - \tau(1-\Theta)\}$ of order ρ. Then*

(i) $f(z_n) = O(1)$ *implies that $f(z)$ is constant,*

(ii) $f(z_n) \to 0$ *as* $|z_n| \to \infty$ *implies that* $f(z) \equiv 0$.

* The notation is due to Milloux (**20**), p. 180. I am indebted to the referee for this suggestion.

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There are a number of important special cases of Theorem 2 which we state as corollaries.

COROLLARY 1. *If, in addition to the assumptions of Theorem* 2, $\Theta = 1$, *then*

$$K(f) \leqslant K(z_n)\frac{d}{D} + \left(D - \frac{d^2}{D}\right)\frac{1}{\rho}.$$

COROLLARY 2. *If, in addition to the assumptions of Theorem* 2, $d = D$, *then*

$$K(f) \leqslant K(z_n) + \left(\frac{d}{\rho} - K(z_n)\right)\tau(1-\Theta) + \frac{d}{\rho}\{\tau(1-\Theta) - \tau^2(1-\Theta)\}.$$

COROLLARY 3. *If both* $\Theta = 1$ *and* $D = d$, *then*

$$K(z_n) = K(f).$$

In the case of non-integral ρ the troublesome convergence condition (ii) is not involved, and Theorem 2 can be refined by going over to a subsequence of $\{z_n\}$ to yield

THEOREM 4. *Suppose that the sequence* $\{z_n\}$ *satisfies* D_ρ *for a non-integral* ρ *and that* $f(z)$ *is an integral function of order* ρ *and type* $K(f)$. *Suppose also that*

$$\underline{\lim}\, r^{-\rho}\log M(r) < \frac{d}{\rho}\{1-\tau(1-\Theta)\}, \tag{2·2}$$

and

$$K(z_n) = \overline{\lim}\,|z_n|^{-\rho}\log|f(z_n)| < \frac{d}{\rho}\{1-\tau(1-\Theta)\}. \tag{2·3}$$

Then

$$K(f) \leqslant K(z_n) + \left\{\frac{d}{\rho} - K(z_n)\right\}\tau(1-\Theta) + \frac{d}{\rho}\{\tau(1-\Theta) - \tau^2(1-\Theta)\}. \tag{2·4}$$

This also has an important special case.

COROLLARY 4. *If, in addition to the hypotheses of Theorem* 4, $\Theta = 1$, *then*

$$K(f) = K(z_n).$$

We note that in Theorem 2 and Corollary 1 the upper bound given for $K(f)$ is related closely to $K(z_n)$ only if $D/d - 1$ is small; if D/d is very large then the theorem asserts little more than

$$K(f) < \frac{D}{\rho}.$$

With this limitation our results cover interpolation with respect to sets of points which roughly have a non-zero *local* lower density for an 'appreciable proportion' of r. If $d = 0$ the interpolation results fail even though $\Theta = 1$, but Theorem 1 then yields

COROLLARY 5. *Suppose that under the conditions of Theorem* 1 $d = 0$ *and* $\Theta = 1$. *Then to every* $\epsilon > 0$ *corresponds an* $R_0(\epsilon)$ *such that if* $R > R_0$ *and* $|z| \leqslant R$

$$\log|\sigma_g(z)| \geqslant -\epsilon r^\rho,$$

except for a set $\mathscr{E}(R)$ *of* z *for which* $m_r(\mathscr{E}) \leqslant \epsilon R$.

It is easy to see that when $\Theta = 1$ the hypothesis of Corollary 4 as far as the lower density of $\{z_n\}$ is concerned is equivalent to (iv)'. It should therefore be possible to prove Corollary 4 by showing that $\{z_n\}$ contains a measurable subsequence of density d (cf. Polya (4), p. 562), which, since $K(z_n)$ is not increased by going over to a subsequence, would essentially reduce Corollary 4 to Pflüger's result. This argument is, however, not available for the case of integral ρ or for $\Theta < 1$ and is not used in this paper.

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The proof of the interpolation theorems depends on establishing an analogue of Whittaker's interpolation formula (Lemma 11), the proof of which requires that $|\sigma_g'(z_n)|$ shall be large enough to ensure the convergence of the cardinal series involved and that $f(z)/\sigma_g(z)$ shall be small on a sequence of circles tending to ∞. This gives a relation between the distribution of z_n and the type of function that can be interpolated, and there is a further relation between the distribution of circles on which $\sigma_g(z)$ is large and the magnitude of $\sigma_g'(z_n)$. It is possible that a different balance between $|\sigma_g(z)|$, $|\sigma_g'(z_n)|$ and $M(r,f)$ from that struck in this paper would also yield interesting results.

3. The proof of Theorem 1 by dissection of the z_n involves a somewhat complicated calculation throughout which certain constant notations, in addition to those of § 2, will be used. Throughout, λ will denote a real number for which $\frac{3}{4} \leqslant \lambda < 1$ usually thought of as close to 1. A 'quadrilateral' will always denote a region intercepted on an annulus by a sector. If A_s, B_s, ... denote annuli, then the quadrilaterals will be denoted $A_{s,t}$, $B_{s,t}$, ... K, K_ρ will be used respectively for an absolute constant and a number depending only on the order ρ concerned; both may vary with each appearance. Finally, for any sequence $\{z_n\}$ involved in the argument $n(\mathrm{D})$ will denote the number of z_n lying in the region D, usually an annulus or quadrilateral, and $n(r)$ the number for which $|z_n| \leqslant r$.

4. *Fundamental lemmas.* We collect here some results that play a fundamental role in the argument but which have themselves only a tenuous connexion with the theory of integral functions. The first is H. Cartan's well-known result concerning the minimum modulus of a polynomial (19).

LEMMA 1. *Let $z_1, \ldots, z_N$ be any N complex numbers and h a positive number. Then the z for which*

$$\left|\prod_{\nu=1}^{N}(z-z_\nu)\right| \leqslant \left(\frac{h}{2e}\right)^N \tag{4·1}$$

can be enclosed in at most N circles, the sum of whose radii does not exceed h.

In proving part (iii) of Theorem 1 we use an argument, in principle the same as that of Pflüger and Lévine, for which we require the following result which is of independent interest.

LEMMA 2. *Let $z_1, \ldots, z_N$ be any complex N numbers and δ_1 a real number such that if $i, j \leqslant N$ and $i \neq j$*

$$|z_i - z_j| \geqslant \delta_1 > 0. \tag{4·2}$$

Then there exists an absolute constant K such that for any i

$$\sum_{i \neq j} \frac{1}{|z_i - z_j|} \leqslant \frac{KN^{\frac{1}{2}}}{\delta_1}. \tag{4·3}$$

In fact, except for the constant K this is a best possible inequality, but this fact is not required here and the proof is therefore omitted.

To prove (4·3) consider the system of annuli B_n, where $n \geqslant 1$, for which

$$n\delta_1 \leqslant |z - z_i| \leqslant (n+1)\,\delta_1.$$

We remark first that for a sufficiently large N_0 (say $N_0 > 36$) any z-region in which $n\delta_1 \leqslant |z| \leqslant (n+1)\,\delta_1$, $\phi_0 \leqslant \arg z \leqslant \phi_0 + \pi/(N_0 n)$ can include at most two of the z_n. For one such z_n excludes all others from the region $|z - z_n| \leqslant \delta_1$, and this circle leaves

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uncovered a portion of the z-region whose diameter is less than δ_1. It follows immediately that B_n can contain at most $4\pi/(\pi/nN_0)$ or $4N_0 n$ of the z_n. Let m_0 be the least integer m such that

$$4N_0 \sum_{\nu=1}^{m} \nu \geqslant N;$$

clearly $m_0 \leqslant N^{\frac{1}{2}}$. Now consider *any* distribution of numbers z_j' in which $|z_j' - z_i| \geqslant \delta_1$ and m_0 annuli B_n each contain $4N_0 n$ of the z_j'. We have at once

$$\sum_j \frac{1}{|z_i - z_j'|} \leqslant 4N_0 \sum_{\nu=1}^{m} \frac{n}{nd} \leqslant \frac{4N_0 N^{\frac{1}{2}}}{d}.$$

Moreover, any admissible $\{z_j\}$ for (4·2) differs from *some* set $\{z_j'\}$ by containing fewer numbers in all, and, in each B_n for which $n \leqslant m_0$, consisting of a subset of the $\{z_j'\}$. It follows immediately that for this $\{z_j'\}$

$$\sum_{i \neq j} \frac{1}{|z_i - z_j|} \leqslant \sum_j \frac{1}{|z_i - z_j'|} \leqslant \frac{4N_0 N^{\frac{1}{2}}}{d}.$$

Our next result is essentially a weak form of Sierpinski's well-known covering lemma (cf. E. C. Titchmarsh (21), p. 356) stated in terms of density instead of measure.

