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ADMISSIBLE CLUSTERING OF AGGREGATOR COMPONENTS: A NECESSARY AND SUFFICIENT STOCHASTIC SEMINONPARAMETRIC TEST FOR WEAK SEPARABILITY

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In aggregation theory, the admissibility condition for clustering components to be aggregated is blockwise weak separability, which also is the condition needed to separate out sectors of the economy. Although weak separability is thereby of central importance in aggregation and index number theory and in econometrics, prior attempts to produce statistical tests of weak separability have performed poorly in Monte Carlo studies. This paper introduces a new class of weak separability tests, which is seminonparametric. Such tests are both based on a necessary and sufficient condition and are fully stochastic, allowing to take into account measurement error. Simulations show that the tests perform well, even for large measurement errors.

Keywords: Weak separability, Seminonparametric tests

1. INTRODUCTION

This paper deals with seminonparametric testing procedures for models of the form

$$U(\mathbf{x}_i^*) = V\left(\mathbf{x}_i^{*(2)}, f\left(\mathbf{x}_i^{*(1)}\right)\right), \quad i = 1, \ldots, T, \tag{1}$$

where $U()$ is a utility function, $V()$ is a macro function, $f()$ is a micro function, $\mathbf{x}_i^* = (x_{i1}^*, x_{i2}^*, \ldots, x_{ik}^*)'$ is a vector of real commodities, and $\mathbf{x}_i^{*(1)}$ and $\mathbf{x}_i^{*(2)}$ are two partitions of $\mathbf{x}_i^*$ such that $\mathbf{x}_i^{*(1)} \cup \mathbf{x}_i^{*(2)} = \mathbf{x}_i^*$ and $\mathbf{x}_i^{*(1)} \cap \mathbf{x}_i^{*(2)} = \varnothing$.

Moreover, the tests considered here also deal with the common situation when $\mathbf{x}_i^*$ is not directly observed by the econometrician. Only $\mathbf{x}_i$ is observed, related to

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$\mathbf{x}_i^*$ by

$$\mathbf{x}_i' = g(\mathbf{p}_i) + \psi_i', \quad i = 1, \ldots, T, \tag{2}$$

where $\mathbf{x}_i^{*\prime} = g(\mathbf{p}_i)$, $g(\mathbf{p}_i)$ is unknown, $\psi_i' = (\psi_{i1}, \psi_{i2}, \ldots, \psi_{ik})$ is a vector of zero mean iid terms with unknown diagonal covariance matrix, and $\mathbf{p}_i$ is the vector of corresponding prices.

Model (1) has been extensively studied, especially within the revealed preference framework. Varian (1983) first, has proposed a fully nonparametric procedure based on the well-known Generalized Axiom of Revealed Preference (GARP). Among others, Swofford and Whitney (1987) and Fisher and Fleissig (1997) have implemented such an approach. Nevertheless, Barnett and Choi (1989) have cast some doubts on the validity of the procedure. On simulated data they have showed that the test was strongly biased toward rejection. Two factors are generally admitted to explain this high rejection rate: (i) The full procedure is nonstochastic. Being constructed as a three-step test of utility maximization, it uses each step the purely deterministic GARP, therefore totally ignoring model (2). Thus, a single violation of the axiom leads to rejection of the null of weak separability, even if caused by purely stochastic causes as measurement error. (ii) Step three of the procedure requires utility and price indices for the subutility to be computed by solving the so-called Afriat inequalities. Nevertheless, the way the inequalities are solved dramatically influences the power of the test [Fleissig and Whitney (2003)]. Moreover, this leads to an only sufficient condition.

Three approaches have tried to correct the initial nonparametric approach. Fleissig and Whitney (2003) suggested a new algorithm to solve the Afriat inequalities. They moreover showed that their test performed well, even if data were measured with small errors. Jones et al. (2005), on the basis of Varian (1985) and Swofford and Whitney (1994), introduced a modified weak separability test that explicitly deals with (2) and incomplete adjustment models. Also, the test is based on a necessary and sufficient condition. At last, on the basis of de Peretti (2005), de Peretti (2007) introduced a stochastic weak separability test. He moreover suggested a necessary and sufficient test under homotheticity.

