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Tripolar shear-Alfvén vortex structures

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Nonlinear structures in the form of coherent tripolar vortices, associated with shear-Alfvén perturbations in the presence of both magnetic and velocity shears, are studied analytically.

1. Introduction

The tripole vortex is one of the fundamental vortical structures that are characteristic features of two-dimensional and quasi-three-dimensional flows. Tripolar vortices consist of a core vortex of elliptical shape and two satellite vortices of opposite vorticity. Tripoles are easily excited in rotating fluids (van Heist and Kloosterziel 1989; van Heist *et al.* 1991), they possess angular momentum, but no linear momentum, and rotate steadily around the centre of the core vortex. Studies of the generation and stability of tripoles in homogeneous two-dimensional flows were reported by Carton *et al.* (1990), Polvani and Carton (1990) and Carton and Legras (1994), who showed that such a structure can appear from the saturated azimuthal instability of a monopolar vortex, and also from the off-axis collision of two dipolar vortices. Also, in high-resolution simulations of two-dimensional turbulence (Legras *et al.* 1988), the emergence of tripolar structures has been observed together with the other fundamental structures – monopoles and dipoles. It should be noted that higher-order structures, such as quadrupolar vortices consisting of a triangularly shaped central vortex and three satellites, seem to be unstable (Morel and Carton 1994; Carnevale and Kloosterziel 1994).

An analytical solution for tripolar structures in a homogeneous two-dimensional (2D) flow was presented by Zhmur (1988), who found it as a special stationary solution imbedded in a background shear flow. For electrostatic drift waves, described by the so-called Hasegawa–Mima equation, a tripolar solution has been obtained explicitly for the case of a background shear flow (Chakrabarti *et al.* 1994). It was also observed that strong monopolar structures, which are a quasistationary solution, will develop into tripole-like structures when propagating essentially perpendicular to the density gradient (Sutyrin *et al.* 1994). For more complex drift waves coupling two fields – the ion-temperature-gradient (ITG, or η_i) modes – it was recently demonstrated that tripolar vortices may exist under special conditions.

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Thus there is ample evidence for the existence of tripolar vortices, both in neutral fluids and in plasmas in the case of electrostatic perturbations. In this paper we show that tripolar-like solutions may also be constructed for electromagnetic waves, such as shear-Alfvén. There, the tripolar structures appear as a solution imbedded in a background shear flow, in the presence of a sheared magnetic field. The tripolar structures arise both in the fluid vorticity and in the magnetic vector potential. They appear with a central core, with the current flowing along the ambient magnetic field, and two side tubes with the current in the opposite direction. While dipolar vortices have previously been found for this shear-Alfvén mode (Jovanović *et al.* 1987; Jovanović and Horton 1994*a*), the present paper is, to the best of our knowledge, the first report of tripoles in such a setting.

2. Basic equations

We study electromagnetic perturbations in a homogeneous plasma with unperturbed density n_0. We assume that both ions and electrons are cold, with the electron pressure being much smaller than the magnetic pressure:

$$\beta \equiv \frac{2n_o T_e}{c^2 \epsilon_0 B_0^2} \ll \frac{m_e}{m_i}. \tag{1}$$

Here m_e and m_i are the electron and ion masses, and T_e is the electron temperature. The plasma is immersed in a sheared magnetic field

$$\mathbf{B}_0 = B_0 \left(\mathbf{e}_z + \mathbf{e}_y \frac{x}{L_s} \right), \tag{2}$$

and moves macroscopically in the y direction with inhomogeneous velocity

$$\mathbf{V}_0 = \mathbf{e}_y V_0 \frac{x - x_0}{L_v}. \tag{3}$$

In the regime of time variation slow compared with the ion gyrofrequency $\mathbf{\Omega}_i$, and weak z dependence

$$\frac{\partial}{\partial t} \ll \mathbf{\Omega}_i, \qquad \frac{\partial}{\partial z} \ll \nabla_\perp, \tag{4}$$

for relatively small perturbations of the electron density and magnetic field,

$$\delta n_e \equiv n_e - n_0 \ll n_0, \tag{5a}$$

$$| \, \delta \mathbf{B} \, | \equiv | \, \mathbf{B} - B_0 \mathbf{e}_z \, | \ll B_0, \tag{5b}$$