LEMMA 3. *Suppose that $\mathscr{E}$ is a set of real $x \geqslant 1$ and that*

$$\lim_{X \to \infty} \frac{\text{meas}\,(\mathscr{E}[1, X])}{X - 1} = \theta > 0.$$

Suppose also that for some $\omega > 0$ there is associated with each x of $\mathscr{E}$ an interval I_x, $[x, (1+\omega)x]$. Then a sequence $\{x_\nu\}$ of x of $\mathscr{E}$ can be chosen such that the I_{x_ν} are non-overlapping and to every $\epsilon > 0$ corresponds $X_0(\epsilon)$ such that if $X > X_0$

$$\sum_{x_\nu \leqslant X} |I_{x_\nu}| = \omega \sum_{x_\nu \leqslant X} x_\nu \geqslant \theta(1-\epsilon)X.$$

The following proof is a trivial modification of the relevant part of Sierpinski's proof.

Let $X_1 = \inf_{\mathscr{E}} x$. Then there exists an x_1 belonging to $\mathscr{E}$ for which $X_1 \leqslant x_1 \leqslant X_1 + \frac{1}{4}\epsilon$. Let I_{x_1} be the associated interval and $\mathscr{E}_1$ the subset of $\mathscr{E}$ for which $x \geqslant (1+\omega)x_1$. If now $X_2 = \inf_{\mathscr{E}_1} x$, there is an x_2 belonging to $\mathscr{E}_1$ for which $X_2 \leqslant x_2 \leqslant X_2 + \frac{1}{8}\epsilon$. Proceeding in this way we can define sequences X_n, x_n such that $X_n \leqslant x_n \leqslant X_n + \epsilon/2^{n+1}$, and associated intervals I_{x_n}. Let J_n be the interval $x_n - \frac{\epsilon}{2^{n+1}} \leqslant x \leqslant x_n$. For any X

$$\mathscr{E}[1, X] \subset \sum_{x_n \leqslant X} I_{x_n} + \sum_{x_n \leqslant X} J_{s_n},$$

and the I_{x_n} are disjoint. It follows that if $X > X_0(\epsilon)$

$$\sum_{x_n \leqslant X} |I_{x_n}| = \omega \sum_{x_n \leqslant X} x_n \geqslant \theta(1 - \tfrac{1}{4}\epsilon)(X-1) - \sum_{\nu=1}^{\infty} \frac{\epsilon}{2^{n+1}} \geqslant \theta(1-\epsilon)X.$$

An immediate consequence is

$$\sum_{(1+\omega)x_n \leqslant X} |I_{x_n}| \geqslant \theta(1-\epsilon)X - \omega X. \tag{4·4}$$

5. The main body of the paper is concerned with the proof of Theorem 1, the remaining results being deduced from it by slight modifications of well-known arguments.

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We first consider $\sigma_g(z)$ for z lying in an annulus $\frac{1}{2}(1+\lambda)R \leqslant |z| \leqslant \frac{1}{2\lambda}(1+\lambda)R$ and dissect its zeros z_n into blocks lying approximately in the regions

$$|z| < \lambda R, \quad \lambda R \leqslant |z| \leqslant \lambda^{-1}R, \quad \lambda^{-1}R < |z| \leqslant (1-\lambda)^{-1/(2\rho)}R, \quad (1-\lambda)^{-1/(2\rho)}R \leqslant |z|,$$

where $1-\lambda$ is small enough to ensure $\lambda > (1-\lambda)^{1/(2\rho)}$. Of these regions the last two yield crude error terms, and the second block can be dealt with by Lemma 1 producing an exceptional set of z. The region $|z| < \lambda R$ yields the main contribution to $|\sigma_g(z)|$, and to deal with it we utilize the special properties of the distribution. The detailed arguments involve a further dissection of the blocks into sectors of small aperture η and annuli of small relative thickness. We assume throughout the following §§ 5 and 6 that $\eta > 0$ and an integer M are chosen depending on λ so that $\eta = 2\pi/M^2$ and $\frac{1}{2}(1-\lambda)^2 \leqslant \eta \leqslant (1-\lambda)^2$, and that η, λ, M are then fixed.

The following lemma embodies some consequences of D_ρ:

LEMMA 4. *If $\{z_n\}$ satisfies D_ρ there exists an $R_1(\lambda)$ such that, if A_s is an annulus $\alpha_s \leqslant |z| \leqslant (1+\mu_s)\alpha_s$, where $(1-\lambda)^4 \leqslant \mu_s \leqslant (1-\lambda)^2$ and $\alpha_s \geqslant R_1$, and $A_{s,t}$ is the 'quadrilateral' cut off by $t\eta \leqslant \arg z < (t+1)\eta$, then*

$$n(A_s) \leqslant D\{1+(1-\lambda)^8\}\{(1+\mu_s^\rho)-1\}\alpha_s^\rho \tag{5·1}$$

$$|n(A_{s,t}) - n(A_{s,0})| \leqslant 2D(1-\lambda)^8\alpha_s^\rho, \tag{5·2}$$

and if $R > R_1$

$$M^2 n(R,0,\eta) \leqslant 2n(R) \tag{5·3}$$

$$n(R) \leqslant 4DR^\rho. \tag{5·4}$$

The proofs are obvious except perhaps for (5·2). We remark that

$$\begin{aligned} |n(A_{s,t}) - n(A_{s,0})| &= \{n[(1+\mu_s)\alpha_s, t\eta, (t+1)\eta] - n[\alpha_s, t\eta, (t+1)\eta]\} \\ &\quad - \{n[(1+\mu_s)\alpha_s, 0, \eta] - n(\alpha_s, 0, \eta)\} \\ &= \{n[(1+\mu_s)\alpha_s, t\eta, (t+1)\eta] - n[(1+\mu_s)\alpha_s, 0, \eta]\} \\ &\quad - \{n[\alpha_s, t\eta, (t+1)\eta] - n(\alpha_s, 0, \eta)\}, \end{aligned}$$

and (5·2) now follows immediately from D_ρ(i) and (5·1).

We next have two lemmas on canonical products which will be applied to deal with error terms. Each one has two parts dealing respectively with the cases $|z_i| \leqslant \lambda R < |z|$ and $|z| < \lambda^{-1}R \leqslant |z_i|$.

LEMMA 5. *Suppose that a canonical product of genus q has its zeros z_n lying in a set of N disjoint annuli A_i, $\alpha_i \leqslant |z| \leqslant \alpha_i(1+\omega_i)$, where $(1-\lambda)^4 \leqslant \omega_i \leqslant (1-\lambda)^2$, such that for some $R > 1$, either*

(i) $$\alpha_N(1+\omega_N) \leqslant \lambda R$$

or

(ii) $$\lambda^{-1}R \leqslant \alpha_1 < \alpha_N \leqslant (1-\lambda)^{-1/(2\rho)}R.$$

Suppose, moreover, that each annulus A_i contains at most $2D(1-\lambda)^6\alpha_i^\rho$, z_n. Then if

$$\tfrac{1}{2}(1+\lambda)R \leqslant |z| \leqslant \frac{1}{2\lambda}(1+\lambda)R,$$

$$|\log|w(z)|| \leqslant K_\rho(1-\lambda)^{\frac{1}{2}}\log\frac{1}{1-\lambda}r^\rho = K_\rho D\epsilon(\lambda)r^\rho.$$

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For $\varpi(z)$ has an $n(u)$ function for which

$$\begin{aligned} n(u) &\leqslant 2D(1-\lambda)^6 \sum_{\alpha_i \leqslant u} \alpha_i^\rho \\ &\leqslant 2D(1-\lambda)^6 u^\rho \sum_{\nu=0}^{\infty} \{1+(1-\lambda)^4\}^{-\rho\nu} \\ &\leqslant K_\rho D(1-\lambda)^2 u^\rho. \end{aligned}$$

