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Null systems and sequence entropy pairs

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Abstract. A measure-preserving transformation (respectively a topological system) is null if the metric (respectively topological) sequence entropy is zero for any sequence. Kushnirenko has shown that an ergodic measure-preserving transformation has a discrete spectrum if and only if it is null. We prove that for a minimal system this statement remains true modulo an almost one-to-one extension. It allows us to show that a scattering system is disjoint from any null minimal system. Moreover, some necessary conditions for a transitive non-minimal system to be null are obtained.

Localizing the notion of sequence entropy, we define sequence entropy pairs and show that there is a maximal null factor for any system. Meanwhile, we define a weaker notion, namely weak mixing pairs. It turns out that a system is weakly mixing if and only if any pair not in the diagonal is a sequence entropy pair if and only if the same holds for a weak mixing pair, answering a question in Blanchard *et al* (F. Blanchard, B. Host and A. Maass, Topological complexity. *Ergod. Th. & Dynam. Sys.*, **20** (2000), 641–662). For a group action we give a direct proof of the fact that the factor induced by the smallest invariant equivalence relation containing the regionally proximal relation is equicontinuous. Furthermore, we show that a non-equicontinuous minimal distal system is not null.

1. *Introduction*

By a *(topological) dynamical system* we mean a compact metric space X with a continuous map T from X onto X. A measure-preserving transformation (respectively a topological system) is null if the metric (respectively topological) sequence entropy is zero for any sequence. Kushnirenko [**18**] has shown that an ergodic measure-preserving transformation T has a discrete spectrum if and only if it is null. Thus, a natural question is to wonder what the situation would be in a topological setting?

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By the well-known Halmos and von Neumann Theorem [**15**], a (topologically) transitive system has a topological discrete spectrum if and only if it is minimal and equicontinuous. Note that it is easy to show that an equicontinuous system is null. It is natural to conjecture that if a minimal system is null then it is also equicontinuous. Unfortunately this is not the case. In [**13**], Goodman obtained a minimal non-equicontinuous system which is null. The question to what extent Kushinirenko's result is valid in a topological setting remains open. We try to answer this question in the present paper.

We prove that for a minimal system Kushnirenko's statement remains true modulo an almost one-to-one extension, i.e. if a minimal system (X, T) is null, then (X, T) is an almost one-to-one extension of an equicontinuous system, which is uniquely ergodic and has a discrete spectrum with respect to a unique measure (in view of the general structure theorem of minimal systems, it is the best result we can get). This allows us to show that a scattering system is disjoint from any null minimal system. Moreover, we show that if a transitive non-minimal system (X, T) is null, then there are non-empty open sets U and V of X such that $N(U, V) = \{n \in \mathbb{Z}_+ \mid U \cap T^{-n}V \neq \emptyset\}$ has a zero upper Banach density.

Localizing the notion of sequence entropy, we define sequence entropy pairs and show that there is a maximal null factor for each system. Meanwhile, we define a weaker notion, namely weak mixing pairs. It turns out that a system is weakly mixing if and only if any pair not in the diagonal is a sequence entropy pair if and only if the same holds for a weak mixing pair, answering Question 3.9 in [**8**]. For a group action we show that the factor induced by the smallest invariant equivalence relation containing weak mixing pairs is equicontinuous and supply a direct proof of the well-known result that the factor induced by the smallest invariant equivalence relation containing the regionally proximal relation is equicontinuous. Furthermore, we show that for a minimal distal system the set of sequence entropy pairs coincides with the regionally proximal relation and, thus, a non-equicontinuous minimal distal system is not null.

The text is organized as follows. In §2, we give the necessary definitions and some general results. To study null minimal systems we find that we need to study null transitive non-minimal systems and this is carried out in §3. In §4, we obtain the results on null minimal systems. Finally in §5 we discuss some properties of null systems.

2. *Preliminary and basic properties*

The topological entropy of T with respect to a finite open cover α is denoted by $h(T, \alpha)$. For an increasing sequence $\mathcal{A} = \{t_1, t_2, \dots\}$ of $\mathbb{Z}_+$, the sequence entropy of T with respect to $\mathcal{A}$ and a finite open cover α is

$$h_{\mathcal{A}}(T, \alpha) = \limsup_{n \to +\infty} \frac{1}{n} \log N\left(\bigvee_{i=1}^{n} T^{-t_i}\alpha\right)$$

and the sequence entropy of T with respect to $\mathcal{A}$ is $h_{\mathcal{A}}(T) = \sup_\alpha h_{\mathcal{A}}(T, \alpha)$.

Definition. Let (X, T) be a dynamical system.

(1) We say that $(x_1, x_2) \in X \times X$ is a sequence entropy pair if $x_1 \neq x_2$ and if whenever U_i are closed mutually disjoint neighborhoods of points x_i, $i = 1, 2$, there exists a sequence $\mathcal{A} \subset \mathbb{Z}_+$ such that $h_{\mathcal{A}}(T, \{U_1^c, U_2^c\}) > 0$.

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(2) We say (X, T) has a uniform positive sequence entropy (for short s.u.p.e.), if every point $(x_1, x_2) \in X \times X$, not in the diagonal $\Delta = \{(x, x) : x \in X\}$, is a sequence entropy pair.

Remark. It is easy to see that (X, T) has s.u.p.e. if and only if for any cover $\mathcal{R} = \{U, V\}$ of X by two non-dense open sets, one has $h_{\mathcal{A}}(T, \mathcal{R}) > 0$ for some sequence $\mathcal{A} \subset \mathbb{Z}_+$.

Denote by $SE(X, T)$ the set of all sequence entropy pairs and by $SE'(X, T)$ its closure. The notion of *entropy pairs* ($E(X, T)$ denotes the set of entropy pairs) was introduced by Blanchard [**7**] and a *uniform positive entropy* (u.p.e.) system, i.e. any pair which is not in the diagonal is an entropy pair, was first studied in [**6**], which was used to study the topological analogue of the K-system. In [**8**] the authors introduced the notion of *complexity pairs* ($\mathrm{Com}(X, T)$ denotes the set of complexity pairs). Recall that the *regionally proximal relation* $RP(X, T)$ is the set of all points $(x_1, x_2) \in X \times X$ such that for each $\epsilon > 0$ and each open neighborhood U_i of x_i, $i = 1, 2$, there are $x_i' \in U_i$, $i = 1, 2$, and $n \in \mathbb{N}$ with $d(T^n(x_1'), T^n(x_2')) < \epsilon$. By [**8, 16**] and easy observations, we have

$$E(X, T) \subset SE(X, T) \subset \mathrm{Com}(X, T) \subset RP(X, T).$$

Moreover, the sequence entropy pairs share several important properties of entropy pairs and complexity pairs (the proofs are similar, see [**7**]). Thus, we have the following proposition.

PROPOSITION 2.1. *Let (X, T) be a dynamical system.*

(1) *If there is an open cover $\mathcal{R} = \{U, V\}$ of X with $h_{\mathcal{A}}(T, \mathcal{R}) > 0$ for some sequence $\mathcal{A} \subset \mathbb{Z}_+$, then there are points $x \in U^c$ and $y \in V^c$ such that (x, y) is a sequence entropy pair.*

(2) *If $h_{\mathcal{A}}(T) > 0$ for some sequence $\mathcal{A} \subset \mathbb{Z}_+$, then $SE'(X, T)$ is a non-empty closed $T \times T$-invariant subset of $X \times X$ containing only sequence entropy pairs and points of Δ.*

(3) *Let $\pi : (Y, S) \to (X, T)$ be a factor map.*
   (a) *If $(x_1, x_2) \in SE(X, T)$, then there exists $(y_1, y_2) \in SE(Y, S)$ with $\pi(y_1) = x_1$ and $\pi(y_2) = x_2$; and*
   (b) *if $(y_1, y_2) \in SE(Y, S)$ and $\pi(y_1) \neq \pi(y_2)$, then $(\pi(y_1), \pi(y_2)) \in SE(X, T)$.*

(4) *Suppose W is a closed T-invariant subset of (X, T). Then if (x_1, x_2) is a sequence entropy pair of $(W, T|_W)$, it is also a sequence entropy pair of (X, T).*

By this proposition, we know that a system (X, T) is null if and only if $SE(X, T) = \emptyset$.

For a dynamical system (X, T), we say (X, T) is *transitive* if, for each pair of non-empty open subsets U and V of X, there is an $n \in \mathbb{N}$ such that $U \cap T^{-n}(V) \neq \emptyset$; however, (X, T) is *weakly mixing* if $(X \times X, T \times T)$ is transitive. The orbit of $x \in X$, $\{x, T(x), T^2(x), \dots\}$, is denoted by $\mathrm{orb}(x, T)$. For a transitive system, $x \in X$ is a *transitive point* if the orbit of x is dense; and it is known that the set of transitive points forms a dense G_δ set. If the set of the transitive points is the whole space X, we then say that (X, T) is *minimal* and, in this case, each point in X is called a *minimal point*.

It is known [**21, 5**] that (X, T) is not weakly mixing if and only if there are two non-empty open sets U and V such that there is no $n \in \mathbb{N}$ such that $U \cap T^{-n}U \neq \emptyset$ and $U \cap T^{-n}V \neq \emptyset$. Motivated by this property, we have the following definition.