neglecting in the low-β limit the compressional component of the magnetic field perturbation, δB_z, and with accuracy to first order in m_e/m_i and in the small parameters defined above, the plasma is described by the standard set of shear-Alfvén equations (Jovanović *et al.* 1987)

$$\left[\frac{\partial}{\partial t} + \frac{1}{B_0} (\mathbf{e}_z \times \boldsymbol{\nabla} \phi) \cdot \boldsymbol{\nabla} \right] \nabla_\perp^2 \phi = -c_A^2 \left[\frac{\partial}{\partial z} - \frac{1}{B_0} (\mathbf{e}_z \times \boldsymbol{\nabla} A_z) \cdot \boldsymbol{\nabla} \right] \nabla_\perp^2 A_z, \tag{6}$$

$$\left[\frac{\partial}{\partial t} + \frac{1}{B_0} (\mathbf{e}_z \times \boldsymbol{\nabla} \phi) \cdot \boldsymbol{\nabla} \right] (1 - \lambda^2 \nabla_\perp^2) A_z = -\frac{\partial \phi}{\partial z}. \tag{7}$$

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Equations (6) and (7) are Hamiltonian, and have sets of conserved quantities, as discussed in the case of shear-Alfvén vortices by Schep *et al.* (1995) and Kuvshinov *et al.* (1998). In the above, $c_A = c\Omega_i/\omega_{pi}$ is the Alfvén speed, $\lambda = c/\omega_{pe}$ is the collisionless skin depth, ω_{pi} and ω_{pe} are the ion and electron plasma frequencies respectively, and we have expressed the electric and magnetic fields in terms of the electrostatic potential ϕ and the z component of the vector potential $\mathbf{A}$:

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \mathbf{e}_z \frac{\partial A_z}{\partial t}, \tag{8a}$$

$$\mathbf{B} = B_0 \mathbf{e}_z - \mathbf{e}_z \times \boldsymbol{\nabla} A_z. \tag{8b}$$

The potentials ϕ and A_z can be written as the sums of corresponding zeroth-order quantities and the perturbations $\delta\phi$ and δA_z

$$\phi = \delta\phi + B_0 V_0 \frac{(x - x_0)^2}{2L_v} + \phi_0, \tag{9a}$$

$$A_z = \delta A_z - B_0 \frac{x^2}{2L_s}, \tag{9b}$$

while the arbitrary constant ϕ_0 will be conveniently adopted later. Equations (6) and (7) are valid in the region where both the total electrostatic potential and the deviation from the constant magnetic field $B_0\mathbf{e}_z$ can be regarded as small quantities (see (5)); that is, for $| x | \ll L_s$ and $| x - x_0 | \ll L_v$. They constitute our basic system describing nonlinear shear-Alfvén perturbations. In the limit $\lambda \to 0$, they reduce to the low-β reduced MHD system (Strauss 1976).

Owing to the dominance of convective nonlinear terms, arising from the $\mathbf{E} \times \mathbf{B}$ drift and the twisting of the magnetic field lines, the shear-Alfvén equations belong to the class of partial differential equations with Poisson-bracket-type nonlinearities, which in many cases possess stationary, spatially localized, coherent nonlinear solutions. In the absence of magnetic and velocity shears, the solution of (6) and (7) has the form of a pair of vortex tubes, rotating in opposite directions, and slightly tilted relative to the magnetic field. The perpendicular scale of such a vortex is of the order of the collisionless skin depth, which requires a two-fluid, rather than a magnetohydrodynamic (MHD) description. In the MHD limit, the vortex becomes singular, i.e. reduces to a point (or filament). In the presence of magnetic shear, a self–organization process, adjusting the profile of the sheared poloidal magnetic field and exciting a localized perpendicular sheared plasma flow, produces a coherent solution in the form of a moving vortex chain. A numerical solution (Jovanović and Vranješ 1994*b*) shows a twin chain of plasma vortices, coupled with a single chain of magnetic islands, whose width is of the order of the collisionless skin depth. The adiabatic evolution of the chain in the presence of small viscosity revealed its finite lifetime. The destruction may occur either directly, or through a sequence of bifurcations that lead to the creation of similar chains, but with decreasing wavelengths, and eventually produces a layer of stochastic magnetic field in the vicinity of the given rational surface.

The present paper is devoted to the study of nonlinear processes associated with shear-Alfvén perturbations in the presence of both magnetic and velocity shears. We demonstrate the existence of coherent vortex structures, whose topology differs substantially from the previously known ones in this system.