We therefore have, if $|z_n| = r_n$, in case (i)

$$\begin{aligned} \sum \left(\frac{r}{r_n} + \ldots + \frac{1}{q}\frac{r^q}{r_n^q}\right) &\leqslant \int_{\frac{1}{2}}^{r} \left(\frac{r}{u} + \ldots + \frac{1}{q}\frac{r^q}{u^q}\right) dn(u) \\ &\leqslant K_\rho q D(1-\lambda)^2 r^\rho + K_\rho D(1-\lambda)^2 q \int_{\frac{1}{2}}^{r} \frac{r^q u^\rho}{u^q} du, \\ &\leqslant K_\rho D(1-\lambda)^2 r^\rho, \end{aligned} \tag{5·5}$$

$$\begin{aligned} \sum_{z_n} \log\left(1+\frac{r}{r_n}\right) &\leqslant \int_{\frac{1}{2}}^{r} \log\left(1+\frac{r}{u}\right) dn(u) \\ &\leqslant K_\rho D(1-\lambda)^2 r^\rho \log 2 + K_\rho D(1-\lambda)^2 r \int_{\frac{1}{2}}^{r} \frac{u^\rho}{u(u+r)} du \\ &\leqslant K_\rho D(1-\lambda)^2 r^\rho; \end{aligned} \tag{5·6}$$

and in case (ii)

$$\begin{aligned} \sum_{z_n} \left(\frac{r}{r_n} + \ldots + \frac{1}{q}\frac{r^q}{r_n^q}\right) &\leqslant \int_{r}^{2(1-\lambda)^{-1/(2\rho)}R} \left(\frac{r}{u} + \ldots + \frac{1}{q}\frac{r^q}{u^q}\right) dn(u) \\ &\leqslant n\{2(1-\lambda)^{-1/(2\rho)}R\} + K_\rho D q (1-\lambda)^2 r \int_{r}^{2(1-\lambda)^{-1/(2\rho)}R} u^{\rho-2} du \\ &\leqslant K_\rho D(1-\lambda) r^\rho, \end{aligned} \tag{5·7}$$

$$\begin{aligned} \sum_{z_n} \log\left(1+\frac{r}{r_n}\right) &\leqslant \int_{r}^{2(1-\lambda)^{-1/(2\rho)}R} \log\left(1+\frac{r}{u}\right) dn(u) \\ &\leqslant n\{2(1-\lambda)^{-1/(2\rho)}R\}\log 2 + K_\rho D(1-\lambda)^2 r \int_{r}^{2(1-\lambda)^{-1/(2\rho)}R} \frac{u^{\rho-1}}{(u+r)} du \\ &\leqslant K_\rho D(1-\lambda) r^\rho. \end{aligned} \tag{5·8}$$

Moreover, we have

$$\log|\varpi(z)| \leqslant \sum_{z_n} \left\{\log\left(1+\frac{r}{r_n}\right) + \frac{r}{r_n} + \ldots + \frac{1}{q}\left(\frac{r}{r_n}\right)^q\right\}, \tag{5·9}$$

$$\log|\varpi(z)| \geqslant \sum_{z_n} \left\{-\log\frac{2}{1-\lambda} - \frac{r}{r_n} - \ldots - \frac{1}{q}\left(\frac{r}{r_n}\right)^q\right\}, \tag{5·10}$$

and the lemma follows from (5·5), (5·6), (5·7), (5·8), (5·9) and (5·10).

LEMMA 6. *Let α_s, μ_s $(s = 1, 2, \ldots)$ be real sequences such that $\mu_s \leqslant (1-\lambda)^2$, the annuli A_s, $\alpha_s \leqslant |z| \leqslant (1+\mu_s)\alpha_s$ are disjoint and either for all s and some $R > 1$*

(i) $$(1+\mu_s)\alpha_s \leqslant \lambda R$$

or

(ii) $$\lambda^{-1}R \leqslant \alpha_s \leqslant (1-\lambda)^{-1/(2\rho)}R.$$

Suppose also that $\varpi(z)$ is a canonical product of genus q whose zeros z_n lie in the 'quadrilaterals' $A_{s,t}$ and are such that

(iii) $$n(A_{s,t}) \leqslant 2D\{(1+\mu_s)^\rho - 1\}\alpha_s^\rho,$$

(iv) $$n(u) \leqslant 2Du^\rho,$$

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and that $\varpi_0(z)$ *is the canonical product of genus* q *having* $n(A_{s,t})$ *zeros concentrated at* $\zeta_{s,t} = \alpha_s \exp it\eta$. *Then if*

$$\tfrac{1}{2}(1+\lambda)R \leqslant |z| \leqslant \frac{1}{2\lambda}(1+\lambda)R,$$

$$|\log|\varpi_0(z)| - \log|\varpi(z)|| \leqslant K_\rho D(1-\lambda)^{\frac{1}{2}} r^\rho.$$

For $$\log\varpi(z) - \log\varpi_0(z) = \sum_{l=1}^{q} \frac{z^l}{l} \sum_{s,t} \sum_{A_{s,t}} \left(\frac{1}{z_n^l} - \frac{1}{\zeta_{s,t}^l}\right) - \sum_{s,t} \sum_{A_{s,t}} \log\left(1 - \frac{z(z_n - \zeta_{s,t})}{\zeta_{s,t}(z_n - z)}\right),$$

and therefore, observing that $\left|\dfrac{z}{z_n - z}\right| \leqslant \dfrac{2}{1-\lambda}$, we obtain, if N is the total number of z_n,

$$|\log\varpi(z) - \log\varpi_0(z)| \leqslant \sum_{l=1}^{q} \frac{r^l}{l} \sum_{s,t} \sum_{A_{s,t}} \frac{l(1-\lambda)^2}{|\zeta_{s,t}|^l} + 4(1-\lambda)N.$$

We now separate the cases. In case (i), since $r/\alpha_s > 1$,

$$|\log\varpi(z) - \log\varpi_0(z)| \leqslant \rho(1-\lambda)^2 r^q \sum_{s,t} \frac{n(A_{s,t})}{\alpha_s^q} + 8D(1-\lambda)r^\rho, \tag{5·11}$$

and we have easily

$$\begin{aligned} \sum_{s,t} \frac{n(A_{s,t})}{\alpha_s^q} &\leqslant 2D \sum_s \{(1+\mu_s)^\rho - 1\}\alpha_s^{\rho-q} \\ &\leqslant K_\rho D \sum_s \{(1+\mu_s)^{\rho-q} - 1\}\alpha_s^{\rho-q} \\ &\leqslant K_\rho D r^{\rho-q}. \end{aligned} \tag{5·12}$$

In case (ii), since $r/\alpha_s \leqslant 1$,

$$\begin{aligned} |\log\varpi(z) - \log\varpi_0(z)| &\leqslant q(1-\lambda)N + 4(1-\lambda)N \\ &\leqslant K_\rho(1-\lambda)\,2D(1-\lambda)^{-\frac{1}{2}} R^\rho \\ &\leqslant K_\rho D(1-\lambda)^{\frac{1}{2}} r^\rho. \end{aligned} \tag{5·13}$$

The lemma results immediately from (5·11), (5·12) and (5·13).

6. We now extract the terms making the principal contribution to $\sigma_g(z)$ and exhibit $\log|\sigma_g(z)|$ in a form from which the asymptotic properties stated in Theorem 1 and the related results required for interpolation can be easily obtained. The complications of the preliminary Theorem A are all necessary for the applications.

THEOREM A. *Suppose that* λ *is a real number for which* $\frac{3}{4} \leqslant \lambda < 1$ *and* $2(1-\lambda)^{1/(2\rho)} < \lambda$, α_s *a real sequence such that* $\alpha_1 \geqslant 1$ *and*

(i) $$\alpha_{s+1} = (1+\mu_s)\alpha_s,$$

where $(1-\lambda)^4 \leqslant \mu_s \leqslant (1-\lambda)^2$, *that* M, η *are chosen as in* § 5, *and that* $A_{s,t}$ *is the quadrilateral* $\alpha_s < |z| \leqslant \alpha_{s+1}$, $t\eta \leqslant \arg z < (t+1)\eta$. *Suppose, moreover, that* z_n, $n = 1, 2, \ldots$, *is a complex sequence such that for all* s *and all* $t \leqslant M^2$ *either*

(ii)(*a*) $$n(A_{s,t}) = 0$$

or

(ii)(*b*) $$|n(A_{s,t}) - n(A_{s,0})| \leqslant 2D(1-\lambda)^3 \alpha_s^\rho,$$

the alternative (*b*) *occurring for any given* s *for a set of* $m_s \geqslant M$ t *occupying the same positions in each group of* M *consecutive* t *defined by* $\nu M\eta \leqslant \arg z < (\nu+1)M\eta$, *and such that*

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$\sum_t \zeta_{s,t}^{-l} = 0$ *when taken over this set of* t *for every* $l \leqslant q$. *Let the* $n(u)$ *function of the* z_n *also satisfy*

(iii) $$n(u) \leqslant 2Du^\rho,$$

(iv) $$n(\alpha_{s+1}) - n(\alpha_s) \leqslant 2D\{(1+\mu_s^\rho) - 1\}\alpha_s^\rho,$$

and suppose that

(v) *if* ρ *is an integer* $\sum_{r_n \leqslant u} z_n^{-p}$ *is convergent to a limit* α *as* $u \to \infty$.

Then if $\sigma_g(z)$ *is defined by* (2·1) *we can choose* $R_2(\lambda)$ *such that, if* $R > R_2(\lambda)$ *and* $|z| \leqslant R$,

(vi) $$\left|\log|\sigma_g(z)| - \sum_{\alpha_{s+1} \leqslant \lambda r} \sum_t n(A_{s,0}) \log\left|1 - \frac{z}{\zeta_{s,t}}\right|\right| \leqslant D\epsilon(\lambda) r^\rho,$$

except for an exceptional set $\mathscr{E}_1(R)$ *of* z *such that* $m_r(\mathscr{E}_1) < 3(1-\lambda)R$, *the inner sum being taken over the set of* t *for which* $n(A_{s,t}) \neq 0$.