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Definition. For a dynamical system (X, T), $(x_1, x_2) \in X \times X \setminus \Delta$ is a weak mixing pair if for any open neighborhood U_i of x_i, $i = 1, 2$, one has $N(U_1, U_1) \cap N(U_1, U_2) \neq \emptyset$, where $N(U, V) = \{n \in \mathbb{Z}_+ \mid \cap T^{-n}(V) \neq \emptyset\}$. The set of weak mixing pairs is denoted by $WM(X, T)$.

Sequence entropy pairs and weak mixing pairs are related by the following proposition.

PROPOSITION 2.2. *For a dynamical system* (X, T),

(1) $SE(X, T) \subset WM(X, T)$*; and*

(2) $WM(X, T) \subset RP(X, T^{-1})$*, if* T *is invertible.*

Proof. For the proof of (1), we follow [**8**]. Assume that $x_1 \neq x_2$ and $(x_1, x_2) \notin WM(X, T)$. Then there are open neighborhoods U_i' of x_i, $i = 1, 2$, such that $N(U_1', U_1') \cap N(U_1', U_2') = \emptyset$. Take closed neighborhoods U_i of x_i with $U_i \subset U_i'$, $i = 1, 2$, and $U_1 \cap U_2 = \emptyset$. Clearly, $N(U_1, U_1) \cap N(U_1, U_2) = \emptyset$. Therefore, $U_1 \cap T^{-n}U_1 = \emptyset$ or $U_1 \cap T^{-n}U_2 = \emptyset$ for any $n \in \mathbb{Z}_+$. Moreover, for any $n \in \mathbb{Z}_+$, we can take $W_n = U_1^c$ or U_2^c such that $U_1 \subset T^{-n}W_n$.

Let $\mathcal{R} = \{U_1^c, U_2^c\}$. For $n \in \mathbb{N}$ and any sequence $\mathcal{A} = \{t_1, t_2, \dots\}$ with $0 \leq t_1 < t_2 < \cdots$, consider for each $x \in X$ the first $i \in \{1, 2, \dots, n\}$ such that $T^{t_i}x \in U_1$, when one exists. We find that $\mathcal{R}_n = \bigvee_{i=1}^n T^{-t_i}\mathcal{R}$ admits a subcover by the sets

$$T^{-t_1}U_1^c \cap \cdots \cap T^{-t_{i-1}}U_1^c \cap T^{-t_i}W_0 \cap T^{-t_{i+1}}W_{t_{i+1}-t_i} \cap \cdots \cap T^{-t_n}W_{t_n-t_i}$$

$i = 1, 2, \dots, n, n+1$. Hence, for all $n \in \mathbb{N}$, $H(\mathcal{R}_n) \leq n+1$ and, therefore, $h_{\mathcal{A}}(T, \mathcal{R}) = 0$. This implies that $(x_1, x_2) \notin SE(X, T)$ and we have proved that $SE(X, T) \subset WM(X, T)$.

Now let $(x_1, x_2) \in WM(X, T)$ and assume that T is invertible. Then for any open neighborhood U_i of x_i, $i = 1, 2$, one has $N(U_1, U_1) \cap N(U_1, U_2) \neq \emptyset$. Thus there exists $n \in \mathbb{Z}_+$ such that $U_1 \cap T^{-n}U_1 \neq \emptyset$ and $U_1 \cap T^{-n}U_2 \neq \emptyset$. This implies that there are $x_1' \in U_1$ and $x_2' \in U_2$ with $T^{-n}x_1', T^{-n}x_2' \in U_1$. Thus, we have $(x_1, x_2) \in RP(X, T^{-1}) \setminus \Delta$. □

We make the following remarks.

(1) Generally $WM(X, T) = RP(X, T) \setminus \Delta$ is not true. Let $X = \{\infty\} \cup \mathbb{Z}$ be the one point compactification of $\mathbb{Z}$. Set $x_\infty = \infty$ and $x_i = i$ for $i \in \mathbb{Z}$. Define $T : X \to X$ such that x_∞ is a fixed point and $T(x_i) = x_{i+1}$, $i \in \mathbb{Z}$. It is easy to see that $RP(X, T) = X \times X$ but $(x_i, x_j) \notin WM(X, T)$ if $i \neq j$.

(2) $WM(X, T)$ is not symmetric generally. Let (X, T) be a transitive system, x_1 be a transitive point and p be a fixed point. For each neighborhood U_1 of p, we have $N(U_1, U_1) = \mathbb{Z}_+$ and, hence, $(p, x_1) \in WM(X, T)$. If, in addition, T is almost equicontinuous [**2**], then $(x_1, p) \notin WM(X, T)$; that is $WM(X, T)$ is not symmetric even for a transitive system.

(3) There are examples where $SE(X, T)$ is a proper subset of $WM(X, T)$ for minimal systems. In fact, in [**13**], Goodman obtained a minimal system whose sequence entropy is zero for any sequence. Thus, $SE(X, T) = \emptyset$. At the same time as the system is a subshift, there is an asymptotic pair (x, y) with $x \neq y$. Thus, $WM(X, T) = RP(X, T) \setminus \Delta \neq \emptyset$ (see Theorem 2.4).

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In §3 and §4, we will give the conditions under which a weak mixing pair is a sequence entropy pair. To prove Theorem 2.1, we need the following proposition.

PROPOSITION 2.3. *Let (X, T) be a dynamical system and $(x_1, x_2) \in X \times X \setminus \Delta$. If for any neighborhoods U_i of x_i $(i = 1, 2)$, there is a sequence $\mathcal{A} = \{t_1, t_2, \dots\}$ of $\mathbb{Z}_+$ such that for any $n \in \mathbb{N}$ and any $s = (s(1), \dots, s(n)) \in \{1, 2\}^n$, $\bigcap_{i=1}^n T^{-t_i} U_{s(i)} \neq \emptyset$, then $(x_1, x_2) \in SE(X, T)$.*

Proof. Let U_i be closed mutually disjoint neighborhood of x_i, $i = 1, 2$. By the assumption, we know that there exists a sequence $0 \leq t_1 < t_2 < t_3 < \cdots$ such that for any $n > 0$ and $s \in \{1, 2\}^n$, we can find $x_s \in \bigcap_{i=1}^n T^{-t_i} U_{s(i)}$.

Let $\mathcal{A} = \{t_1, t_2, t_3, \dots\}$ and let $X_n = \{x_s \mid s \in \{1, 2\}^n\}$. Note that for every $s \in \{1, 2\}^n$, one has $\left|\bigcap_{i=1}^n T^{-t_i} U_{s(i)}^c \cap X_n\right| = 1$. Combining this fact and $|X_n| = 2^n$, one gets $N\left(\bigvee_{i=1}^n T^{-t_i}\mathcal{U}\right) = 2^n$, where $\mathcal{U} = \{U_1^c, U_2^c\}$.

Hence,

$$h_{\mathcal{A}}(T, \mathcal{U}) = \limsup_{n \to +\infty} \frac{1}{n} H\left(\bigvee_{i=1}^n T^{-t_i}\mathcal{U}\right) = \log 2$$

and, thus, $(x_1, x_2) \in SE(X, T)$. □

With this preparation, we can now prove Theorem 2.1.

THEOREM 2.1. *Let (X, T) be a dynamical system. Then the following statements are equivalent:*

(1) *(X, T) is weakly mixing;*
(2) *$WM(X, T) = X \times X \setminus \Delta$;*
(3) *(X, T) is s.u.p.e.*

Proof. By Proposition 2.2, (3) implies (2) and by [**21**] or [**5**] (2) implies (1). It remains to show (1) implies (3).

Now let (X, T) be weakly mixing. Then we have the following fact $(*)$: for non-empty open sets U_1, U_2, V_1 and V_2 of X there are non-empty open sets U_3 and V_3 such that $N(U_1, V_1) \cap N(U_2, V_2) \supset N(U_3, V_3)$ (see [**11**, Proposition II.3]).

Let $(x_1, x_2) \in X \times X \setminus \Delta$ and U_i be an open neighborhood of x_i, $i = 1, 2$. We are going to find $0 < t_1 < t_2 < \cdots$ such that for each $n \in \mathbb{N}$ and $s \in \{1, 2\}^n$, $U_{s(0)} \cap T^{-t_1} U_{s(1)} \cap \cdots \cap T^{-t_{n-1}} U_{s(n-1)} \neq \emptyset$.

By the fact $(*)$ there is $t_1 \in N(U_1, U_1) \cap N(U_1, U_2) \cap N(U_2, U_1) \cap N(U_2, U_2)$. Assume that we have found $0 < t_1 < \cdots < t_l$ with the required property. By the fact $(*)$,

$$\bigcap_{s \in \{1,2\}^{l+1},\, i=1,2} N(U_{s(0)} \cap T^{-t_1} U_{s(1)} \cap \cdots \cap T^{-t_l} U_{s(l)}, U_i) \neq \emptyset.$$

Let t_{l+1} be any number in this intersection with $t_{l+1} > t_l$. Then $t_1, \dots, t_{l+1}$ have the required property.