Nevertheless, these three approaches are not totally satisfactory. The first approach clearly remains nonstochastic. It does not explicitly deal with measurement error, i.e., not allowing test of the significance of the violations of the GARP.[1] The second one is extremely computationally burdensome, preventing its use for large data sets. Moreover, it uses the Varian (1985) lower bound test, leading to a quite unclear decision rule. At last, even if the de Peretti (2007) approach allows test of the significance of the deviation from weak separability, it remains within the Afriat inequalities framework under the general nonhomothetic case. It then produces an only sufficient condition.

In this paper, we introduce a new class of weak separability tests. With regard to the above works, the test we want to consider differs in two ways. First, it is seminonparametric in the sense that it uses nonparametric tests to check the maximization assumptions but parametric ones to test the weak separability

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condition. Nevertheless, it should be noted that the procedure remains parametric flexible form free, in the sense that no estimation of $f()$ is required. Moreover, the test is defined beyond the Afriat inequalities framework and produces a necessary and sufficient condition. Second, the whole procedure is fully stochastic. Based on (2), the violations of GARP are tested for their significance. Also, the separability condition we use allows for measurement error or small optimization errors. Results from Monte Carlo simulations show that the procedure performs well, even for large measurement errors.

This paper is structured as follows. Standard deterministic nonparametric tests for separability are discussed in Section 2. In Section 3, we introduce an alternate test for weak separability, replacing the standard Afriat inequalities–based condition by a necessary and sufficient one. Section 4 extends the approach to deal with measurement error. In Section 5, we perform Monte Carlo simulations to analyze the power of the test. Section 6 concludes.

2. PRELIMINARIES

We first assume that data are perfectly observed. Let $\mathbf{X}(=\mathbf{X}^*)$ be a $(T \times k)$ matrix of observed real quantities, where T denotes the number of observations and k the number of goods. Let $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{ik})'$ be the ith row of the matrix, $i = 1, \ldots, T$. Similarly, define $\mathbf{P}$ as a $(T \times k)$ matrix of corresponding prices, and let $\mathbf{p}_i = (p_{i1}, p_{i2}, \ldots, p_{ik})'$ be the ith row of the matrix, $i = 1, \ldots, T$. Now consider two partitions of $\mathbf{X}$, the $(T \times a)$ $\mathbf{X}^{(1)}$ matrix, $a \in \{1, \ldots, k-1\}$, with $\mathbf{x}_i^{(1)} = (x_{i1}, x_{i2}, \ldots, x_{ia})'$, and the $(T \times (k-a))$ $\mathbf{X}^{(2)}$ matrix with $\mathbf{x}_i^{(2)} = (x_{i(a+1)}, x_{i(a+2)}, \ldots, x_{ik})'$. Let $\mathbf{P}^{(1)}$ and $\mathbf{P}^{(2)}$ be the corresponding associated price matrices, with $\mathbf{p}_i^{(1)} = (p_{i1}, p_{i2}, \ldots, p_{ia})'$ and $\mathbf{p}_i^{(2)} = (p_{i(a+1)}, p_{i(a+2)}, \ldots, p_{ik})'$. Define the weak separability of the first a columns of $\mathbf{X}$, that is, of $\mathbf{X}^{(1)}$, as follows:

DEFINITION 1. *There is weak separability if there exists a utility function (3) rationalizing the data, and if the latter admits a rewriting (4),*

$$U_i = U(\mathbf{x}_i), \quad i = 1, \ldots, T, \tag{3}$$

$$U_i = V\left(\mathbf{x}_i^{(2)}, f\left(\mathbf{x}_i^{(1)}\right)\right), \quad i = 1, \ldots, T, \tag{4}$$

where $U()$ is the overall utility function; $V()$ is a strictly increasing function, known as the macrofunction; and $f()$ is the subutility function, or the microfunction. It is also the aggregator function if homothetic.