We begin with the annulus $\frac{1}{2}(1+\lambda)R \leqslant |z| \leqslant \frac{1}{2\lambda}(1+\lambda)R$ for large R and choose integers $P(R)$, $Q(R)$, $L(R)$ such that

$$\alpha_{P+1} \leqslant \lambda R \leqslant \alpha_{P+2}, \quad \alpha_{Q-1} \leqslant \lambda^{-1}R \leqslant \alpha_Q, \quad \tfrac{1}{2}(1-\lambda)^{-1/(2\rho)}R \leqslant \alpha_{L+1} \leqslant (1-\lambda)^{-1/(2\rho)}R$$

and write

$$\sigma_g(z) = \left\{F(z)\exp\left(\frac{p-q}{p}z^p \sum_{r_n \leqslant \alpha_{L+1}} z_n^{-p}\right)\right\} \prod_{r_n \leqslant \alpha_{P+1}} E\left(\frac{z}{z_n}, q\right) \prod_{\alpha_{P+1} < r_n \leqslant \alpha_Q} E\left(\frac{z}{z_n}, q\right)$$
$$\prod_{\alpha_Q < r_n \leqslant \alpha_{L+1}} E\left(\frac{z}{z_n}, q\right) \prod_{r_n > \alpha_{L+1}} E\left(\frac{z}{z_n}, p\right)$$
$$= \Pi_1(z)\,\Pi_2(z)\,\Pi_3(z)\,\Pi_4(z)\,\Pi_5(z).$$

It follows immediately from the definition of $F(z)$ in § 2 and (v) that we can choose $R_3(\lambda) > 2\alpha_1$ such that if $R \geqslant R_3(\lambda)$

$$|\log|\Pi_1(z)|| \leqslant (1-\lambda)r^\rho. \tag{6·1}$$

LEMMA 7. *There exists a* K_ρ *such that*

$$|\log|\Pi_5(z)|| \leqslant K_\rho D(1-\lambda)^{(p+1-\rho)/(2\rho)} r^\rho.$$

For
$$|\log|\Pi_5(z)|| \leqslant \sum_{r_n > \alpha_{L+1}} \sum_{\nu=p+1}^{\infty} \frac{r^\nu}{\nu r_n^\nu} \leqslant K_\rho \sum_{r_n > \alpha_{L+1}} \frac{r^{p+1}}{r_n^{p+1}}$$
$$\leqslant K_\rho \int_{\frac{1}{2}(1-\lambda)^{-1/(2\rho)}R} \left(\frac{r}{u}\right)^{p+1} dn(u),$$

and by (iii) and the choice of α_L it follows that

$$|\log|\Pi_5(z)|| \leqslant K_\rho D(1-\lambda)^{(p+1-\rho)/(2\rho)} r^\rho.$$

LEMMA 8. *If* $R > 2\alpha_1$ *and* $\frac{1}{2}(1+\lambda)R \leqslant |z| \leqslant \frac{1}{2\lambda}(1+\lambda)R$, *then for some* K_ρ

$$|\log|\Pi_3(z)|| \leqslant K_\rho D(1-\lambda)\log\frac{1}{(1-\lambda)} r^\rho,$$

except for a set $\mathscr{E}(R)$ *of* z *for which* $m_r(\mathscr{E}) \leqslant (1-\lambda)^2 R$.

For by hypothesis

$$n(\alpha_Q) - n(\alpha_{P+1}) \leqslant 2D(\alpha_Q^\rho - \alpha_P^\rho) \leqslant K_\rho D(1-\lambda)\, r^\rho.$$

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We therefore have since $r/r_n \leqslant \lambda^{-2}$

$$\left|\log\left|\prod_{\alpha_{P+1}<r_n\leqslant\alpha_Q}\exp\left(\frac{z}{z_n}+\ldots+\frac{1}{q}\frac{z^q}{z_n^q}\right)\right|\right|\leqslant K_\rho D(1-\lambda)\lambda^{-2q}r^\rho\leqslant K_\rho D(1-\lambda)r^\rho, \quad (6\cdot2)$$

and $$\log\left|\prod_{\alpha_{P+1}<r_n\leqslant\alpha_Q}\left(1-\frac{z}{z_n}\right)\right|\leqslant K_\rho D(1-\lambda)r^\rho\log(1+\lambda^{-2})\leqslant K_\rho D(1-\lambda)r^\rho. \quad (6\cdot3)$$

Moreover, by Lemma 1 with $h=(1-\lambda)^2R$ since $r_n\leqslant 2R$,

$$\log\left|\prod_{\alpha_{P+1}<r_n\leqslant\alpha_Q}\left(1-\frac{z}{z_n}\right)\right|\geqslant\{n(\alpha_Q)-n(\alpha_{P+1})\}[2\log(1-\lambda)+\log R-\log 2e-\log 2R]$$
$$\geqslant -K_\rho D(1-\lambda)r^\rho\log\frac{1}{1-\lambda}, \quad (6\cdot4)$$

except for z that can be included in $n(\alpha_Q)-n(\alpha_{P+1})$ circles the sum of whose radii does not exceed $(1-\lambda)^2R$. The lemma follows from (6·2), (6·3) and (6·4).

LEMMA 9. *If $R>2\alpha_1$, for some K_ρ*

$$|\log|\Pi_4(z)||\leqslant K_\rho D(1-\lambda)^{\frac{1}{2}}\log\frac{1}{1-\lambda}r^\rho.$$

By hypothesis we can write $\Pi_4=\varpi_1(\zeta)\varpi_2(\zeta)(\varpi_3(\zeta))^{-1}$, where ϖ_2, ϖ_3 are canonical products of genus q for which $n(A_{s,t})\leqslant 2D(1-\lambda)^8\alpha_s^\rho$,

and ϖ_1 is of genus q and has $n(A_{s,0})$ zeros in the $A_{s,t}$ satisfying (ii) (b) and none in the others. It follows that by Lemma 5

$$|\log|\varpi_2(z)||+|\log|\varpi_3(z)||\leqslant DK_\rho(1-\lambda)^{\frac{1}{2}}\log\frac{1}{(1-\lambda)}r^\rho. \quad (6\cdot5)$$

Let $\varpi_0(z)$ be the canonical product of genus q having $n(A_{s,0})$ zeros at $\zeta_{s,t}=\alpha_s\exp i\eta t$, where $Q\leqslant s\leqslant L$ for each s,t for which Π_4 has $n(A_{s,t})\neq 0$. By Lemma 6, case (ii), we then have $$|\log|\varpi_1(z)|-\log|\varpi_0(z)||\leqslant DK_\rho(1-\lambda)^{\frac{1}{2}}r^\rho. \quad (6\cdot6)$$

Finally, by condition (ii) of Theorem A,

$$\log|\varpi_0(z)|=\sum_s n(A_{s,0})\sum_t{}'\log\left|1-\frac{z}{\zeta_{s,t}}\right|,$$

the summation being taken over the set of s,t for which $n(A_{s,t})\neq 0$. By the restrictions imposed on this set, for each s

$$\prod_t{}'\left(1-\frac{z}{\zeta_{s,t}}\right)=\prod_{\nu=1}^{m_s/M}\left\{1-\chi_\nu\left(\frac{z}{\alpha_s}\right)^M\right\},$$

where $|\chi_\nu|=1$ and therefore since $\left(\frac{r}{\alpha_s}\right)^M\leqslant\left(\frac{1+\lambda}{2}\right)^{2/(1-\lambda)}\leqslant\frac{1}{2}$,

$$|\log|\varpi_0(z)||\leqslant\sum_s\frac{m_s}{M}n(A_{s,0})\log 2. \quad (6\cdot7)$$

Moreover, it follows by summing (ii) (b) over the t involved for each s that

$$\sum_s m_s n(A_{s,0})\leqslant n(\alpha_L)+2D(1-\lambda)^8\sum_{\substack{t\\ s\leqslant L}}\alpha_s^\rho\leqslant K_\rho D(1-\lambda)^{-\frac{1}{2}}r^\rho. \quad (6\cdot8)$$

It follows from (6·7) and (6·8) recalling that $M\geqslant\frac{\sqrt{2/\pi}}{1-\lambda}$ that

$$|\log|\varpi_0(z)||\leqslant K_\rho D(1-\lambda)^{\frac{1}{2}}r^\rho, \quad (6\cdot9)$$

and the lemma follows from (6·5), (6·6) and (6·9).

