Some definite integrals involving Legendre functions. By Mr W. N. BAILEY, Trinity College.

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1. The integral

$$\int_0^\theta P_{n-1}(\cos\phi)\, P_n\{\cos(\theta-\phi)\}\, d\phi = \frac{\sin n\theta}{n}, \tag{1·1}$$

where n is a positive integer, was obtained recently by Van der Pol* by operational methods.

If we replace ϕ by t/n and θ by z/n, and make $n \to \infty$, we obtain the formula

$$\int_0^z J_0(t)\, J_0(z-t)\, dt = \sin z, \tag{1·2}$$

which is a particular case of a known formula†.

In attempting to discover an integral involving Legendre functions which would similarly give, as a limiting form, Bateman's integral‡

$$\int_0^z J_\mu(t)\, J_\nu(z-t)\, \frac{dt}{t} = \frac{J_{\mu+\nu}(z)}{\mu}, \tag{1·3}$$

I was led to the formula

$$\int_0^z P_n\{\cos(z-t)\}\, P_n^{-m}(\cos t)\, \frac{dt}{\sin t} = \frac{P_n^{-m}(\cos z)}{m}, \tag{1·4}$$

where n and m are positive integers. This formula gives, as the limiting form, the particular case of Bateman's integral when $\nu = 0$ and μ is a positive integer. In this paper I prove the formula (1·4), and obtain integrals involving Legendre functions which are analogous to Kapteyn's trigonometrical integrals§ involving Bessel functions.

2. We first prove (1·4) in the particular case when $m = n$, in which case it takes the form

$$\int_0^z P_n(\cos t)\, \sin^{n-1}(z-t)\, dt = \frac{\sin^n z}{n}. \tag{2·1}$$

* *Phil. Mag.* 8 (1929), 861–898 (70).

† See G. N. Watson, *Theory of Bessel Functions*, § 12·2 (4).

‡ G. N. Watson, *loc. cit.* § 12·2 (3).

§ G. N. Watson, *loc. cit.* § 12·21.

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Denoting the integral on the left by I_n, we have

$$I_{n-1}\sin z = \int_0^z P_{n-1}(\cos t)\sin^{n-2}(z-t)\{\sin(z-t)\cos t + \sin t\cos(z-t)\}\,dt.$$

Assuming that $n>1$, and integrating by parts, we obtain

$$\int_0^z P_{n-1}(\cos t)\sin^{n-2}(z-t)\cos(z-t)\sin t\,dt = \frac{1}{n-1}\int_0^z \sin^{n-1}(z-t)\frac{d}{dt}\{P_{n-1}(\cos t)\sin t\}\,dt.$$

Thus

$$\begin{aligned}(n-1)I_{n-1}\sin z &= \int_0^z \sin^{n-1}(z-t)\{n\cos t\,P_{n-1}(\cos t) - \sin^2 t\,P'_{n-1}(\cos t)\}\,dt\\ &= \int_0^z \sin^{n-1}(z-t)[\cos t\{n\cos t\,P_n(\cos t) + \sin^2 t\,P_n'(\cos t)\}\\ &\qquad - \sin^2 t\{\cos t\,P_n'(\cos t) - nP_n(\cos t)\}]\,dt\\ &= n\int_0^z \sin^{n-1}(z-t)\,P_n(\cos t)\,dt = nI_n.\end{aligned}$$

But evidently (2·1) is true when $n=1$, and hence it is true for all positive integral values of n.

If we differentiate (2·1) with regard to z, we obtain

$$(2\cdot2)\qquad \int_0^z P_n\{\cos(z-t)\}\sin^{n-2}t\cos t\,dt = \frac{\sin^{n-1}z\cos z}{n-1},$$

and this is the form which (1·4) assumes when $m=n-1$.