By Proposition 2.3, $(x_1, x_2) \in SE(X, T)$. Thus (X, T) has s.u.p.e. □

The proof of the following theorem is similar to that of Proposition 6 [**7**] (using Theorem 2.1) and, in §4, we will strengthen it by showing that a scattering system is disjoint from any null minimal system.

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THEOREM 2.2. *Let (X, T) be a weakly mixing system. Then (X, T) is disjoint from any null minimal system.*

For a dynamical system the topological Pinsker factor, i.e. the maximal factor with zero topological entropy, exists [**9**]. Similarly we have (following [**9**]) the following theorem.

THEOREM 2.3. *For a dynamical system (X, T), the smallest closed invariant equivalence relation containing $SE(X, T)$ induces the maximal null factor.*

For a minimal system (of $\mathbb{Z}_+$-actions) the regionally proximal relation is a closed invariant equivalence relation which coincides with the set of complexity pairs up to the diagonal [**8, 16**]. As the notion of weak mixing pairs can be generalized to the group actions on compact Hausdorff spaces, we will discuss its relation with the regionally proximal relation in the general setting. By a *flow* we mean (X, T, ϕ) or (X, T), where X is a compact Hausdorff space, T is a topological group and $\phi : T \times X \to X$ is continuous with $\phi(e, x) = x$ and $\phi(t_1, \phi(t_2, x)) = \phi(t_1 t_2, x)$. For simplicity, denote $\phi(t, x)$ by tx. Let (X, T) and (Y, T) be two flows. A continuous surjective map $\pi : X \to Y$ is a *homomorphism* or a *factor map* if $\pi(tx) = t\pi(x)$ for each $x \in X$ and $t \in T$. Let $\mathcal{U}_X$ be the unique uniformity on X inducing the topology of X, which is the collection of all neighborhoods of the diagonal. Each $\alpha \in \mathcal{U}_X$ is called an *index*. A flow (X, T, ϕ) is *minimal* if $\{\phi(t, x) \mid t \in T\}$ is dense for each $x \in X$, and it is *equicontinuous* if for each $\alpha \in \mathcal{U}_X$ there is $\beta \in \mathcal{U}_X$ such that for each $(x, y) \in \beta$ and $t \in T$, one has $(tx, ty) \in \alpha$.

Let $WM(X, T, \phi)$ be the collection of points $(x_1, x_2) \setminus \Delta$ such that for each neighborhood U_i of x_i, there is a $t \in T$ such that $U_1 \cap t(U_1) \neq \emptyset$ and $U_1 \cap t(U_2) \neq \emptyset$. Let $P(X, T, \phi) = \bigcap_{\alpha \in \mathcal{U}_X} T\alpha$ and $RP(X, T, \phi) = \bigcap_{\alpha \in \mathcal{U}_X} \overline{T\alpha}$.

For $(x_1, x_2) \in RP(X, T, \phi)$ let $L(x_1, x_2)$ be the subset of X such that for each x_3 in the set, each neighborhood U_i of x_i, $i = 1, 2, 3$, there are $x, y \in U_3$ and $t \in T$ with $tx \in U_1$ and $ty \in U_2$. It is easy to see that $L(x_1, x_2)$ is non-empty, closed and invariant. We have the following simple observation.

PROPOSITION 2.4. *Let (X, T) be a flow and $(x_1, x_2) \in RP(X, T, \phi)$. Then $(x_1, x_2) \in WM(X, T, \phi)$ if and only if $x_1 \in L(x_1, x_2)$.*

THEOREM 2.4. *Let (X, T) be a minimal flow. Then*

(1) *$WM(X, T, \phi) = RP(X, T, \phi) \setminus \Delta$ and*

(2) *if $T = \mathbb{Z}$, then $WM(X, T) = WM(X, T, \phi)$ and $RP(X, T) = RP(X, T, \phi)$.*

Proof. (1) It is easy to see that $WM(X, T, \phi) \subset RP(X, T, \phi)$ and it remains to show $WM(X, T, \phi) \supset RP(X, T, \phi) \setminus \Delta$. Let $(x_1, x_2) \in RP(X, T, \phi) \setminus \Delta$. Then there is an $x \in L(x_1, x_2)$. As (X, T) is minimal, we have $X = L(x_1, x_2)$. In particular, $x_1 \in L(x_1, x_2)$. Thus, $(x_1, x_2) \in WM(X, T, \phi)$.

(2) It is clear that $WM(X, T) \subset WM(X, T, \phi)$. Let $(x_1, x_2) \in WM(X, T, \phi)$. Then for each disjoint open neighborhood U_i of x_i, $i = 1, 2$, there is an $n \in \mathbb{Z} \setminus \{0\}$ such that $U_1 \cap T^{-n}(U_1) \neq \emptyset$ and $U_1 \cap T^{-n}(U_2) \neq \emptyset$, i.e. there exists an $(x_1, x_2) \in U_1 \times U_1$ such that $(T^n(x_1), T^n(x_2)) \in U_1 \times U_2$. We claim that there is an $m > 0$ with $U_1 \cap T^{-m}(U_1) \neq \emptyset$ and $U_1 \cap T^{-m}(U_2) \neq \emptyset$. In fact, if $n > 0$, let $m = n$. We now assume that $n < 0$. As the set of minimal points of $T \times T$ is dense in $X \times X$, there is a minimal point $(y_1, y_2) \in U_1 \times U_1$

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such that $(T^n(y_1), T^n(y_2)) \in U_1 \times U_2$. Since (y_1, y_2) is a minimal point, there is an $n_1 > |n|$ such that $(T^{n_1}(y_1), T^{n_1}(y_2)) \in U_1 \times U_1$ and $(T^n(T^{n_1}y_1), T^n(T^{n_1}y_2)) \in U_1 \times U_2$. Let $m = n_1 + n$. Then m satisfies the required property. To sum up we have proved $WM(X, T) \supset WM(X, T, \phi)$ and, hence, $WM(X, T) = WM(X, T, \phi)$.

The other equality can be proved in the same way. □

We now show a simple and useful lemma.

LEMMA 2.1. *Let (X, T) be a flow. Then (X, T) is not equicontinuous if and only if there are $x \in X$, $x_\beta \to x$ and $t_\beta \in T$ such that $t_\beta x \to x'$, $t_\beta x_\beta \to x''$, $x' \neq x''$ and $(x', x'') \in RP(X, T, \phi)$.*

Proof. ($\Rightarrow$) Assume that (X, T) is not equicontinuous. Then there is an index α such that for each index β there is a $t \in T$ with $t\beta \not\subset \alpha$; that is, for each β, there are $(z_\beta, y_\beta) \in \beta$, $t_\beta \in T$ with $(t_\beta z_\beta, t_\beta y_\beta) \notin \alpha$. Let α_1 be an index with $\alpha_1^2 \subset \alpha$. Assume that $\lim z_\beta = x = \lim y_\beta$ (if necessary pass to a subnet). Then we have $(t_\beta x, t_\beta x_\beta) \notin \alpha_1$, where $x_\beta = z_\beta$ or y_β. Passing to a subnet if necessary, we assume that $t_\beta x \to x'$ and $t_\beta x_\beta \to x''$. Then it is clear that $x' \neq x''$, $(x', x'') \in RP(X, T, \phi)$.

($\Leftarrow$) It is sufficient to prove that if (X, T) is equicontinuous, then $RP(X, T, \phi) = \Delta$. Suppose (X, T) is equicontinuous. Then for each $\alpha_0 \in \mathcal{U}_X$, there is a $\beta_0 \in \mathcal{U}_X$ such that $T\beta_0 \subset \alpha_0$. Hence, $RP(X, T, \phi) = \bigcap_{\alpha \in \mathcal{U}_X} \overline{T\alpha} \subset \overline{T\beta_0} \subset \overline{\alpha_0}$. As α_0 is arbitrary, $RP(X, T, \phi) = \Delta$. □

Generally speaking, a pair in the regionally proximal relation cannot be lifted by factor maps. That is, if $\pi : (X, T, \phi_X) \to (Y, T, \phi_Y)$ is a factor map and $(y_1, y_2) \in RP(Y, T, \phi_Y)$, sometimes we cannot find an $(x_1, x_2) \in RP(X, T, \phi_X)$ such that $\pi(x_i) = y_i$, $i = 1, 2$. But the special pair which we have in Lemma 2.1 does have a lifting property. This allows us to give simple proofs of some known theorems.

COROLLARY 2.1.

(1) *The factor of an equicontinuous flow is equicontinuous.*

(2) *Let (X, T) be a flow. Then the factor (Y, T) of (X, T) induced by the smallest invariant equivalence relation containing $RP(X, T, \phi)$ is the maximal equicontinuous factor.*

Proof. (1) Let $\pi : (X, T) \to (Y, T)$ be a factor map with (X, T) equicontinuous. Assume that (Y, T) is not equicontinuous. By Lemma 2.1, there are $y \in Y$, $y_\beta \to y$ and $t_\beta \in T$ such that $t_\beta y \to y'$, $t_\beta y_\beta \to y''$ and $y' \neq y''$. Take $x_\beta \in X$ with $\pi(x_\beta) = y_\beta$. Suppose that $x_\beta \to x$, $t_\beta x \to x'$ and $t_\beta x_\beta \to x''$ (if necessary pass to a subnet). It is clear that $\pi(x) = y$, $\pi(x') = y'$ and $\pi(x'') = y''$. Moreover, $x' \neq x''$. This implies that (X, T) is not equicontinuous, a contradiction.