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It remains to deal with $\Pi_2(z)$. As with Π_4 we write

$$\Pi_2(z) = \varpi_4(z)\,\varpi_5(z)\,(\varpi_6(z))^{-1},$$

where $\varpi_5\varpi_6$ have $\quad n(A_{s,t}) \leqslant 2D(1-\lambda)^8\alpha_s^\rho,$

and $\varpi_4(\zeta)$ has $n(A_{s,t}) = 0$ or $n(A_{s,t}) = n(A_{s,0})$, according to whether t is in case (a) or (b) for $\sigma_g(z)$. It follows from Lemma 5 that

$$|\log|\varpi_5(z)|| + |\log|\varpi_6(z)|| \leqslant 2DK_\rho(1-\lambda)^{\frac{1}{2}}\log\frac{1}{1-\lambda}r^\rho. \tag{6·10}$$

Let $\varpi_7(z)$ be a canonical product of genus q having $n(A_{s,0})$ zeros at those $\zeta_{s,t}$ for which $n(A_{s,t}) \neq 0$. By Lemma 6

$$|\log|\varpi_4(z)| - \log|\varpi_7(z)|| \leqslant K_\rho D(1-\lambda)^{\frac{1}{2}}r^\rho, \tag{6·11}$$

and obviously $\quad \log|\varpi_7(z)| = \sum_{\alpha_{s+1}\leqslant\lambda R} n(A_{s,0})\sum_t{}'\log\left|1-\frac{z}{\zeta_{s,t}}\right|. \tag{6·12}$

Combining Lemmas 7, 8 and 9 and (6·1), (6·10), (6·11) and (6·12), we obtain that if $R > R_3(\lambda)$ and $\frac{1}{2}(1+\lambda)R \leqslant |z| \leqslant \frac{1}{2\lambda}(1+\lambda)R$

$$\left|\log|\sigma_g(z)| - \sum_{\alpha_{s+1}\leqslant\lambda R} n(A_{s,0})\sum_t{}'\log\left|1-\frac{z}{\zeta_{s,t}}\right|\right| \leqslant K_\rho D(1-\lambda)^{\frac{1}{2}}\log\frac{1}{1-\lambda}r^\rho, \tag{6·13}$$

except for a set of z of radial measure not exceeding $(1-\lambda)^2R$. Moreover,

$$\left|\sum_{\lambda r<\alpha_{s+1}\leqslant\lambda R} n(A_{s,0})\sum_t{}'\log\left|1-\frac{z}{\zeta_{s,t}}\right|\right| \leqslant \log\frac{2}{1-\lambda}\sum_{\lambda r<\alpha_{s+1}\leqslant\lambda R} m_s n(A_{s,0}), \tag{6·14}$$

and by (ii) (b) and (iv)

$$\begin{aligned}\sum_{\lambda r<\alpha_{s+1}\leqslant\lambda R} m_s n(A_{s,0}) &\leqslant \sum_{\lambda r<\alpha_{s+1}\leqslant\lambda R}(n(A_s) + D(1-\lambda)^6\alpha_s^\rho)\\ &\leqslant K_\rho D\sum_{\lambda r<\alpha_{s+1}\leqslant\lambda R}(\alpha_{s+1}^\rho - \alpha_s^\rho)\\ &\leqslant K_\rho D r^\rho\left\{1-\left(\frac{\lambda}{1+(1-\lambda)^2}\right)^\rho\right\}\\ &\leqslant K_\rho D(1-\lambda)r^\rho.\end{aligned} \tag{6·15}$$

Combining (6·13), (6·14) and (6·15) it follows that if $R > R_3(\lambda)$ (vi) holds for

$$\tfrac{1}{2}(1+\lambda)R \leqslant |z| \leqslant \frac{1}{2\lambda}(1+\lambda)R,$$

except for z in a set of radial measure not exceeding $(1-\lambda)^2R$.

The whole argument can be repeated with R replaced by $\left(\frac{1+\lambda}{2}\right)^\nu R$ provided $\left(\frac{1+\lambda}{2}\right)^\nu R \geqslant R_3(\lambda)$. It follows that (vi) holds for $|z| \leqslant R$ with an exceptional set $\mathscr{E}_1(R)$ for which

$$\mu_r(\mathscr{E}_1) \leqslant R_3(\lambda) + R(1-\lambda)^2\sum_{\nu=0}^{\infty}\left(\frac{1+\lambda}{2}\right)^\nu \leqslant R_3(\lambda) + 2(1-\lambda)R.$$

Theorem A follows on taking $R_2(\lambda) = \dfrac{R_3(\lambda)}{(1-\lambda)}$.

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7. *Proof of Theorem* 1. We assume that for each λ for which $1>\lambda\geqslant\frac{3}{4}$, $\lambda\geqslant 2(1-\lambda)^{1/(2\rho)}$, η M are chosen as in § 5 and Theorem A. Writing temporarily $\xi=(1-\lambda)^2$, $\delta=(1-\lambda)^4$, we can now apply Lemma 3 with $\epsilon=(1-\lambda)^8$, $\omega=\xi+\delta+\xi\delta$ and $\mathscr{E}$ the set of $r>2R_1(\lambda)$ for which

$$n\{(1+\xi)\,r\}-n(r)\geqslant d\{1-(1-\lambda)^8\}\{(1+\xi)^\rho-1\}r^\rho, \tag{7·1}$$

obtaining a sequence β_s belonging to $\mathscr{E}$ and an $R_4(\lambda)>R_1(\lambda)$ such that the intervals $(\beta_s,\beta_s(1+\xi)(1+\delta))$ are non-overlapping and

$$(\xi+\delta+\xi\delta)\sum_{\beta_{s+1}\leqslant R}\beta_s\geqslant\Theta\{1-(1-\lambda)^8\}R-K_\rho(1-\lambda)^2R, \tag{7·2}$$

if $R>R_4(\lambda)$. We take the sequence of annuli $B_s(\beta_s,(1+\xi)\beta_s)$ as part of the sequence of annuli on Theorem A and complete it by dividing the complementary annuli in $|z|\geqslant R_1$ into annuli $C_s(\gamma_s,(1+\mu_s)\gamma_s)$ of relative breadth μ_s such that $\delta\leqslant\mu_s\leqslant 2\delta$. The conditions of Theorem A are now satisfied by the combined sequences β_s, γ_s, with $m_s=M^2$ for all s. It follows that if $R>R_2(\lambda)$ and $|z|\leqslant R$

$$\begin{aligned}\left|\log|\sigma_g(z)|-\sum_{\beta_{s+1}\leqslant\lambda r}n(B_{s,0})\log\left|1-\left(\frac{z}{\beta_s}\right)^{M^2}\right|-\sum_{\gamma_{s+1}\leqslant\lambda r}n(C_{s,0})\log\left|1-\left(\frac{z}{\gamma_s}\right)^{M^2}\right|\right|&\\ \leqslant D\epsilon(\lambda)\,r^\rho+\sum_{r_n\leqslant R_1}|\log|E_n(z,q)||&\\ \leqslant D\epsilon(\lambda)\,r^\rho,&\end{aligned} \tag{7·3}$$

if $R>R_5(\lambda)>R_2(\lambda)$ except for z of radial measure not exceeding $3(1-\lambda)R$.

Since throughout the sums

$$\left(\frac{r}{\beta_s}\right)^{M^2},\left(\frac{r}{\gamma_s}\right)^{M^2}\geqslant\left(\frac{1+\lambda}{2\lambda}\right)^{M^2}\geqslant 1+M(1-\lambda)\geqslant 2,$$

it follows that if σ is a β_s or γ_s

$$\left|\log\left|1-\left(\frac{z}{\sigma}\right)^{M^2}\right|-M^2\log\frac{r}{\sigma}\right|\leqslant\log 2, \tag{7·4}$$

and we conclude from (7·3) and (7·4) that

$$\begin{aligned}\left|\log|\sigma_g(z)|-M^2\sum_{\beta_{s+1}\leqslant\lambda r}n(B_{s,0})\log\frac{r}{r_s}-M^2\sum_{\gamma_{s+1}\leqslant\lambda r}n(C_{s,0})\log\frac{r}{\gamma_s}\right|&\\ \leqslant D\epsilon(\lambda)\,r^\rho+\log 2n(\lambda r,0,\eta)&\\ \leqslant D\epsilon(\lambda)\,r^\rho&\end{aligned} \tag{7·5}$$

by (5·3). Moreover, by (5·2), (5·3) and (7·1)

$$\begin{aligned}d\{1-K_\rho(1-\lambda)^2\}\rho\xi\beta_s^\rho\leqslant M^2n(B_{s,0})\leqslant D\{1+K_\rho(1-\lambda)^2\}\rho\xi\beta_s^\rho,&\\ M^2n(C_{s,0})\leqslant D\{1+K_\rho(1-\lambda)^2\}\rho\mu_s\gamma_s^\rho,&\end{aligned} \tag{7·6}$$

and thus, if $R>R_5$ except for the above exceptional set,

$$\log|\sigma_g(z)|\leqslant D\{1+K_\rho(1-\lambda)^2\}\rho\left[\sum_{\beta_s}\xi\beta_s^\rho\log\frac{r}{\beta_s}+\sum_{\gamma_s}\mu_s\gamma_s^\rho\log\frac{r}{r_s}\right]+D\epsilon(\lambda)r^\rho, \tag{7·7}$$

$$\log|\sigma_g(z)|\geqslant d\{1-K_\rho(1-\lambda)^2\}\rho\sum_{\beta_s}\xi\beta_s^\rho\log\frac{r}{r_s}-D\epsilon(\lambda)r^\rho. \tag{7·8}$$