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3. Stationary solution

In this section we construct a stationary nonlinear solution of the shear-Alfvén equations (6) and (7) in the presence of macroscopic shear flow and magnetic shear. It is convenient to rewrite our basic equations in the reference frame travelling with velocity $u\mathbf{e}_y$, and to make use of the dimensionless quantities

$$x' = \frac{x}{\lambda}, \qquad y' = \frac{y - ut}{\lambda}, \qquad z' = \frac{uz}{c_A\lambda}, \qquad t' = \frac{ut}{\lambda}, \tag{10}$$

and

$$\varphi(x', y') \equiv \frac{\phi - B_0 ux}{B_0 u\lambda} = \delta\varphi - x' + \frac{(x' - x'_0)^2}{2b} + \varphi_0, \tag{11a}$$

$$\psi(x', y') \equiv \frac{c_A A_z}{B_0 u\lambda} = \delta\psi - \frac{x'^2}{2a}, \tag{11b}$$

where

$$a = \frac{uL_s}{c_A\lambda}, \qquad b = \frac{uL_v}{V_0\lambda}.$$

Thus, our basic equations take the simple form

$$\left[\frac{\partial}{\partial t'} + \left(\mathbf{e}_z \times \boldsymbol{\nabla}'\varphi\right)\cdot\boldsymbol{\nabla}'\right]\nabla'^2_\perp\varphi + \left[\frac{\partial}{\partial z'} - \left(\mathbf{e}_z \times \boldsymbol{\nabla}'\psi\right)\cdot\boldsymbol{\nabla}'\right]\nabla'^2_\perp\psi = 0, \tag{12}$$

$$\left[\frac{\partial}{\partial t'} + \left(\mathbf{e}_z \times \boldsymbol{\nabla}'\varphi\right)\cdot\boldsymbol{\nabla}'\right](1 - \nabla'^2_\perp)\psi + \frac{\partial\varphi}{\partial z'} = 0. \tag{13}$$

For the sake of simplicity, primes indicating dimensionless quantities will be omitted in the rest of the paper.

It is well known that the above equations possess a two-dimensional solution that is time-stationary in the moving reference frame, $\partial/\partial t = 0$, in the form of an elongated structure tilted by a (small) angle α relative to the z axes, $\partial/\partial z = \alpha\,\partial/\partial y$. Conveniently adopting $\alpha = 0$, i.e. for structures that are parallel to the background magnetic field, at $x = 0$, we may readily integrate (13) once:

$$(1 - \nabla^2_\perp)\psi = F(\varphi), \tag{14}$$

where F is an arbitrary function of its argument. Substituting the above expression into (12), in the same fashion we obtain

$$\nabla^2_\perp\varphi - F'(\varphi)\,\nabla^2_\perp\psi = G(\varphi), \tag{15}$$

where F' is the first derivative with respect to its argument, and G is another arbitrary function.

In the standard vortex scenario, $F(\varphi)$ and $G(\varphi)$ are taken to be linear functions,

$$F(\varphi) = F_0 + F_1\varphi, \tag{16a}$$

$$G(\varphi) = G_0 + G_1\varphi, \tag{16b}$$

allowing for different values of the constants $F^{\rm out}_{0,1}$, $G^{\rm out}_{0,1}$ and $F^{\rm in}_{0,1}$, $G^{\rm in}_{0,1}$ outside and inside the vortex core respectively. The core is taken to be a circle in the (x, y) plane, with radius r_0.

From (11), and for localized perturbations $\delta\varphi$ and $\delta\psi$ that vanish at infinity, one can see that outside the vortex core such simple linear functions are possible only if their velocity relative to the surrounding plasma is zero at $x = 0$, which is

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equivalent to $x_0 = -b$, and with the following values of the integration constants:

$$F_0^{\rm out} = \frac{1}{a}, \qquad F_1^{\rm out} = -\frac{b}{a}, \tag{17a}$$

$$G_0^{\rm out} = \frac{1}{b} - \frac{b}{a^2}, \qquad G_1^{\rm out} = 0, \tag{17b}$$

where $\varphi_0 = -\frac{1}{2}b$ has been adopted.