We can now easily prove (1·4) by induction. For, using the formulae

$$(2\cdot3)\quad 2\frac{d}{dz}\{P_n^{-m}(\cos z)\} = P_n^{-m+1}(\cos z) - (n+m+1)(n-m)\,P_n^{-m-1}(\cos z),$$

$$(2\cdot4)\quad 2\frac{d}{dz}\left\{\frac{mP_n^{-m}(\cos z)}{\sin z}\right\} = \frac{(m-1)\,P_n^{-m+1}(\cos z)}{\sin z} - (n+m+1)(n-m)\frac{(m+1)\,P_n^{-m-1}(\cos z)}{\sin z},$$

which are easily derived from the known recurrence formulae, and differentiating (1·4) in the form

$$\int_0^z P_n(\cos t)\frac{P_n^{-m}\{\cos(z-t)\}}{\sin(z-t)}\,dt = \frac{P_n^{-m}(\cos z)}{m},$$

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we find that, if (1·4) is true when m has the values M and $M-1$, it is also true when $m = M+1$, and also that, if (1·4) is true when m has the values M and $M+1$, it is also true when $m = M-1$. The complete result therefore follows, since we know that (1·4) is true when m has the values n and $n-1$.

It is possible that the formula is true for all positive values of m, but if so a different method would evidently have to be adopted to prove it in the more general case.

3. Corresponding to Kapteyn's integrals

$$(3\cdot1) \qquad \int_0^z \cos(z-t)\, J_0(t)\, dt = z J_0(z),$$

$$(3\cdot2) \qquad \int_0^z \sin(z-t)\, J_0(t)\, dt = z J_1(z),$$

$$(3\cdot3) \qquad \int_0^z \sin(z-t)\, J_1(t)\, dt = \sin z - z J_0(z),$$

we have the formulae

$$(3\cdot4) \quad \int_0^z \cos n(z-t)\, P_{n+1}(\cos t)\, dt = \sin z\, P_n(\cos z),$$

$$(3\cdot5) \quad \int_0^z \sin n(z-t)\, P_{n+1}(\cos t)\, dt = n \sin z\, P_n^{-1}(\cos z),$$

$$(3\cdot6) \quad \int_0^z \sin n(z-t)\, P_{n+1}^{-1}(\cos t)\, dt = \frac{\sin nz - n \sin z\, P_n(\cos z)}{(n+1)(n+2)}$$

To prove (3·4), let y denote the integral on the left. Then

$$\frac{dy}{dz} = P_{n+1}(\cos z) - n \int_0^z \sin n(z-t)\, P_{n+1}(\cos t)\, dt,$$

and

$$\frac{d^2y}{dz^2} + n^2 y = -\sin z\, P'_{n+1}(\cos z).$$

It is easily verified, by using the recurrence formulae for Legendre functions, that this differential equation is satisfied by

$$\sin z\, P_n(\cos z),$$

and hence

$$y = A \cos nz + B \sin nz + \sin z\, P_n(\cos z),$$

where A and B are constants. But when $z = 0$, $y = 0$ and $\frac{dy}{dz} = 1$, and so $A = B = 0$, and (3·4) is proved.

By differentiating (3·4) with regard to z, it is not difficult to deduce (3·5).

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We can prove (3·6) by using methods similar to those used in proving (3·4). We can also deduce (3·6) from (3·4) in the following way.

$$\begin{aligned}\sin nz &= -\int_0^z \frac{d}{dt}[\sin n(z-t)\, P_{n+1}(\cos t)]\, dt \\ &= \int_0^z [(n+1)(n+2)\sin n(z-t)\, P_{n+1}^{-1}(\cos t) \\ &\qquad + n\cos n(z-t)\, P_{n+1}(\cos t)]\, dt \\ &= (n+1)(n+2)\int_0^z \sin n(z-t)\, P_{n+1}^{-1}(\cos t)\, dt \\ &\qquad + n\sin z\, P_n(\cos z),\end{aligned}$$

and the result follows.