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We complete the argument by appealing to the following

LEMMA 10. *Suppose that* ρ, r, λ, ξ_i $(i = 1, 2, \ldots, N)$ *are positive numbers such that* $(1-\lambda)^4 \leqslant \xi_i \leqslant (1-\lambda)^2$, $(1+\xi_i)^\rho \leqslant 2$, $\frac{3}{4} \leqslant \lambda < 1$, *and that* $r_1, \ldots, r_N$ *are* N *positive numbers such that*

$$r_{i+1} \geqslant (1+\xi_i) r_i, \quad (1+\xi_N) r_N \leqslant \lambda r.$$

Then if I_i *is the interval* $r_i \leqslant u \leqslant (1+\xi_i) r_i$ *there exists* K_ρ *such that*

$$\left| \rho \sum_i \xi_i r_i^\rho \log \frac{r}{r_i} - \rho \int_{\Sigma I_i} x^{\rho-1} \log \frac{r}{x} dx \right| \leqslant K_\rho (1-\lambda)^2 \sum_i \xi_i r_i^\rho \log \frac{r}{r_i} + K_\rho (1-\lambda)^2 r^\rho.$$

For we have

$$\left| \xi_i r_i^\rho \log \frac{r}{r_i} - \int_{I_i} x^{\rho-1} \log \frac{r}{x} dx \right| \leqslant \xi_i^2 r_i^2 \max_{I_i} \left| \frac{d}{du} \left(u^{\rho-1} \log \frac{r}{u} \right) \right|$$

$$\leqslant K_\rho \xi_i^2 r_i^\rho \log r/r_i + K_\rho \xi_i^2 r_i^\rho,$$

and since

$$\rho \Sigma \xi_i^2 r_i^\rho \leqslant (1-\lambda)^2 K_\rho \sum_i \{(1+\xi_i)^\rho - 1\} r_i^\rho \leqslant K_\rho (1-\lambda)^2 r^\rho,$$

we obtain the result of the lemma on summing over i.

It follows immediately that if Σ, I are the sum and integral occurring in the lemma and $(1-\lambda)$ is sufficiently small that

$$|\Sigma - I| \leqslant K_\rho (1-\lambda)^2 I + K_\rho (1-\lambda)^2 r^\rho. \tag{7·9}$$

Applying (7·9) to (7·7) and (7·8) with $\xi_i = \xi$, μ_i in the appropriate sums and taking account of (7·5), it follows that if $R > R_5(\lambda)$

$$\log |\sigma_g(z)| \leqslant D\{1 + K_\rho (1-\lambda)^2\} \rho \int_0^r x^{\rho-1} \log \frac{r}{x} dx + D\epsilon(\lambda) r^\rho, \tag{7·10}$$

and since $\sum_{\beta_{s+1} \leqslant \lambda r} B_s$ covers a set of measure at least equal to $\frac{\lambda r}{1+(1-\lambda)^2}[\Theta - K_\rho (1-\lambda)^2]$ in $0 \leqslant x \leqslant \lambda r$

$$\log |\sigma_g(z)| \geqslant d\{1 - K_\rho (1-\lambda)^2\} \rho \int_{\Sigma B_s} x^{\rho-1} \log \frac{r}{x} dx - D\epsilon(\lambda) r^\rho$$

$$\geqslant d\{1 - K_\rho (1-\lambda)^2\} \rho [1 - \tau\{1 - \Theta + \epsilon(\lambda)\}] \int_0^r x^{\rho-1} \log \frac{r}{x} dx - D\epsilon(\lambda) r^\rho, \tag{7·11}$$

except for a set of z of radial measure not exceeding $3(1-\lambda) R$. Writing temporarily $\frac{d}{\rho}\{1 - \tau(1-\Theta)\} = d'$ and letting λ tend to 1, we conclude that for any $\eta > 0$ if $R > R'(\eta)$ and $|z| \leqslant R$

$$\left(\frac{D}{\rho} + \eta \right) r^\rho \geqslant \log |\sigma_g(z)| \geqslant (d' - \eta) r^\rho, \tag{7·12}$$

except for a set $\mathscr{E}$ of z such that $\mu_r(\mathscr{E}) \leqslant \eta R$.

It follows that we can choose a sequence R_n $(n = 1, 2, \ldots)$, where $R_n \to \infty$ with n such that

$$R_{n+1} \leqslant R_n (1 + 2\eta),$$

and if $|z| = R_n$ (7·12) holds. For each n let A_n be the annulis, $R_n \leqslant |z| \leqslant R_{n+1}$ and C_n its boundary. Applying the maximum modulus theorem to $\sigma_g(z)$ in A_n we have if $|z| \geqslant R_1$

$$\log |\sigma_g(z)| \leqslant \left(\frac{D}{\rho} + \eta \right) (1 + 2\eta)^\rho r^\rho. \tag{7·13}$$

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As a temporary notation write $\zeta_1, \ldots, \zeta_M$, where $M = M(n)$ for the zeros z_ν in A_n and define $H(z)$ by

$$H(z) = \frac{\sigma_g(z)}{\prod_{i=1}^{M} (z-\zeta_i)}$$

if $z \neq \zeta_i$,

$$H(\zeta_i) = \frac{\sigma_g'(\zeta_i)}{\prod_{i \neq j} (\zeta_i - \zeta_j)}.$$

Thus defined $H(z)$ is regular and non-vanishing in A_n, and therefore by the minimum modulus theorem

$$|H(\zeta_i)| \geqslant \min_{C_n} |H(z)| \geqslant \min_{C_n} |\sigma_g(z)| \min \frac{1}{|\prod_j (z-\zeta_j)|},$$

that is,

$$\begin{aligned} |\sigma_g'(\zeta_i)| = \left|H(\zeta_i) \prod_{i \neq j} (\zeta_i - \zeta_j)\right| &\geqslant \min_{C_n} \frac{1}{|z-\zeta_i|} \min_{C_n} \prod_{i \neq j} \left|\frac{\zeta_i - \zeta_j}{z-\zeta_j}\right| \exp(d'-\eta) R_n^\rho \\ &\geqslant \frac{1}{4R_n} \left(\max_{C_n} \prod_{j \neq i} \left|1 - \frac{z-\zeta_j}{\zeta_i - \zeta_j}\right|\right)^{-1} \exp\{d' - \epsilon(\eta)\} R_n^\rho \\ &\geqslant \frac{1}{4R_n} \exp\left\{\{d'-\epsilon(\eta)\} R_n^\rho - 4R_n \sum_{i \neq j} \frac{1}{|\zeta_i - \zeta_j|}\right\}. \end{aligned} \tag{7·14}$$

But if necessary by rechoosing R_1 we may assume that

$$n(A_n) \leqslant 4D\rho\eta R_n^\rho,$$

and applying Lemma 2 with $N \leqslant 4D\rho\eta R_n^\rho$ and $\delta_1 \geqslant h \min(R_{n+1}^{1-\frac{1}{2}\rho}, R_n^{1-\frac{1}{2}\rho})$, we have

$$\sum_{i \neq j} \frac{1}{|\zeta_i - \zeta_j|} \leqslant \frac{K}{h} (4D\rho\eta)^{\frac{1}{2}} R_n^{\rho-1} \{1 + (1+2\eta)^{\frac{1}{2}\rho - 1}\} = \epsilon(\eta) R_n^\rho, \tag{7·15}$$

and hence if $R_{n+1} \geqslant |z_i| \geqslant R_n$ and $n \geqslant 1$,

$$\begin{aligned} |\sigma_g'(z_i)| &\geqslant \exp[\{d' - \epsilon(\eta)\} R_n^\rho - \log 4R_n] \\ &\geqslant \exp\{d' - \epsilon(\eta)\} |z_n|^\rho. \end{aligned} \tag{7·16}$$

Theorem 1 follows from (7·12), (7·13) and (7·16) on rechoosing η.

8. *Interpolation lemmas.* We now embody the essentials of the interpolation argument in two preliminary results of which the first is merely a slightly generalized version of a theorem of Ganapathy Iyer ((12), Theorem 2), and the second combines the same author's argument with an earlier one due to Whittaker.

For brevity in this section only we shall say that a sequence $\{z_n\}$ belongs to $\mathcal{G}(D^*, d^*, \eta, \rho)$ if there exists an integral function $\sigma_g(z)$ vanishing for all z_n and to any $\epsilon > 0$ corresponds an $r_0(\epsilon)$ such that

$$\log |\sigma_g(z)| \leqslant (D^* + \epsilon) r^\rho, \tag{8·1}$$

$$\log |\sigma_g(z)| \geqslant (d^* - \epsilon) r^\rho, \tag{8·2}$$

if $R > r_0(\epsilon)$ and $|z| \leqslant R$ except for a set of z of radial measure not exceeding nR;

$$\log |\sigma_g'(z_n)| \geqslant (d^* - \epsilon) |z_n|^\rho \tag{8·3}$$

if $|z_n| \geqslant r_0$.