With the linear expressions (16) for the functions F and G on the whole (x, y) plane, (14) and (15) decouple to give

$$(\nabla_\perp^2 + \kappa_1^2)(\nabla_\perp^2 + \kappa_2^2)\left(\delta\varphi + \frac{x^2}{2b} + \frac{G_0}{G_1}\right) = 0, \tag{18}$$

where the characteristic wavenumbers $\kappa_{1,2}$ are related to the slopes F_1 and G_1 through

$$\kappa_{1,2}^2 = \tfrac{1}{2}\{-(F_1^2 - G_1 - 1) \pm [(F_1^2 - G_1 - 1)^2 - 4G_1]^{1/2}\}. \tag{19}$$

Equation (18) separates variables in the cylindrical frame $r = \left(x^2 + y^2\right)^{1/2}$, $\theta = \arctan(y/x)$, with the usual radial solution in the form of Bessel functions. Noting the presence of the term $x^2 = \frac{1}{2}r^2(1+\cos 2\theta)$, we assume a solution that contains only the zeroth and second cylindrical harmonics (i.e. as a superposition of a monopole and a quadrupole): $\delta\varphi = \delta\varphi_0(r) + \delta\varphi_2(r)\cos 2\theta$.

One may readily verify that $\kappa_2^{\rm out} = 0$. Then, a spatially localized solution is obtained for $\rho^2 \equiv -\kappa_1^{\rm out^2} = 1 - b^2/a^2 > 0$, and consequently, outside the vortex core ($r > r_0$) we have

$$\delta\varphi_0^{\rm out}(r) = c_1^{\rm out} K_0(\rho r), \tag{20a}$$

$$\delta\varphi_2^{\rm out}(r) = d_1^{\rm out} K_2(\rho r) + \frac{d_2^{\rm out}}{r^2}. \tag{20b}$$

Likewise, within the core ($r < r_0$) the solution is given by

$$\delta\varphi_0^{\rm in}(r) = c_1^{\rm in} J_0(\kappa_1 r) + c_2^{\rm in} J_0(\kappa_2 r) - \frac{r^2}{4b} - g_0, \tag{21a}$$

$$\delta\varphi_2^{\rm in}(r) = d_1^{\rm in} J_2(\kappa_1 r) + d_2^{\rm in} J_2(\kappa_2 r) - \frac{r^2}{4b}, \tag{21b}$$

where $g_0 = G_0^{\rm in}/G_1^{\rm in}$. The corresponding perturbations of the vector potentials can be expressed as

$$\delta\psi_0^{\rm out}(r) = \frac{c_1^{\rm out}}{(1+\rho^2)^{1/2}} K_0(\rho r), \tag{22a}$$

$$\delta\psi_2^{\rm out}(r) = \frac{d_1^{\rm out}}{(1+\rho^2)^{1/2}} K_2(\rho r) + (1+\rho^2)^{1/2}\frac{d_2^{\rm out}}{r^2}, \tag{22b}$$

and

$$\delta\psi_0^{\rm in}(r) = c_1^{\rm in}\left(\frac{1+\kappa_2^2}{1+\kappa_1^2}\right)^{1/2} J_0(\kappa_1 r) + c_2^{\rm in}\left(\frac{1+\kappa_1^2}{1+\kappa_2^2}\right)^{1/2} J_0(\kappa_2 r) + \frac{r^2}{4a} - h_0, \tag{23a}$$

$$\delta\psi_2^{\rm in}(r) = d_1^{\rm in}\left(\frac{1+\kappa_2^2}{1+\kappa_1^2}\right)^{1/2} J_2(\kappa_1 r) + d_2^{\rm in}\left(\frac{1+\kappa_1^2}{1+\kappa_2^2}\right)^{1/2} J_2(\kappa_2 r) + \frac{r^2}{4a}, \tag{23b}$$

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where $h_0 = F_1^{\rm in} G_0^{\rm in}/G_1^{\rm in} - F_0^{\rm in}$. In these expressions, $J_{0,2}$ and $K_{0,2}$ are Bessel and modified Bessel functions respectively, of the given order, and $c_{1,2}^{\rm in,out}$ and $d_{1,2}^{\rm in,out}$ are integration constants (the boundary conditions). For the sake of simplicity, we have used the notation $\kappa_{1,2}$ instead of $\kappa_{1,2}^{\rm in}$.