(2) We only need to show that (Y, T) is equicontinuous. Let $\pi : (X, T) \to (Y, T)$ be the factor map. Assume that (Y, T) is not equicontinuous. Then following the argument in (1), we know that there are $y \in Y$, $y_\beta \to y$ and $t_\beta \in T$ such that $t_\beta y \to y'$, $t_\beta y_\beta \to y''$ and $y' \neq y''$. Moreover, there are $x \in X$, $x_\beta \to x$ and $t_\beta \in T$ such that $t_\beta x \to x'$, $t_\beta x_\beta \to x''$ with $\pi(x) = y$, $\pi(x') = y'$ and $\pi(x'') = y''$. This implies that $x' \neq x''$.

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It is clear that $(x', x'') \in RP(X, T, \phi)$. By the definition of π, we have $\pi(x') = \pi(x'')$, i.e. $y' = y''$, a contradiction. □

With the help of Lemma 2.1, we can prove the next theorem.

THEOREM 2.5. *Let (X, T) be a flow.*

(1) *$WM(X, T, \phi) = \emptyset$ if and only if (X, T) is equicontinuous.*

(2) *The factor (Y, T) of (X, T) induced by the smallest invariant equivalence relation containing $WM(X, T, \phi)$ is equicontinuous.*

Proof (1) It remains to show that if $WM(X, T, \phi) = \emptyset$ then (X, T) is equicontinuous. Assume that (X, T) is not equicontinuous. By Lemma 2.1, there are $x \in X$, $x_\beta \to x$ and $t_\beta \in T$ such that $t_\beta x \to x'$, $t_\beta x_\beta \to x''$, $x' \neq x''$ and $(x', x'') \in RP(X, T, \phi)$. It is easy to see that $x \in L(x', x'')$. As x' is in the orbit closure of x, we have $x' \in L(x', x'')$. This implies that $(x', x'') \in WM(X, T, \phi)$, a contradiction. This ends the proof of (1).

(2) Let $\pi : X \to Y$ be the factor map. Assume that (Y, T) is not equicontinuous. Then, by Lemma 2.1, there are y, $y_\beta \to y$ and t_β such that $t_\beta y \to y'$ and $t_\beta y_\beta \to y''$ with $y' \neq y''$. Take $x_\beta \in X$ with $\pi(x_\beta) = y_\beta$. Assume that $\lim x_\beta = x$, $\lim t_\beta x = x'$ and $\lim t_\beta x_\beta = x''$. Then $\pi(x) = y$, $\pi(x') = \lim t_\beta y = y'$ and $\pi(x'') = \lim t_\beta y_\beta = y''$. It is clear that $x' \neq x''$. By the argument in (1), we have $x' \in L(x', x'')$. This implies that $(x', x'') \in MW(X, T, \phi)$. By the definition of π, we have $y' = \pi(x') = \pi(x'') = y''$, a contradiction. Thus, (Y, T) is equicontinuous. □

Remark. In [**4**], the author gave a proof of the fact that the smallest closed invariant equivalence relation containing $RP(X, T, \phi)$ induces the maximal equicontinuous factor using a characterization of equicontinuous systems, namely (X, T) is equicontinuous if and only if each element in the Ellis semi-group is a homeomorphism. Corollary 2.1 shows that this can be done directly without using the Ellis semi-group theorem.

3. *Transitive non-minimal cases*

Generally speaking, it is difficult to check whether a pair (x, y) is in $SE(X, T)$ but it is relatively easy to check whether it is in $WM(X, T)$. In §3 and §4, we will give the conditions under which a weak mixing pair is a sequence entropy pair.

Call a continuous map $f : X \to Y$ between two topological spaces X and Y *semi-open* if the image of every non-empty open set of X has a non-empty interior. We have the following very useful lemma.

LEMMA 3.1. *Let (X, T) be a dynamical system, $(x_1, x_2) \in X \times X \setminus \Delta$ and $A = \overline{\text{orb}((x_1, x_2), T \times T)}$. If $\pi_1 : A \to X$ is semi-open, then $(x_1, x_2) \in SE(X, T)$ if and only if $(x_1, x_2) \in WM(X, T)$, where π_1 is the projection to the first coordinate.*

Proof. By Proposition 2.2, $SE(X, T) \subset WM(X, T)$. Now suppose $(x_1, x_2) \in WM(X, T)$ and $\pi_1 : A \to X$ is semi-open. We are going to show that $(x_1, x_2) \in SE(X, T)$. By Propositions 2.3 it remains to prove for any neighborhoods U_1, U_2 of x_1, x_2 in X, there exists a sequence $0 \leq t_1 < t_2 < t_3 < \cdots$ such that for any $l > 0$ and $s \in \{1, 2\}^l$, $\bigcap_{i=1}^{l} T^{-t_i} U_{s(i)} \neq \emptyset$. The sequence is constructed by induction and, in fact, we have the following claim.

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CLAIM. *For any neighborhoods U_1, U_2 of x_1, x_2 in X, there exists a sequence $0 \le t_1 < t_2 < t_3 < \cdots$ such that for any $l > 0$ and $s \in \{1,2\}^l$, we can find $M_s \in \mathbb{N}$ with $T^{M_s}(x_1) \in \bigcap_{i=1}^{l} T^{-t_i} U_{s(i)}$ and $T^{M_s}(x_j) \in U_j$, $j = 1, 2$.*

Proof of claim. Fix open neighborhoods U_1, U_2 of x_1, x_2 in X. Since $\pi_1 : A \to X$ is semi-open, $W_1 = \mathrm{int}(\pi_1((U_1 \times U_2) \cap A))$ is a non-empty open set of X and $W_1 \subset U_1$. Let $W_2 = U_2$.

As $(W_1 \times W_2) \cap A$ is a non-empty open set of A and the orbit of (x_1, x_2) is dense in A, there exists an $n \in \mathbb{N}$ such that $(T \times T)^n(x_1, x_2) \in (W_1 \times W_2) \cap A$. Since $(x_1, x_2) \in WM(X, T)$, it follows that $(T^n(x_1), T^n(x_2)) \in WM(X, T)$. Therefore, $N(W_1, W_1) \cap N(W_1, W_2) \neq \emptyset$. So there exists a $t_1 \ge 0$ such that $W_1 \cap T^{-t_1} W_1 \neq \emptyset$ and $W_1 \cap T^{-t_1} W_2 \neq \emptyset$. Since the orbit of (x_1, x_2) is dense in A, there exist $M_1, M_2 \in \mathbb{N}$ such that $(T^{M_i}(x_1), T^{M_i}(x_2)) \in ((W_1 \cap T^{-t_1} W_i) \times U_2) \cap A$, $i = 1, 2$. Thus $T^{M_i}(x_1) \in U_1 \cap T^{-t_1} U_i$ and $T^{M_i}(x_j) \in U_j$ for $i, j = 1, 2$.

Now suppose $0 \le t_1 < t_2 < \cdots < t_l$, $l \ge 1$ have been defined and satisfy that, for any $s \in \{1,2\}^l$, there exists an $M_s \in \mathbb{N}$ such that $T^{M_s}(x_1) \in \bigcap_{i=1}^{l} T^{-t_i} U_{s(i)}$ and $T^{M_s}(x_j) \in U_j$, $j = 1, 2$.

We are going to define t_{l+1}. Take $\delta > 0$ such that for any $z_1, z_2 \in X$ with $d(z_1, x_1) < \delta$ and $d(z_2, x_2) < \delta$, one has $T^{M_s}(z_1) \in \bigcap_{i=1}^{l} T^{-t_i} U_{s(i)}$ and $T^{M_s}(z_j) \in U_j$, $j = 1, 2$ for any $s \in \{1,2\}^l$. Let $U_1^\delta = \{z \in X \mid d(z, x_1) < \delta\}$, $U_2^\delta = \{z \in X \mid d(z, x_2) < \delta\}$.

Since $\pi_1 : A \to X$ is semi-open, $W_1^\delta = \mathrm{int}(\pi_1((U_1^\delta \times U_2^\delta) \cap A))$ is a non-empty open set and $W_1^\delta \subset U_1^\delta$. Let $W_2^\delta = U_2^\delta$. Using the same argument as before we have $N(W_1^\delta, W_1^\delta) \cap N(W_1^\delta, W_2^\delta) \neq \emptyset$. Without loss of generality (letting δ be as small as necessary), we may assume that $N(W_1^\delta, W_1^\delta) \cap N(W_1^\delta, W_2^\delta) \subset \{t_1+1, t_1+2, \dots\}$. So there exists a $t_{l+1} > t_l$ such that $W_1^\delta \cap T^{-t_{l+1}} W_i^\delta \neq \emptyset$, $i = 1, 2$. We can choose $P_1, P_2 \in \mathbb{N}$ such that $T^{P_i}(x_1) \in W_1^\delta \cap T^{-t_{l+1}} W_i^\delta$ and $T^{P_i}(x_j) \in U_j^\delta$ for $i, j = 1, 2$.