An integral of a somewhat similar type can be obtained by differentiating (1·1) with regard to θ. The result is

$$(3\cdot7)\qquad \int_0^\theta P_{n-1}(\cos\phi)\, P_n^{-1}\{\cos(\theta-\phi)\}\, d\phi = \frac{P_{n-1}(\cos\theta) - \cos n\theta}{n(n+1)},$$

the limiting form of which is

$$\int_0^z J_0(z)\, J_1(z-t)\, dt = J_0(z) - \cos z.$$

This is a particular case of a known formula*.

Kapteyn's trigonometrical integrals have recently been generalized†, and one might therefore expect that (3·4), (3·5) and (3·6) can also be generalized. It is no doubt true that there are formulae expressing the more general integrals in finite form, but the coefficients are apparently not capable of being expressed in a simple form.

4. In conclusion I note a formula of a different type. Since

$$\begin{aligned}&\left(\frac{d}{dz}\right)^r [(1-z^2)^{-\frac{1}{2}m} P_n^{-m}(z)] \\ &\qquad = (-1)^r (n+m+1)_r\, (m-n)_r\, (1-z^2)^{-\frac{1}{2}(m+r)}\, P_n^{-m-r}(z),\end{aligned}$$

we have, by Taylor's theorem,

$$(4\cdot1)\qquad \{1-(z+h)^2\}^{-\frac{1}{2}m} P_n^{-m}(z+h) = \sum_{r=0}^{\infty} \frac{(-h)^r (n+m+1)_r\, (m-n)_r}{r!} (1-z^2)^{-\frac{1}{2}(m+r)}\, P_n^{-m-r}(z).$$

* G. N. Watson, *loc. cit.* § 12·2 (4).

† W. N. Bailey, "Some integrals of Kapteyn's type involving Bessel functions", *Proc. London Math. Soc.* (2), 30 (1930), 422–424. See also W. N. Bailey, "Some definite integrals involving Bessel functions", *Proc. London Math. Soc.* (2), 31 (1930), 200–208.

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Now put $z = \cos\theta$, $z + h = \cos(\theta + \phi)$, and we obtain

$$\sin^{-m}(\theta+\phi)\,P_n^{-m}\{\cos(\theta+\phi)\} = \sum_{r=0}^{\infty} \frac{\{2\sin\tfrac{1}{2}\phi\sin(\theta+\tfrac{1}{2}\phi)\}^r (n+m+1)_r\,(m-n)_r}{r!} \times \sin^{-m-r}\theta\,P_n^{-m-r}(\cos\theta).$$

Let $\phi \to -\theta$, and we derive the formula

$$(4{\cdot}2)\qquad \frac{\sin^m\theta}{2^m m!} = \sum_{r=0}^{\infty} \frac{(-1)^r (n+m+1)_r\,(m-n)_r}{r!}\,P_n^{-m-r}(\cos\theta)\tan^r\tfrac{1}{2}\theta.$$

This is the analogue of the formula*

$$(4{\cdot}3)\qquad \frac{z^m}{2^m m!} = \sum_{r=0}^{\infty} \frac{(\frac{1}{2}z)^r}{r!}\,J_{m+r}(z).$$

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It is perhaps worth while noting that Van der Pol's formula can easily be deduced from (1·4). For, denoting the integral on the left of (1·1) by y, we find, after using the differential equation satisfied by $P_n\{\cos(\theta-\phi)\}$, that

$$\frac{d^2y}{d\theta^2} + n^2 y = -\sin\theta P'_{n-1}(\cos\theta) + \int_0^\theta P_{n-1}(\cos\phi)\,P'_{n-1}\{\cos(\theta-\phi)\}\,d\phi.$$

Van der Pol's formula follows immediately if we can prove that the right hand side of this equation is zero. This is true if

$$\int_0^\theta P_{n-1}(\cos\phi)\,\frac{P_{n-1}^{-1}\{\cos(\theta-\phi)\}}{\sin(\theta-\phi)}\,d\phi = P_{n-1}^{-1}(\cos\theta),$$

and this is the particular case of (1·4) when $m = 1$.

* G. N. Watson, *loc. cit.* § 5·22 (7).

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