4. Nonlinear dispersion relation

Since the analytical expressions for the potentials $\delta\varphi$ and $\delta\psi$ are different inside and outside the vortex core, the following continuity conditions must be satisfied at the vortex-core edge in order to have a physically viable solution at $r = r_0$:

(i) continuity of the functions $F(\varphi)$ and $G(\varphi)$, in order to avoid singularities in the original equations (12) and (13);

(ii) continuity of the potentials $\delta\varphi$ and $\delta\psi$, which ensures finiteness of the electric and magnetic fields;

(iii) continuity of $(\partial/\partial\theta)\,\delta\varphi$ and $(\partial/\partial r)\,\delta\psi$, which is equivalent to the usual physical continuity conditions for the parallel component of the electric field and of the perpendicular component of the magnetic field;

(iv) continuity of $(\partial/\partial r)\,\delta\psi$ and $(\partial/\partial\theta)\,\delta\psi$, which ensures the absence of surface charges and currents at $r = r_0$.

Making use of (16), (17) and (19), we obtain from these continuity conditions

$$(1+\kappa_1^2)^{1/2}(1+\kappa_2^2)^{1/2}\left[\delta\varphi_0^{\rm in}(r_0) + \frac{r_0^2}{4b} + g_0\right] - h_0 = (1-\rho^2)^{1/2}\delta\varphi_0^{\rm out}(r_0) - \frac{r_0^2}{4a} + \frac{1}{a}, \quad (24)$$

$$\kappa_1^2\kappa_2^2\left[\delta\varphi_0^{\rm in}(r_0) + \frac{r_0^2}{4b} + g_0\right] = \frac{\rho^2}{b}, \quad (25)$$

$$\delta\varphi_2^{\rm in}(r_0) + \frac{r_0^2}{4b} = \delta\varphi_2^{\rm out}(r_0) + \frac{r_0^2}{4b} = 0, \quad (26)$$

and

$$\delta\varphi_i^{\rm in}(r_0) = \delta\varphi_i^{\rm out}(r_0), \quad (27)$$

$$\delta\psi_i^{\rm in}(r_0) = \delta\psi_i^{\rm out}(r_0), \quad (28)$$

$$\frac{d}{dr_0}\delta\varphi_i^{\rm in}(r_0) = \frac{d}{dr_0}\delta\varphi_i^{\rm out}(r_0), \quad (29)$$

$$\frac{d}{dr_0}\delta\psi_i^{\rm in}(r_0) = \frac{d}{dr_0}\delta\psi_i^{\rm out}(r_0), \quad (30)$$

$$i = 0, 2.$$

With the use of (20)–(23), and noting that (27), for $i = 2$, is satisfied by (26), it is possible to determine all the remaining parameters $c_1^{\rm in}$, $c_2^{\rm in}$, $d_1^{\rm in}$, $d_2^{\rm in}$, $c_1^{\rm out}$, $d_1^{\rm out}$, $d_2^{\rm out}$, g_0 and h_0 from the above equations. We shall further simplify these expressions by adopting the notation

$$\Lambda = \left(\frac{1+\kappa_2^2}{1+\kappa_1^2}\right)^{1/2}, \quad \mathscr{J}_{i,1} = J_i(\kappa_1 r_0), \quad \mathscr{J}_{i,2} = J_i(\kappa_2 r_0), \quad \mathscr{K}_i = K_i(\rho r_0),$$

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$$\mathscr{J}'_{i,1} = \frac{d}{dr_0} J_i(\kappa_1 r_0), \quad \mathscr{J}'_{i,2} = \frac{d}{dr_0} J_i(\kappa_2 r_0), \quad \mathscr{K}'_i = \frac{d}{dr_0} K_i(\rho r_0), \quad i = 0, 2.$$

Conveniently subtracting (28) for $i = 0$ from (24),

$$c_1^{\text{in}} \mathscr{J}_{0,1} \kappa_1^2 \Lambda + c_2^{\text{in}} \mathscr{J}_{0,2} \frac{\kappa_2^2}{\Lambda} + c_1^{\text{out}} \frac{\mathscr{K}_0 \rho}{(1-\rho^2)^{1/2}} = 0, \tag{31}$$

and rewriting, for $i = 0$, (29) and (30) as

$$c_1^{\text{in}} \mathscr{J}'_{0,1} + c_2^{\text{in}} \mathscr{J}'_{0,2} - c_1^{\text{out}} \mathscr{K}'_0 = 0, \tag{32}$$

$$c_1^{\text{in}} \mathscr{J}'_{0,1} \Lambda + c_2^{\text{in}} \frac{\mathscr{J}'_{0,2}}{\Lambda} - c_1^{\text{out}} \frac{\mathscr{K}'_0}{(1-\rho^2)^{1/2}} = 0, \tag{33}$$

we obtain a system of three homogeneous algebraic equations for the coefficients c_1^{in}, c_2^{in} and c_1^{out}.