For any $r \in \{1,2\}^{l+1}$, let $s \in \{1,2\}^l$ with $s(i) = r(i)$, $i = 1, 2, \dots, l$ and let $r(l+1) = k$. Put $M_r = M_s + P_k$. Then

$$T^{M_r}(x_1) = T^{M_s}(T^{P_k}(x_1)) \in \bigcap_{i=1}^{l} T^{-t_i} U_{s(i)}.$$

As $T^{t_{l+1}} T^{P_k}(x_1) \in W_k^\delta$, we have $T^{t_{l+1}} T^{M_r}(x_1) = T^{M_s} T^{t_{l+1}} T^{P_k}(x_1) \in U_k$. So

$$T^{M_r}(x_1) \in \left(\bigcap_{i=1}^{l} T^{-t_i} U_{s(i)} \right) \cap (T^{-t_{l+1}} U_k) = \bigcap_{i=1}^{l+1} T^{-t_i} U_{r(i)}.$$

Moreover, $T^{M_r}(x_j) = T^{M_s} T^{P_k}(x_j) \in U_j$. This ends the proof of the claim. □

□

Applying Lemma 3.1 we have the following corollary.

COROLLARY 3.1. *Let (X, T) be a transitive dynamical system, x_1 a transitive point of T and p a fixed point of T. Then $(x_1, p) \in SE(X, T)$ if and only if $(x_1, p) \in WM(X, T)$.*

Proof. Set $A = \overline{\mathrm{orb}(x_1, p)}$. Then $\pi_1 : A \to X$ is the identity and, hence, is semi-open. Applying Lemma 3.1 we get the corollary. □

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Let S be a subset of $\mathbb{Z}_+$. The *upper Banach density* of S is

$$BD^\star(S) = \limsup_{|I|\to+\infty} \frac{|S \cap I|}{|I|}$$

where I ranges over intervals of $\mathbb{Z}_+$. We say the *lower Banach density* of S is one if, for each $a < 1$, there is an N such that for any subinterval I of $\mathbb{Z}_+$ with $|I| \geq N$, we have $|S \cap I| \geq a|I|$. Let (X, T) be a dynamical system. We shall say that (X, T) is *Banach-transitive* if for any non-empty open subsets U, V of X, $N(U, V)$ has a positive upper Banach density. For $x \in X$ and $U \subset X$, let $N(x, U) = \{n \in \mathbb{Z}_+ : T^n(x) \in U\}$. Note that a dynamical system is an E-system if it is transitive and there is an invariant measure with full support. It is known [**14**] that if (X, T) is an E-system, then $N(U, V)$ is syndetic for any non-empty open subsets U, V of X and, thus, is Banach-transitive. Note that an infinite subset A of $\mathbb{Z}_+$ is *syndetic* if it has bounded gaps and it is *thick* if it contains arbitrary long intervals of integers. It is easy to see that the intersection of a syndetic set and a thick set is not empty. Now we can prove Theorem 3.1.

THEOREM 3.1. *Let (X, T) be a dynamical system. If (X, T) is non-minimal and Banach-transitive, then $SE(X, T) \neq \emptyset$.*

Proof. Without loss of generality, we assume that (X, T) has a fixed point p (using Proposition 2.1). Let x_1 be a transitive point and U_1 and U_2 be open neighborhoods of x_1 and p, respectively. There are two cases.

Case 1. (X, T) is an E-system. As (X, T) is an E-system, $N(U_1, U_1)$ is syndetic [**14**]. As p is a fixed point, $N(x_1, U_2)$ is thick. Thus, $N(U_1, U_2) = N(x_1, U_2) - N(x_1, U_1)$ is a thick set. Combining these facts, one has $N(U_1, U_1) \cap N(U_1, U_2) \neq \emptyset$. By Corollary 3.1, we know that $(x_1, p) \in SE(X, T)$.

Case 2. (X, T) is not an E-system. Without loss of generality, we assume that (X, T) is uniquely ergodic and the unique ergodic measure is δ_p (collapsing the closure of the union of the supports of all invariant measures and using Proposition 2.1). We have the following claim.

CLAIM. *The lower Banach density of $N(x_1, U_2)$ is 1.*

Proof of claim. If this is not the case, then $N(x_1, U_2^c)$ has a positive upper Banach density. Thus, there exist $k \in \mathbb{N}$ and $a_k < b_k$ such that

$$\lim_{k\to+\infty} \frac{1}{b_k - a_k} |N(x_1, U_2^c) \cap \{a_k, a_k + 1, \ldots, b_k - 1\}| > 0 \quad \text{and} \quad \lim_{k\to+\infty} (b_k - a_k) = +\infty.$$

Now, set

$$\mu_k = \frac{1}{b_k - a_k} \sum_{i=a_k}^{b_k-1} \delta_{T^i x_1}.$$

Let $\mu = \lim_{i\to+\infty} \mu_{k_i}$ be a limit point of $\{\mu_k\}_{k=1}^{+\infty}$ in the weak*-topology. Clearly, μ is an

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invariant measure of (X, T) and

$$\mu(U_2^c) \geq \lim_{i\to+\infty} \mu_{k_i}(U_2^c) = \lim_{i\to+\infty} \frac{1}{b_{k_i} - a_{k_i}} \sum_{i=a_{k_i}}^{b_{k_i}-1} \delta_{T^i x_1}(U_2^c)$$
$$= \lim_{i\to+\infty} \frac{|N(x_1, U_2^c) \cap \{a_{k_i}, a_{k_i}+1, \ldots, b_{k_i}-1\}|}{b_{k_i} - a_{k_i}} > 0.$$

Thus $\mu \neq \delta_p$ which contradicts the unique ergodicity of (X, T). This finishes the proof of the claim. □

As (X, T) is Banach-transitive, $N(U_1, U_1)$ has a positive upper Banach density. By the claim, the lower Banach density of $N(x_1, U_2)$ is 1. This implies that the lower Banach density of $N(U_1, U_2) = N(x_1, U_2) - N(x_1, U_1)$ is 1. Thus, $N(U_1, U_1) \cap N(U_1, U_2) \neq \emptyset$. By Corollary 3.1, we know that $(x_1, p) \in SE(X, T)$. □

We remark that though we could not supply a transitive non-minimal null system, we conjecture that such a system exists.

A dynamical system (X, T) is *topologically ergodic* if for each pair of non-empty open sets U and V, $N(U, V)$ is syndetic and a dynamical system is *extremely scattering*, if its product with any topologically ergodic system is transitive. It is proved in [**17**] that an extremely scattering system is Banach-transitive. Thus we have the following corollary.

COROLLARY 3.2. *If a dynamical system is non-trivial and extremely scattering, then $SE(X, T) \neq \emptyset$ and, thus, the maximal null factor is trivial.*

Proof. If (X, T) is minimal, then it is weakly mixing and $SE(X, T) \neq \emptyset$ by Theorem 2.1. If it is not minimal, then $SE(X, T) \neq \emptyset$ by Theorem 3.1. □

4. *Minimal cases*

In this section we consider $\mathbb{Z}$-actions. To get the results for surjective systems (X, T), we need to consider the natural extension $(\tilde{X}, \tilde{T})$ of (X, T). As the minimality of (X, T) implies the minimality of $(\tilde{X}, \tilde{T})$, for any sequence $\mathcal{A}$, $h_{\mathcal{A}}(T) = h_{\mathcal{A}}(\tilde{T})$ [**13**] and the projection to the first coordinate is almost one to one [**19**], Theorems 4.3, 4.5 and Corollary 4.3 are also valid for surjective systems.

First let us give some basic definitions in topological dynamical systems. Let (X, d) be a compact metric space and $T : X \to X$ a homeomorphism. (X, T) is *two-sided transitive* if there is an $x \in X$ such that $\{T^n(x) \mid n \in \mathbb{Z}\}$ is dense in X. It is easy to see that if x is a two-sided recurrent point (i.e. there are $n_i \in \mathbb{Z}$, $|n_i| \to +\infty$, with $T^{n_i}(x) \to x$), then two-sided transitivity implies transitivity. Two-sided minimality is defined in the same way; and it is equivalent to minimality. Two points x_1, x_2 of X are called *proximal* if there exists a sequence $n_i \to \infty$ ($n_i \in \mathbb{Z}$) such that $d(T^{n_i}(x_1), T^{n_i}(x_2)) \to 0$. x_1, x_2 are called *distal* if they are not proximal.

Recall that a continuous map $\pi : X \to Y$ is called a homomorphism of systems (X, T) and (Y, S) if it is onto and $\pi T = S\pi$. We say (X, T) is an *extension* of (Y, S). If π is also injective, then it is called an *isomorphism*. An extension $\pi : X \to Y$ is called *proximal* (respectively *distal*) if $\pi(x_1) = \pi(x_2)$ implies that x_1 and x_2 are proximal

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(respectively distal). It is called *equicontinuous* if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\pi(x_1) = \pi(x_2)$ and $d(x_1, x_2) < \delta$ imply $d(T^n(x_1), T^n(x_2)) < \epsilon$ for any $n \in \mathbb{Z}$. An equicontinuous extension is also called an *isometric extension*. The extension is *highly proximal* if for every non-empty open set U of X and every point y of Y, there exists an $n \in \mathbb{Z}$ such that $T^n(\pi^{-1}(y)) \subset U$. If $\pi : X \to Y$ is highly proximal then X, Y are necessarily minimal and π is proximal. It is well known that when X and Y are metric spaces, π is highly proximal if and only if it is almost one to one, i.e. there exists a dense G_δ set $X_0 \subset X$ such that $\pi^{-1}\pi(x) = \{x\}$ for any $x \in X_0$. Finally, the extension is *weak mixing* if the subsystem $R_\pi = \{(x_1, x_2) \mid \pi(x_1) = \pi(x_2)\}$ is two-sided transitive under $T \times T$.