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The first interpolation lemma is then

LEMMA 11. *Suppose that* $\{z_n\}$ *belongs to* $\mathscr{G}(D^*, d^*, \eta, \rho)$ *and that* $f(z)$ *is an integral function of type* $K(f)$ *and order* ρ *for which*

$$\underline{\lim}\, r^{-\rho} \log M(r,f) = \sigma < d^*(1+2\eta)^{-\rho}, \tag{8·4}$$

$$\overline{\lim}\, |z_n|^{-\rho} \log |f(z_n)| = \beta < d^*; \tag{8·5}$$

then

(i) *for all* z

$$f(z) = \sigma_g(z) \sum_{z_n} \frac{f(z_n)}{\sigma_g'(z_n)(z-z_n)}, \tag{8·6}$$

the series being absolutely and uniformly convergent in any bounded closed z *region not including a* z_n;

(ii)

$$K(f) \leqslant \left(\beta \frac{d^*}{D^*} + D^* - \frac{d^{*2}}{D^*}\right)(1+2\eta)^{\rho}. \tag{8·7}$$

Choose ϵ, δ and $l<1$ so that $0<\delta<d^*-\beta$, $D^* l^{\rho} = (d^*-\beta-\delta)$, and

$$\epsilon < \tfrac{1}{4} \min\{d^* - \sigma(1+2\eta)^{\rho}, \tfrac{1}{4}\delta\}.$$

Then there exists $R''(\eta, \delta)$ such that if $R > R''$ and $|z| \leqslant R$ (8·2) holds and also

$$(d^* - \epsilon) l^{\rho} r^{\rho} \leqslant \log |\sigma_g(lz)| \leqslant (D^* + \epsilon) l^{\rho} r^{\rho} \tag{8·8}$$

except for a set of z of radial measure not exceeding $2\eta R$, and (8·1) and (8·3) hold if $|z|, |z_n| \geqslant R''$.

If now h_1 is such that $\Sigma |z_n|^{-h_1} < \infty$, then it follows immediately that for any $\eta' > \eta$ there exists a sequence of circles C_n, $|z| = R_n$, where $R_{n+1} \leqslant (1+2\eta') R_n$ on which (8·1), (8·2), (8·3) and (8·8) hold and $|z - z_n| \geqslant |z_n|^{-h_1}$ for all z_n. Moreover, if $\eta' - \eta$ is sufficiently small we can select a subsequence C_{n_ν} on which we also have

$$\log M(R_{n_\nu}, f) \leqslant \{\sigma(1+2\eta')^{\rho} + \epsilon\} R_{n_\nu}^{\rho}. \tag{8·9}$$

Now consider for sufficiently large ν

$$\begin{aligned} I_\nu &= \frac{1}{2\pi i} \int_{C_{n_\nu}} \frac{x^m f(x)}{\sigma_g(x)(x-z)}\, dx \\ &= \frac{z^m f(z)}{\sigma_g(z)} - \sum_{r_n < R_{n_\nu}} \frac{z_n^m f(z_n)}{\sigma_g'(z_n)(z-z_n)}. \end{aligned} \tag{8·10}$$

If $\min |z - z_n| = \delta_1 > 0$, for large n

$$\left| \frac{z_n^m f(z_n)}{\sigma_g'(z_n)(z-z_n)} \right| \leqslant \frac{r_n^m}{\delta_1} \left| \frac{f(z_n)}{\sigma_g'(z_n)} \right| \leqslant \frac{r_n^m}{\delta_1} \exp(\beta + \epsilon - d^* + \epsilon) r_n^{\rho},$$

and therefore the complete (ungrouped) series on the right of (8·10) is absolutely and uniformly convergent for z in any bounded closed region not containing a z_n. Similarly, for large ν if $\eta' - \eta$ is sufficiently small

$$|I_\nu| \leqslant \frac{R_{n_\nu}^{m+1}}{\frac{1}{2} R_{n_\nu}} \exp\{\sigma(1+2\eta')^{\rho} + \epsilon - d^* + \epsilon\} R_{n_\nu}^{\rho} \leqslant 2 R_{n_\nu}^m \exp - \epsilon R_{n_\nu}^{\rho},$$

and therefore $I_\nu \to 0$ as $\nu \to \infty$. We conclude that for each integer m if $z \neq z_i$

$$\frac{f(z) z^m}{\sigma_g(z)} = \sum_{n=1}^{\infty} \frac{z_n^m f(z_n)}{\sigma_g'(z_n)(z-z)_n}. \tag{8·11}$$

The first part of the lemma is the case $m = 0$ of (8·11).

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Since, moreover, $\sigma_g(lz)$ is an integral function of order ρ and type at most

$$D^*l^\rho = d^* - \beta - \delta,$$

it has a Taylor expansion valid for all z

$$\sigma_g(lz) = \sum_{\nu=0}^{\infty} c_\nu z^\nu,$$

and it is well known that these facts imply that

$$\overline{\lim_r} r^{-\rho} \log \left(\sum_m |c_m| r^m\right) \leqslant d^* - \beta - \delta. \tag{8·12}$$

It follows from (8·3), (8·5) and (8·8) and the choice of ϵ that the double series

$$\sum_{m,n} \frac{|c_m| r_n^m |f(z_n)|}{|\sigma_g'(z_n)|\,|z - z_n|} = \sum_n \sum_m \frac{|f(z_n)| r_n^m |c_m|}{|\sigma_g'(z_n)|\,|z - z_n|}$$

is uniformly convergent for z in any bounded closed region not including a z_n. Inverting the order of summation in the double series without moduli we conclude that if $z \neq z_i$

$$\frac{f(z)\,\sigma_g(lz)}{\sigma_g(z)} = \frac{f(z)}{\sigma_g(z)} \sum_{m=0}^{\infty} c_m z^m = \sum_{n=0}^{\infty} \frac{f(z_n)\,\sigma_g(lz_n)}{\sigma_g'(z_n)\,(z - z_n)}. \tag{8·13}$$

In particular, if $|z| = R_\nu$,

$$\begin{aligned} |f(z)| &\leqslant \left|\frac{\sigma_g(z)}{\sigma_g(lz)}\right| \sum_n \frac{|f(z_n)|\,|\sigma_g(lz_n)|\, r_n^{h_1}}{|\sigma_g'(z_n)|} \\ &\leqslant \left|\frac{\sigma_g(z)}{\sigma_g(lz)}\right| K(l, h_1) \\ &\leqslant K(l, h_1) \exp\left(D^* - d^* l^\rho + \delta\right) R_\nu^\rho; \end{aligned}$$

and therefore by the maximum modulus theorem applied to the annulus $R_{\nu-1} \leqslant |z| \leqslant R_\nu$

$$M(r, f) \leqslant K(l, h_1) \exp\left(D^* - \frac{d^*}{D^*}(d^* - \beta - \delta) + \delta\right)(1 + 2\eta')^\rho r^\rho,$$

or

$$K(f) \leqslant \left(D^* - \frac{d^*}{D^*}(d^* - \beta - \delta) + \delta\right)(1 + 2\eta')^\rho.$$

Lemma 11, part (ii), follows on letting $\delta \to 0$ and η' to η.

The second interpolation result is

LEMMA 12. *Suppose that $\{z_n\}$ belongs to $\mathcal{G}(D^*, d^*, \eta, \rho)$ and that $f(z)$ is an integral function of type $K(f) < d^*$ of order ρ. Then if $f(z_n) = O(1)$, $f(z)$ is necessarily constant.*

We can argue essentially as in the proof of Lemma 11 obtaining (8·11). At the next stage, in virtue of the extra hypothesis $f(z_n) = O(1)$ we can replace $\sigma_g(lz)$ by any integral function $g(z)$ of type $K(g) < d^*$, obtaining instead of (8·13)

$$\frac{f(z)\,g(z)}{\sigma_g(z)} = \sum_{n=1}^{\infty} \frac{g(z_n) f(z_n)}{\sigma_g'(z_n)\,(z - z_n)}. \tag{8·14}$$

In particular, we can take $g(z) = z^m f(z)$ for any integer m and obtain

$$\frac{f(z)^2 z^m}{\sigma_g(z)} = \sum_{n=1}^{\infty} \frac{f(z_n)^2 z_n^m}{\sigma_g'(z_n)\,(z - z_n)}.$$

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Since $(f(z_n))^2 = O(1)$ we can repeat the last step and obtain, ultimately setting $m = 0$, for any integer N

$$f(z)^N = \sigma_g(z) \sum_{n=1}^{\infty} \frac{f(z_n)^N}{\sigma_g'(z_n)(z-z_n)},$$

the series converging uniformly in any bounded closed region not containing a z_n. If $\min|z-z_\nu| = \Delta_z > 0$

$$|f(z)| \leqslant |\sigma_g(z)|^{1/N} \sup|f(z_n)| \left(\sum_{n=1}^{\infty} \frac{1}{|\sigma_g'(z_n)|\Delta_z}\right)^{1/N},$$

and therefore letting $N \to \infty$ $\quad |f(z)| \leqslant \sup|f(z_n)|$.