Similarly, (26), and (28)–(30) for $i = 2$, form a homogeneous system for the parameters d_1^{in}, d_2^{in}, d_1^{out} and d_2^{out}:

$$d_1^{\text{in}} \mathscr{J}_{2,1} + d_2^{\text{in}} \mathscr{J}_{2,2} = 0, \tag{34}$$

$$d_1^{\text{in}} \mathscr{J}_{2,1} \Lambda + d_2^{\text{in}} \frac{\mathscr{J}_{2,2}}{\Lambda} - d_1^{\text{out}} \frac{\mathscr{K}_2}{(1-\rho^2)^{1/2}} - d_2^{\text{out}} \frac{(1-\rho^2)^{1/2}}{r_0^2} = 0, \tag{35}$$

$$d_1^{\text{in}} \mathscr{J}'_{2,1} + d_2^{\text{in}} \mathscr{J}'_{2,2} - d_1^{\text{out}} \mathscr{K}'_2 + d_2^{\text{out}} \frac{2}{r_0^3} = 0, \tag{36}$$

$$d_1^{\text{in}} \mathscr{J}'_{2,1} \Lambda + d_2^{\text{in}} \frac{\mathscr{J}'_{2,2}}{\Lambda} - d_1^{\text{out}} \frac{\mathscr{K}'_2}{(1-\rho^2)^{1/2}} + d_2^{\text{out}} (1-\rho^2)^{1/2} \frac{2}{r_0^3} = 0. \tag{37}$$

The remaining equations are inhomogeneous. Thus (25) and (26) can be rewritten as

$$\kappa_1^2 \kappa_2^2 (c_1^{\text{in}} \mathscr{J}_{0,1} + c_2^{\text{in}} \mathscr{J}_{0,2}) = \frac{\rho^2}{b}, \tag{38}$$

$$d_1^{\text{out}} \mathscr{K}_2 + \frac{d_2^{\text{out}}}{r_0^2} + \frac{r_0^2}{4b} = 0, \tag{39}$$

while (24), and (27) for $i = 0$, take the form

$$(1+\kappa_1^2)^{1/2} (1+\kappa_2^2)^{1/2} (c_1^{\text{in}} \mathscr{J}_{0,1} + c_2^{\text{in}} \mathscr{J}_{0,2}) - h_0 = (1-\rho^2)^{1/2} c_1^{\text{out}} \mathscr{K}_0 - \frac{r_0^2}{4a} + \frac{1}{a}, \tag{40}$$

$$c_1^{\text{in}} \mathscr{J}_{0,1} + c_2^{\text{in}} \mathscr{J}_{0,2} - \frac{r_0^2}{4b} - g_0 = c_1^{\text{out}} \mathscr{K}_0, \tag{41}$$

and will be used only to determine the constants g_0 and h_0.

One can readily see from the above equations that the trivial solutions $c_1^{\text{in,out}} = c_2^{\text{in,out}} = 0$ and $d_1^{\text{in,out}} = d_2^{\text{in,out}} = 0$ are forbidden if $r_0 \neq 0$ and $\rho \neq 0$. In other words, any stationary vortex-type solution of our original equations must contain both the monopole and quadrupole components.

The required non-trivial (i.e. non-zero) solutions of the homogeneous systems of equations (31)–(33) and (34)–(37) exist only if the corresponding determinants of the systems reduce to zero. This yields

$$\mathscr{D}_0 = 0, \qquad \mathscr{D}_2 = 0, \tag{42}$$

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where

$$\mathscr{D}_0 = \begin{vmatrix} \mathscr{J}_{0,1}\kappa_1^2\Lambda & \mathscr{J}_{0,2}\kappa_2^2/\Lambda & \mathscr{K}_0\rho^2/\left(1-\rho^2\right)^{1/2} \\ \mathscr{J}'_{0,1} & \mathscr{J}'_{0,2} & -\mathscr{K}_0 \\ \mathscr{J}'_{0,1}\Lambda & \mathscr{J}'_{0,2}/\Lambda & -\mathscr{K}'_0/\left(1-\rho^2\right)^{1/2} \end{vmatrix}, \tag{43}$$