The following theorem is the structure theorem for minimal systems [**4**, Theorem 14.30].

THEOREM 4.1. *Given a compact metric minimal system (X, T), there exist a countable ordinal η and a canonically defined commutative diagram of minimal systems (a PI tower):*

$$\begin{array}{ccccccccccccc}
X=X_0 & \xleftarrow{id} & X_0 & \xleftarrow{\phi_1} & X_1 & \longleftarrow & \cdots X_\nu & \xleftarrow{id} & X_\nu & \xleftarrow{\phi_{\nu+1}} & X_{\nu+1} & \longleftarrow & \cdots X_\eta=X_\infty \\
\downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_\nu} & & \downarrow{\scriptstyle \sigma_{\nu+1}} & & \downarrow{\scriptstyle \pi_{\nu+1}} & & \downarrow{\scriptstyle \pi_\infty} \\
\{pt\}=Y_0 & \xleftarrow[\rho_1]{} & Z_1 & \xleftarrow[\psi_1]{} & Y_1 & \longleftarrow & \cdots Y_\nu & \xleftarrow[\rho_{\nu+1}]{} & Z_{\nu+1} & \xleftarrow[\psi_{\nu+1}]{} & Y_{\nu+1} & \longleftarrow & \cdots Y_\eta=Y_\infty
\end{array}$$

where for each $\nu \leq \eta$, ρ_ν is equicontinuous, ϕ_ν and ψ_ν are proximal and π_∞ is weakly mixing. For a limit ordinal ν, X_ν, Y_ν, π_ν etc. are the inverse limits of $X_\lambda, Y_\lambda, \pi_\lambda$, etc. for $\lambda < \nu$. $\{pt\}$ denotes the trivial system.

We say that an extension $\phi : X \to Y$ of a minimal system is a *strictly PI extension* if there is a countable ordinal η and a family of minimal systems X_λ $(0 \leq \lambda \leq \eta)$ such that

(i) $X_0 = Y$ and $X_\eta = X$;
(ii) for every $\lambda < \eta$, there exists an extension $\phi_\lambda : X_{\lambda+1} \to X_\lambda$ which is either proximal or isometric;
(iii) for a limit ordinal $\nu \leq \eta$ the system X_ν is the inverse limit of the systems X_λ $(\lambda < \nu)$; and
(iv) ϕ is the transfinite composition of extensions $\phi_\lambda (\lambda < \eta)$.

An extension $\phi : X \to Y$ of minimal systems is called a *PI extension* if there exists a proximal extension $\psi : Z \to X$ of a minimal system such that $\phi \circ \psi : Z \to Y$ is a strictly PI extension. If Y is the trivial system and $\phi : X \to Y$ is a (strictly) PI-extension, then X is called a *(strictly) PI system.*

If in these definitions proximal is replaced by highly proximal, then we get the notions of *strictly HPI extensions (systems)* and *HPI extensions (systems).*

The following lemma, belonging to Bronstein, can be found in [**4**, Theorem 14.31].

LEMMA 4.1. *Let (X, T) be a minimal dynamical system. Then X is PI if and only if it satisfies the following property: whenever W is a closed invariant subset of $X \times X$ which is two-sided transitive and has a dense subset of minimal points, then W is minimal.*

Recall that the definitions $WM(X, T)$ and $RP(X, T)$ for $\mathbb{Z}_+$- and $\mathbb{Z}$-actions are the same under the minimality assumption (Theorem 2.4).

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THEOREM 4.2. *Any minimal null system (X, T) is PI.*

Proof. If (X, T) is not PI, then there exists a closed invariant subset W of $X \times X$ which is a non-minimal transitive system and which has a dense subset of minimal points by Lemma 4.1 (in this case, two-sided transitivity implies transitivity). Since $(W, T \times T)$ contains a dense set of minimal points, it is Banach-transitive. By Theorem 3.1, we know that $SE(W, T \times T) \neq \emptyset$; say $((x_1, y_1), (x_2, y_2)) \in SE(W, T \times T)$. As $(x_1, y_1) \neq (x_2, y_2)$, we have that $(x_1, x_2) \in SE(X, T)$ or $(y_1, y_2) \in SE(X, T)$ (Proposition 2.1(3)(b)). This contradicts the assumption $SE(X, T) = \emptyset$. □

In order to get further properties of null minimal systems, we need to consider the relationships between sequence entropy pairs and isometric extension. First we give a result concerning the conditions under which a weak mixing pair is a sequence entropy pair, which is a corollary of Lemma 3.1.

COROLLARY 4.1. *Let $\pi : (X, T) \to (Y, T)$ be a distal extension of minimal dynamical systems and let $x_1, x_2 \in X$ with $x_1 \neq x_2$ and $\pi(x_1) = \pi(x_2)$. Then $(x_1, x_2) \in SE(X, T)$ if and only if $(x_1, x_2) \in WM(X, T)$.*

Proof. Since π is distal, (x_1, x_2) is a minimal point of $(X \times X, T \times T)$ [**4**, Theorem 5.6]. Therefore, $A = \overline{\text{orb}((x_1, x_2), T \times T)}$ is a minimal set. Since a homomorphism between minimal systems is semi-open [**4**, Theorem 1.15], $\pi_1 : A \to X$ is semi-open and the corollary follows from Lemma 3.1. □

The following is a corollary of Corollary 4.1.

COROLLARY 4.2. *If (X, T) is a minimal distal system, then $SE(X, T) = RP(X, T) \setminus \Delta$.*

Proof. As (X, T) is distal, $\pi : X \to Y$ is a distal extension, where (Y, S) is the trivial system. Thus, $SE(X, T) = WM(X, T) = RP(X, T) \setminus \Delta$, following Corollary 4.1 and Theorem 2.4. □

To prove our main theorem we need some knowledge of the algebraic theory of minimal dynamical systems (for details see [**4**, ch. 10]). Let M be the universal minimal system for $\mathbb{Z}$-actions (M can be taken as a minimal left ideal of the Stone–Čech compactification $\beta\mathbb{Z}$ of the discrete group $\mathbb{Z}$). M has a semi-group structure inherited from $\beta\mathbb{Z}$ and it has an action (it may not be continuous) on every minimal system. Let u be an idempotent (i.e. $u^2 = u$) of M and $G = uM$. Then G is a group. For any minimal dynamical system (X, T), choose $x \in X$ such that $ux = x$. The pointed system (X, x) is called a *minimal ambit*. The Ellis group of an ambit (X, x) is defined to be $G(X, x) = \{g \in G \mid gx = x\}$. The following lemma can be found in [**4**, Theorem 10.1].

LEMMA 4.2.

(1) *Let $\pi : (X, x) \to (Y, y)$ be a homomorphism of minimal ambits. Then $G(X, x) = G(Y, y)$ if and only if π is a proximal extension.*
(2) *Let (X, x) and (Y, y) be minimal ambits with Y distal. Then $G(X, x) \subset G(Y, y)$ if and only if there is a homomorphism $\pi : X \to Y$ with $\pi(x) = y$.*

Now we are ready to prove our main theorem.

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THEOREM 4.3. *Let (X, T) be a null minimal system. Then (X, T) is an almost one-to-one extension of an equicontinuous system.*

Proof. We consider the PI tower of (X, T) as constructed in Theorem 4.1. As (X, T) is a null minimal system, Theorem 4.2 implies (X, T) is a PI system and, therefore, π_∞ in the PI tower is an isomorphism [**4**, Theorem 14.23].

We divide the proof into the following steps.

Step 1. Z_1 is the maximal equicontinuous factor of X_∞. From the construction of the PI tower we know that Z_1 is the maximal equicontinuous factor of X. As the composition of proximal extensions is proximal, by Lemma 4.2, X_∞ and X have the same Ellis group. Suppose Z is the maximal equicontinuous factor of X_∞. By Lemma 4.2 again, Z is also a factor of X and, therefore, $Z = Z_1$.