We thus have $f(z)$ bounded and therefore constant.

9. *Proofs of Theorems* 2 *and* 3. The proofs are now immediate. By Theorem 1 we can take $D^* = \frac{D}{\rho}$, $d^* = \frac{d}{\rho}\{1-\tau(1-\Theta)\}$ in Lemma 11 and η as small as we please. We thus obtain

$$K(f) \leqslant K(z_n)\frac{d}{D}\{1-\tau(1-\Theta)\} + \frac{D}{\rho} - \frac{d^2}{D\rho}\{1-\tau(1-\Theta)\}^2,$$

that is, Theorem 2.

Theorem 3, part (i), is an immediate consequence of Lemma 12 with

$$d^* = \frac{d}{\rho}\{1-\tau(1-\Theta)\}.$$

Further, if $f(z)$ is constant and $f(z_n) \to 0$ on a sequence of distinct z_n, then $f(z) \equiv 0$.

Corollaries 1, 2 and 3 are trivial special cases of Theorem 2.

10. *Proof of Theorem* 4. For Theorem 4 we require the full force of Theorem A with $m_s < M^2$. For each λ such that $\lambda > 2(1-\lambda)^{1/(2\rho)}$ choose M, η as in Theorem A. The argument follows that of Theorem 1 fairly closely, and we omit some details. Corresponding to each λ we can construct as in Theorem 1 a sequence $\beta_s(\lambda)$, non-overlapping annuli $B_s, \beta_s < |z| \leqslant (1+\xi)\beta_s$ and quadrilaterals $B_{s,t}$, $\beta_s < |z| \leqslant (1+\xi)\beta_s$, $t\eta \leqslant \arg z < (t+1)\eta$ such that

$$|n(B_{s,t}) - n(B_{s,0}^*)| \leqslant 2D(1-\lambda)^8\beta_s^\rho, \tag{10·1}$$

$$n(B_s) \leqslant D\{1+(1-\lambda)^8\}\{(1+\xi)^\rho - 1\}\beta_s^\rho, \tag{10·2}$$

$$n(B_s) \geqslant d\{1-(1-\lambda)^8\}\{(1+\xi)^\rho - 1\}\beta_s^\rho, \tag{10·3}$$

and if $u \geqslant \beta_1$

$$n(u) \leqslant 2Du^\rho; \tag{10·4}$$

while the β_s satisfy (7·2). Write $D_s = n(B_s)/\{(1+\xi)^\rho - 1\}\beta_s^\rho$, so that clearly

$$\{1-(1-\lambda)^8\} \leqslant D_s \leqslant D\{1+(1-\lambda)^8\}.$$

We now define a subsequence $z_n(\lambda)$ of the z_n as follows. First omit all z_n that do not lie in a B_s. For the remainder, if $|D_s - d| \leqslant (1-\lambda)$ transfer all the z_n in B_s to the new sequence; if this is not the case let p_s be an integer such that

$$\frac{p_s}{M} \leqslant \frac{D_s - d}{D_s} \leqslant \frac{p_s}{M} + (1-\lambda).$$

Since $M(1-\lambda) \geqslant (2\pi)^{\frac{1}{2}}$ this is possible. Now omit a set of 'quadrilaterals' from B_s in such a way that

(i) there are p_s omitted quadrilaterals in each group of M consecutive ones occupying $\nu M\eta \leqslant \arg z < (\nu+1)M\eta$ for some integer ν and they occupy the same relative positions in each group;

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Since $p_s < M$ (ii) is possible and the number of quadrilaterals retained in each annulus is at least M. The sequence $z_n(\lambda)$ is completed by the z_n from the retained quadrilaterals, and it is easy to see that it satisfies the conditions of Theorem A. We can therefore choose $R_6(\lambda)$ such that if $R > R_6(\lambda)$ and $|z| \leqslant R$

$$\left|\log|\sigma_g(z)| - \sum_{\beta_{s+1}\leqslant\lambda r}{}' \sum_t n(B_{s,0}) \log\left|1-\frac{z}{\zeta_{s,t}}\right|\right| \leqslant D\epsilon(\lambda) r^\rho, \tag{10·5}$$

except for a set of z of radial measure not exceeding $3(1-\lambda)R$; the Σ' indicating that t corresponding to omitted quadrilaterals are omitted. Equation (10·5) implies that

$$\left|\log|\sigma_g(z)| - \sum_{\beta_{s+1}\leqslant\lambda r} n(B_{s,0}) \sum_{\nu=1}^{M-p_s} \log\left|1-\chi_\nu\frac{z^M}{\beta_s^M}\right|\right| \leqslant D\epsilon(\lambda) r^\rho,$$

where $|\chi_\nu| = 1$, and, since $\left(\frac{r}{\beta_s}\right)^M \geqslant \left(\frac{1+\lambda}{2\lambda}\right)^M \geqslant 2$ and consequently

$$\left|\log\left|1-\chi_\nu\left(\frac{z}{\beta_s}\right)^M\right| - M\log\frac{r}{\beta_s}\right| \leqslant \log 2,$$

we arrive at

$$\left|\log|\sigma_g(z)| - M\sum_s (M-p_s)\, n(B_{s,0}) \log\frac{r}{\beta_s}\right| \leqslant D\epsilon(\lambda) r^\rho + M\log 2 \Sigma n(B_{s,0}) \leqslant D\epsilon(\lambda) r^\rho, \tag{10·6}$$

by an argument used in the similar situation in Theorem 1.

Moreover, by (10·1) if $q_s = M - p_s$

$$\left|Mq_s n(B_{s,0}) - \frac{q_s}{M} D_s\{(1+\xi)^\rho - 1\}\beta_s^\rho\right| \leqslant K_\rho D(1-\lambda)^6 \beta_s^\rho \leqslant K_\rho D(1-\lambda)\{(1+\xi)^\rho - 1\}\beta_s^\rho, \tag{10·7}$$

and by choice of p_s

$$\left|\frac{D_s q_s}{M} - d\right| \leqslant (1-\lambda). \tag{10·8}$$

Combining (10·6), (10·7) and (10·8) we have

$$\begin{aligned}\left|\log|\sigma_g(z)| - d\sum_s \{(1+\xi)^\rho - 1\}\beta_s^\rho \log\frac{r}{\beta_s}\right| &\leqslant K_\rho D(1-\lambda) \sum_{\beta_{s+1}\leqslant\lambda r} \{(1+\xi)^\rho - 1\}\beta_s^\rho \log\frac{2}{1-\lambda} + D\epsilon(\lambda) r^\rho \\ &\leqslant D\epsilon(\lambda) r^\rho. \end{aligned} \tag{10·9}$$

By a use of Lemma 10 similar to that in Theorem 1 we conclude that we can choose $R_7(\lambda)$, $\epsilon'(\lambda)$, where $\epsilon'(\lambda) \to 0$ as $\lambda \to 1$, such that if $R > R_7$ and $|z| \leqslant R$

$$\log|\sigma_g(z)| \geqslant \left\{\frac{d}{\rho}\{1-\tau(1-\Theta)\} - \epsilon'(\lambda)\right\} r^\rho = \mathfrak{d}(\lambda) r^\rho, \tag{10·10}$$

$$\log|\sigma_g(z)| \leqslant \left(\frac{d}{\rho} + \epsilon'(\lambda)\right) r^\rho = \mathscr{D}(\lambda) r^\rho, \tag{10·11}$$

except for a set of z of radial measure not exceeding $3(1-\lambda)R$; and if $|z_n| \geqslant R_7$

$$\log|\sigma_g(z_n)| \geqslant \mathfrak{d}(\lambda)|z_n|^\rho. \tag{10·12}$$

It now follows from Lemma 11 that for λ sufficiently near 1 if $f(z)$ satisfies (2·2) and (2·3)

$$K(f) \leqslant \left\{K\{z_n(\lambda)\}\frac{\mathfrak{d}(\lambda)}{\mathscr{D}(\lambda)} + \mathscr{D}(\lambda) - \frac{\mathfrak{d}^2(\lambda)}{\mathscr{D}(\lambda)}\right\}\{1+6(1-\lambda)\}^\rho, \tag{10·13}$$

and since $K(z_n) \geqslant K(z_n(\lambda))$, Theorem 4 follows from (10·13) on letting λ tend to 1.

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Corollary 4 is a trivial special case.

In conclusion, we observe that the principal idea of this paper embodied in Theorem A and its proof could probably be combined with Pflüger's technique to yield still more general results in which, in the notation of § 2 (i), $n(r, S_1)/(\phi_1 - \theta_1)$ varies with the orientation of the sector.

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