$$\mathscr{D}_2 = \begin{vmatrix} \mathscr{J}_{2,1} & \mathscr{J}_{2,2} & 0 & 0 \\ \mathscr{J}_{2,1}\Lambda & \mathscr{J}_{2,2}/\Lambda & -\mathscr{K}_2/\left(1-\rho^2\right)^{1/2} & -\left(1-\rho^2\right)^{1/2}/r_0^2 \\ \mathscr{J}'_{2,1} & \mathscr{J}'_{2,2} & -\mathscr{K}'_2 & 2/r_0^3 \\ \mathscr{J}'_{2,1}\Lambda & \mathscr{J}'_{2,2}/\Lambda & -\mathscr{K}'_2/\left(1-\rho^2\right)^{1/2} & 2\left(1-\rho^2\right)^{1/2}/r_0^3 \end{vmatrix}. \tag{44}$$

The above equations may be interpreted as nonlinear dispersion relations for the 'inside' wavenumbers κ_1 and κ_2.

As discussed above, any stationary solution of the complete nonlinear system (6) and (7), constructed in the present scenario, is a superposition of monopole and quadrupole components. However, their dispersion equation factorizes into two separate dispersion equations that must be satisfied simultaneously, and whose structure is such that the existence of more than one vortical mode is not allowed, as was the case with dipoles (Jovanović and Horton 1994*a*).

The axis of these vortices is parallel to the local magnetic field, and the vortex propagation velocity is zero in the frame where the null lines of the background magnetic and velocity fields coincide. For every background plasma configuration, defined by the ratio of shear lengths a/b, there is only one free vortex parameter, namely the core radius r_0. With an appropriate choice of the radius r_0, satisfying the additional constraint

$$\mathscr{D}_3 = \begin{vmatrix} \mathscr{J}_{2,1}\Lambda & \mathscr{J}_{2,2}/\Lambda & -\mathscr{K}_2/\left(1-\rho^2\right)^{1/2} \\ \mathscr{J}'_{2,1} & \mathscr{J}'_{2,2} & -\mathscr{K}'_2 \\ \mathscr{J}'_{2,1}\Lambda & \mathscr{J}'_{2,2}/\Lambda & -\mathscr{K}'_2/\left(1-\rho^2\right)^{1/2} \end{vmatrix} = 0, \tag{45}$$

it is possible to obtain a strongly localized vortex, with $d_2^{\mathrm{out}} = 0$, whose quadrupole component is, in the outer region $r > r_0$, given in terms of only the Bessel function K_2.

5. Conclusions

We have constructed explicit analytical solutions of the shear-Alfvén equations corresponding to tripolar vortices, propagating in an inhomogeneous background plasma with linear and parallel magnetic and velocity field shears. In the reference frame where the null lines of the background magnetic and velocity fields coincide, the propagation velocity of these tripolar vortices vanishes.

These vortices consist of a monopole and a quadrupole component, and obey a 'dispersion equation' that is factorized into two separate dispersion equations for the monopole and the quadrupole components respectively. Aside from the degenerate cases where the core vortex radius vanishes, or the magnetic and velocity shear lengths are equal, both components must be present and the two dispersion equations (42) must be satisfied simultaneously.

The x^2 dependence of the background potentials ϕ and A_z can be seen as a special case of the superposition of monopolar, x^2+y^2, and quadrupolar, x^2-y^2, terms with equal coefficients. Similar superpositions with unequal coefficients describe either

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hyperbolic X-point configurations with variable angles between the separatrices or elliptical O-point configurations. In all of these background configurations, the current density and the vorticity are constant and the equilibrium condition is satisfied if the asymptotes (in the hyperbolic case) or the principal axes (in the elliptical case) of ϕ and A_z coincide, i.e. if the two potentials have the same $(x-y)$ dependence aside from a scale factor and thus the velocity and the poloidal field lines are 'parallel'. The procedure developed in this paper can be easily extended to such configurations, leading to vortex structures localized around the X point or the O point. These vortices do not propagate. They obey the dispersion equations (42), but contain a different mixture of monopole and quadrupole components, which mirrors that of the background configuration.

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