Step 2. The isometric extensions $\rho_\nu (\nu \geq 2)$ are isomorphisms. Suppose $\rho_{\nu+1}$ is not isomorphism for some $\nu \geq 1$. Then there exist $z_1 \neq z_2 \in Z_{\nu+1}$ such that $\rho_{\nu+1}(z_1) = \rho_{\nu+1}(z_2)$. The maximal equicontinuous factor of $Z_{\nu+1}$ is also Z_1 as $Z_{\nu+1}$ is a factor of X_∞. It is well known that the regionally proximal relation of a minimal system is a closed invariant equivalent relation. So $(z_1, z_2) \in RP(Z_{\nu+1})$ as z_1 and z_2 have the same image in Z_1. Therefore, $(z_1, z_2) \in WM(Z_{\nu+1})$ as $WM(Z_{\nu+1}) = RP(Z_{\nu+1}) \setminus \Delta$. Then, by Corollary 4.1 and the fact that an isometric extension is always distal, we have $(z_1, z_2) \in SE(Z_{\nu+1})$. Let (x_1', x_2') be a preimage of (z_1, z_2) in $X_\infty \times X_\infty$ such that $(z_1, z_2) \in SE(X_\infty)$ (Proposition 2.1). Clearly x_1', x_2' are distal as z_1, z_2 are distal. Therefore, the images x_1, x_2 of x_1', x_2' in X are different and $(x_1, x_2) \in SE(X)$. This contradicts the assumption $SE(X) = \emptyset$.

Step 3. X is a proximal extension of Z_1. As all ρ_ν $(\nu \geq 2)$ are isomorphism, X_∞ is a proximal extension of both Z_1 and X. Therefore, the systems X, X_∞ and Z_1 have the same Ellis group. By Lemma 4.2, $\sigma_1 : X \to Z_1$ is a proximal extension.

Step 4. $\sigma_1 : X \to Z_1$ is an HPI extension. If $\sigma_1 : X \to Z_1$ is not an HPI extension then by Theorem 4.8 of Woude [**23**] there exists a closed invariant subset $A \subset R_{\sigma_1} = \{(x, y) \in X \times X \mid \sigma_1(x) = \sigma_1(y)\}$ such that $(A, T \times T)$ is a two-sided transitive non-minimal system and the projections $\pi_i : A \to X$ are semi-open for $i = 1, 2$. It is clear that, in this case, $(A, T \times T)$ is transitive. Thus, there exists $(x_1, x_2) \in A$, $x_1 \neq x_2$ and (x_1, x_2) is a transitive point. Since σ_1 is a proximal extension of a equicontinuous flow, we have

$$R_{\sigma_1} \subset Prox(X, T) \subset RP(X, T) = WM(X, T) \cup \Delta.$$

Therefore, (x_1, x_2) is a weak mixing pair and, by Lemma 3.1, it is a sequence entropy pair. This contradicts the assumption that X is a null system.

Step 5. $\sigma_1 : X \to Z_1$ is an almost one-to-one extension. As $\sigma_1 : X \to Z_1$ is an HPI extension and Z_1 is equicontinuous, we know X is an HPI-system. As we do for the PI-tower, we can construct a canonical HPI tower for every minimal system. As X is an HPI system, the canonical HPI tower of X is the same as its PI-tower and all the extensions ϕ_ν, ψ_ν $(1 \leq \nu \leq \eta)$ are highly proximal (see [**24**, Remark VI.4.21(3)]). In particular, π_∞ is an isomorphism. Thus steps 1 and 2 can be applied in this case and since the highly proximal extension is kept under transfinite composition, we have that X_∞ is a highly proximal extension of both X and Z_1. As all the spaces are assumed to be metric, these

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two extensions are almost one to one and, therefore, $\sigma_1 : X \to Z_1$ is almost one to one. This finishes the proof. □

In the following we show that in some other cases, a proximal extension is almost one to one. To do this, we need the following lemma.

LEMMA 4.3. *Let $\pi : X \to Y$ be a proximal extension of minimal systems, $W_n = \{(x_1, \ldots, x_n) \mid \pi(x_1) = \cdots = \pi(x_n), x_i \in X\}$ for $n \in \mathbb{N}$. Then,*
(1) *W_n has only one minimal set $\Delta_n = \{(x, \ldots, x) \mid x \in X\}$; and*
(2) *if there is a $y \in Y$ such that $\pi^{-1}(y)$ is finite, then π is almost one to one.*

Proof. (1) It is easy to see that W_2 contains only one minimal set Δ_2. Assume that W_i contains only one minimal set Δ_i for $1 \leq i \leq n$. Let $(x_1, \ldots, x_{n+1}) \in W_{n+1}$ be a minimal point. Then (x_i, x_j) are proximal and, thus, $(x_i, x_j) \in W_2$. As (x_i, x_j) is a minimal point, we have $x_i = x_j$. Thus, W_{n+1} contains only one minimal set Δ_{n+1}.

(2) Suppose that $\pi^{-1}(y) = \{x_1, x_2, \ldots, x_n\}$. By (1) W_n contains only one minimal set. We have that, for each $\epsilon > 0$, there are $m = m(\epsilon)$ and $x = x(\epsilon)$ such that $d(T^m(x_i), x) < \epsilon$ for each $1 \leq i \leq n$. Thus, π is a highly proximal extension by the definition and, therefore, it is almost one to one. □

We say a dynamical system is *semi-distal* if every recurrent point in $X \times X$ (i.e. points (x, y) with $n_i \in \mathbb{N}$, $n_i \to \infty$ with $(T \times T)^{n_i}(x, y) \to (x, y)$) is minimal [**1**]. Now we have the next corollary.

COROLLARY 4.3. *Let (X, T) be a minimal system and (Y, S) be equicontinuous and $\pi : X \to Y$ be a proximal extension. If (X, T) is semi-distal (in particular if π is asymptotic) or there is a $y \in Y$ such that $\pi^{-1}(y)$ is finite, then π is almost one to one.*

Proof. Without loss of generality we assume that X is not finite. Thus, X is uncountable. First we assume that X is semi-distal.

Recall [**23**] that a system is HPI if and only if each two-sided transitive sub-system $W \subset X \times X$ with each projection to X semi-open is minimal. Now assume that $W \subset X \times X$ is a two-sided transitive subsystem with each projection to X semi-open. It is clear that, in this case, $(W, T \times T)$ is transitive. Let (x_1, x_2) be a transitive point of $(W, T \times T)$. As (X, T) is semi-distal, (x_1, x_2) is a minimal point. Thus, W is minimal. This proves that (X, T) is HPI. The argument in Theorem 4.3 can be applied here and we get that π is almost one to one.

Now we assume that $\pi^{-1}(y)$ is finite for some $y \in Y$. By Lemma 4.3, π is almost one to one. □

In [**13**], the author showed that for a finite dimensional system the restricted variational principle holds for sequence entropy. Though his result cannot be applied in our situation, we still can prove the following theorem.

THEOREM 4.4. *Let (X, T) be a null minimal system. Then (X, T) is uniquely ergodic.*

Proof. By Theorem 4.3 we assume that Y is equicontinuous and $\pi : X \to Y$ is proximal. Let ν be the unique measure on Y.

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Let μ be an invariant measure on X and consider the disintegration of μ over ν. That is, for a.e. $y \in Y$ we have a measure μ_y on X such that $\mathrm{supp}(\mu_y) \subset \pi^{-1}(y)$ (see [**22**] or [**12**]) and $\mu = \int_{y\in Y} \mu_y \, d\nu$. We claim that for a.e. $y \in Y$, μ_y is concentrated on one point.

Proof of the claim. Let $W = \{(x, y) \in X \times X \mid \pi(x) = \pi(y)\}$. As $\mathrm{supp}(\mu_y) \subset \pi^{-1}(y)$, a.e. $y \in Y$ we have $\mathrm{supp}(\mu_y \times \mu_y) \subset \pi^{-1}(y) \times \pi^{-1}(y) \subset W$, a.e. $y \in Y$. Thus,

$$\mu \times_Y \mu(W) = \int_{y\in Y} \mu_y \times \mu_y(W) \, d\nu = 1,$$

where $\mu \times_Y \mu$ is a measure defined on Borel sets of $X \times X$ such that if A is a Borel set of $X \times X$, then

$$\mu \times_Y \mu(A) = \int_Y \mu_y \times \mu_y(A) \, d\nu(y).$$

If $\mu \times_Y \mu(\Delta) \neq 1$, then $\mu \times_Y \mu(W \setminus \Delta) > 0$. Thus, there is an ergodic measure w on $(W, T \times T)$ such that $w(W \setminus \Delta) > 0$. As Δ is the only minimal set of $(W, T \times T)$ (Lemma 4.3(1)), $\mathrm{supp}(w)$ is an E-system which is not minimal. By Theorem 3.1, $SE(\mathrm{supp}(w), T \times T) \neq \emptyset$, a contradiction. Thus, we have $\mu \times_Y \mu(\Delta) = 1$. Since

$$\mu \times_Y \mu(\Delta) = \int_{y\in Y} \mu_y \times \mu_y(\Delta) \, d\nu = 1$$

we have that $\mu_y \times \mu_y(\Delta) = 1$, a.e. $y \in Y$. A simple calculation shows that μ_y is concentrated on one point, say $c(y)$. This ends the proof of the claim. □

We now show (X, T) is uniquely ergodic. Assume that μ^1 and μ^2 are two invariant measures on X. Let $\mu = \frac{1}{2}(\mu^1 + \mu^2)$. By the claim, μ^i_y is concentrated on $c^i(y)$, a.e. $y \in Y$. Thus, μ_y is concentrated on $c^1(y)$ and $c^2(y)$, a.e. $y \in Y$. By the claim, we have $c^1(y) = c^2(y)$, a.e. $y \in Y$. Thus, $\mu^1 = \mu^2$. This proves that (X, T) is uniquely ergodic. □

A dynamical system is *scattering* if it is weakly disjoint from all minimal systems [**8**]. It is known [**3, 17**] that weak mixing is strictly stronger than scattering. Now we show the following theorem.

THEOREM 4.5. *A scattering system is disjoint from any minimal null system.*

Proof. It is a direct consequence of Theorem 4.3 and [**3**, Theorem 3.7] which states that if (X_1, T_1) is an almost one-to-one extension of a distal system, then a transitive system (X, T) is disjoint from (X_1, T_1) if and only if they are weakly disjoint. □

5. *Some properties of null systems*

In this section we give some properties of null systems. Note that the notion of the nullness of a system is valid for any continuous map of a compact Hausdorff space and Proposition 2.1 remains true in this more general situation.

Let (X, T) and (X', T') be two dynamical systems. It is known [**20**] that $h_{\mathcal{A}}(X \times X', T \times T') = h_{\mathcal{A}}(X, T) + h_{\mathcal{A}}(X', T')$ does not hold in general. In contrast to this fact, we have the following proposition.

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PROPOSITION 5.1. *The property of nullness of a dynamical system is stable under factor maps, arbitrary products and inverse limits.*

Proof. This is a simple application of Proposition 2.1. □

In the following we analyze the null minimal systems obtained by Goodman in [**13**] in order to see how far they are from equicontinuous systems. We begin by briefly reviewing the theory of the substitution minimal system on two symbols. For details, see [**10**].

Let Σ_2 be the set of all bisequences $x = \ldots x_{-1}x_0x_1\ldots$ with $x_i \in \{0, 1\}$, $i \in \mathbb{Z}$. A compatible metric is given by $d(x, y) = 1/(1 + k)$, where $k = \min\{|n| \mid x_n \neq y_n\}$, $x, y \in \Sigma_2$. The shift map $\sigma \mid \Sigma_2 \to \Sigma_2$ is defined by $(\sigma x)_n = x_{n+1}$ for all $n \in \mathbb{Z}$. Note that in this section $\mathrm{orb}(x, \sigma)$ denotes the set $\{\sigma^i(x) \mid i \in \mathbb{Z}\}$.

If n is a positive integer, an n-block C will be a finite sequence $C = c_1c_2\ldots c_n$, with $c_i \in \{0, 1\}$. A *substitution of length* $n (n \geq 2)$ is a map θ from $\{0, 1\}$ into the n-blocks, i.e. θ is a pair of n-blocks $\theta(0) = a = a_0a_1\ldots a_{n-1}$, $\theta(1) = b = b_0b_1\ldots b_{n-1}$. Set $H = \{00, 01, 10, 11\}$ and let $\lambda_\theta : H \to H$ be given by $00 \to a_{n-1}a_0, 01 \to a_{n-1}b_0, 10 \to b_{n-1}a_0, 11 \to b_{n-1}b_0$. It is easy to see that the set of the fixed points of λ_θ^2 is not empty. Let pq be a fixed point of λ_θ^2, then there is a unique sequence $w^{pq} \in \Sigma_2$ such that $\theta^2 w^{pq} = w^{pq}$ and $w_{-1}^{pq}w_0^{pq} = pq$. Let $O_\theta = \overline{\mathrm{Orb}(w^{pq}, \sigma)}$ be the orbit closure of w^{pq} in Σ_2. We call (O_θ, σ) a *substitution minimal system*.

θ is called *finite* if any of the following conditions holds:

(1) $a_i = b_i$ for all i;

(2) $a_i = 0$ for all i or $b_i = 1$ for all i;

(3) $a_i = 1$ for all i and $b_i = 0$ for all i; and

(4) n is odd and $a = \bar{b} = 0101\ldots010$ or $1010\ldots101$, where $\bar{b}$ is the mirror image of b, i.e. $\bar{b}$ is obtained from b by replacing all the zeros in b by ones and all the ones by zeros.

θ is called *continuous* if it is not finite and if $a = \bar{b}$. θ is called *discrete* if it is neither finite nor continuous. If θ is finite, then the only minimal sets in O_θ are periodic orbits which is uninteresting as minimal sets. So in this paper we always assume θ is discrete.

By $Z(n), (n \geq 2)$, we denote the additive group of n-adic integers: $Z(n) = \left\{z = \sum_{i=0}^{\infty} z_i n^i \mid z_i = 0, 1, \ldots, n-1\right\}$. A metric in $Z(n)$ is given by $d'(z, z') = (\min\{i \mid z_i \neq z_{i'}\} + 1)^{-1}$, where $z = \sum_{i=0}^{\infty} z_i n^i$, $z' = \sum_{i=0}^{\infty} z_i' n^i \in Z(n)$. $(Z(n), d')$ is a compact metric group. A map $\tau : Z(n) \to Z(n)$ is defined by $\tau(z) = z + 1$. Clearly, τ is an isometric homeomorphism.

A basic fact is that there is a homomorphism $\pi : (O_\theta, \sigma) \to (Z(n), \tau)$ mapping w^{pq} to 0. Set $J_k = \{m \mid 0 \leq m \leq n^k - 1, \theta^k(0)$ and $\theta^k(1)$ disagree at place $m\}$, $J_\infty = \left\{\sum_{i=0}^{\infty} z_i n^i \mid z_i \in J_1 \text{ for all } i\right\}$, $E = \cup\{\tau^k J_\infty \mid k \in \mathbb{Z}\}$, $Z^\star = Z(n) - E$. Then we have $Z^\star = \{z \in Z(n) \mid \pi^{-1}(z)$ is a single point$\}$. If $z \in E$, then $\pi^{-1}(z)$ consists of exactly two points unless $z \in \mathrm{orb}(0, \tau)$ and a and b disagree at both endpoints. In this case, $\pi^{-1}(z)$ consists of four points (see [**10**]).

LEMMA 5.1. *Let (O_θ, σ) be a substitution minimal flow, where θ is a substitution of constant length n $(n \geq 3)$ on the symbols 0 and 1. Let $\theta(0) = a_0a_1\ldots a_{n-2}a_{n-1}$ and*

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$\theta(1) = a_0a_1 \dots a_{n-2}a'_{n-1}$ with $a_{n-1} \neq a'_{n-1}$, then (O_θ, σ) is null and $\pi : (O_\theta, \sigma) \to (Z(n), \tau)$ is an asymptotic extension. Moreover, $RP(O_\theta, \sigma) \setminus \Delta$ consists of one orbit under $\sigma \times \sigma$.

Proof. By Proposition 6.2 of [**13**], (O_θ, σ) is null. Since $J_1 = \{n-1\}$, as before $J_\infty = \left\{\sum_{i=0}^{\infty}(n-1)n^i\right\}$, $E = \left\{\sum_{i=0}^{\infty}(n-1)n^i + t \mid t \in \mathbb{Z}\right\} = \text{orb}(0, \tau)$, $Z^\star = Z(n) - E = Z(n) - \text{orb}(0, \tau)$. For $z \in Z(n) - \text{orb}(0, \tau)$, $\pi^{-1}(z)$ is a single point. For $z \in \text{orb}(0, \tau)$, $\pi^{-1}(z)$ consists of exactly two points. Letting $\pi^{-1}(0) = \{w, w'\}$, we have $\pi^{-1}(\tau^k(0)) = \{\sigma^k(w), \sigma^k(w')\}$ for $k \in \mathbb{Z}$. Since (O_θ, σ) is a subshift, there exists an $x \neq x'$ such that (x, x') is an asymptotic pair. By the previous discussion, we have $\{x, x'\} = \{\sigma^j(w), \sigma^j(w')\}$ for some $j \in \mathbb{Z}$. This shows that $\{\sigma^k(w), \sigma^k(w')\}$ is an asymptotic pair for all $k \in \mathbb{Z}$.

Therefore, $\pi : (O_\theta, \sigma) \to (Z(n), \tau)$ is an asymptotic extension and $RP(O_\theta, \sigma) \setminus \Delta$ consists of one orbit under $\sigma \times \sigma$. □

COROLLARY 5.1. *There is a null minimal system which is an asymptotic extension of an equicontinuous system such that $PR(X, T) \setminus \Delta$ is uncountable.*

Proof. Let $p_1, p_2, \dots$ be all odd prime numbers and let (X_i, T_i) be the system in Lemma 5.1 with $n = p_i$ for each $i \in \mathbb{N}$. It is easy to see that $\Pi_i(X_i, T_i)$ is minimal and the corollary follows. □

Finally we have some open problems:

(1) Is there a null transitive system which is two-scattering?

Remark. A dynamical system is two-scattering if $\text{Com}(X, T) \cup \Delta = X \times X$. Note that $E(X, T) \subset SE(X, T) \subset \text{Com}(X, T)$ and any weakly mixing system with zero topological entropy will be an example such that $E(X, T) = \emptyset$ and $SE(X, T) \cup \Delta = X \times X$. It will be interesting to get an example such that $SE(X, T) = \emptyset$ and $\text{Com}(X, T) \cup \Delta = X \times X$. If such a system exists, it is non-minimal (minimal two-scattering implies weak mixing) and non-Banach-transitive. Thus, we may assume that it has only one minimal set which is a fixed point.

(2) Does a transitive non-minimal null system exist?

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