Following Deaton and Muellbauer (1980), note that the weak separability implies that the marginal rate of substitution between any two goods of the separable group is independent of the goods outside the group, i.e.,

$$\partial \left(\frac{\partial U(\mathbf{x}_i)/\partial x_{ij}}{\partial U(\mathbf{x}_i)/\partial x_{il}} \right) \Big/ \partial x_{im} = 0 \tag{5}$$

for $j, l = 1, \ldots, a$; $j \neq l$; $m = a+1, \ldots, k$; and $i = 1, \ldots, T$.

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On the basis of Varian (1982), Varian (1983), developed a procedure to test for weak separability that ignores condition (5). To meet the weak separability criterion, three conditions must be fulfilled, that is, $U()$, $f()$, and $V()$ must exist. Testing for weak separability therefore reduces to a three-step test of utility maximization, each step using the well-known GARP, introduced hereafter.

Define the three following binary relations: $\mathbf{x}_i$ is said to be strictly directly revealed preferred to $\mathbf{x}_j$ if $\mathbf{p}_i \cdot \mathbf{x}_i > \mathbf{p}_i \cdot \mathbf{x}_j$, written $\mathbf{x}_i P^0 \mathbf{x}_j$; $\mathbf{x}_i$ is said to be directly revealed preferred to $\mathbf{x}_j$ if $\mathbf{p}_i \cdot \mathbf{x}_i \geq \mathbf{p}_i \cdot \mathbf{x}_j$, written $\mathbf{x}_i R^0 \mathbf{x}_j$; finally, $\mathbf{x}_i$ is said to be revealed preferred to $\mathbf{x}_j$ if $\mathbf{x}_i R^0 \mathbf{x}_m$,$\mathbf{x}_m R^0 \mathbf{x}_k, \ldots, \mathbf{x}_p R^0 \mathbf{x}_j$, written $\mathbf{x}_i R \mathbf{x}_j$, where R is the transitive closure of R^0. Using the above definitions, GARP is defined as follows:

DEFINITION 2 (GARP). *For a couple of observations* (i, j) $i \in \{1, \ldots, T\}$, $j \in \{1, \ldots, T\}$: $\mathbf{x}_i R \mathbf{x}_j \Rightarrow \mathbf{p}_j \cdot \mathbf{x}_j \leq \mathbf{p}_j \cdot \mathbf{x}_i$ *(or* $\mathbf{x}_i R \mathbf{x}_j$ *implies not* $\mathbf{p}_j \cdot \mathbf{x}_j > \mathbf{p}_j \cdot \mathbf{x}_i$*).*

Using GARP, Varian (1982) proved that the following.

THEOREM 1 [Varian (1982)]. *For a set* $\{(\mathbf{x}_i, \mathbf{p}_i)\}_{i=1}^T$*, the three following conditions are equivalent:*

(i) *There exists a locally nonsatiated utility function* $U()$ *that rationalizes the data.*
(ii) *There exist strictly positive utility indices* U_i *and marginal income indices* λ_i *that satisfy* $\forall i \epsilon \{1, \ldots, T\}$ $\forall j \epsilon \{1, \ldots, T\}$ *the Afriat inequalities:*

$$U_i \leqslant U_j + \lambda_j (\mathbf{p}_j \cdot \mathbf{x}_i - \mathbf{p}_j \cdot \mathbf{x}_j). \tag{6}$$

(iii) *The data satisfy GARP.*

Testing for the weak separability of $\mathbf{X}^{(1)}$ is now straightforward. It amounts to checking whether the three following conditions hold.

CONDITION 1. *GARP holds for* $\{(\mathbf{x}_i, \mathbf{p}_i)\}_{i=1}^T$*, that is,* $U()$ *exists.*

CONDITION 2. *GARP holds for* $\{(\mathbf{x}_i^{(1)}, \mathbf{p}_i^{(1)})\}_{i=1}^T$*, that is,* $f()$ *exists.*

CONDITION 3. *GARP holds for* $\{((\mathbf{x}_i^{(2)}, U_i), (\mathbf{p}_i^{(2)}, \lambda_i^{-1}))\}_{i=1}^T$*, where* U_i *and* λ_i *are strictly positive indices satisfying (6) for* $\{(\mathbf{x}_i^{(1)}, \mathbf{p}_i^{(1)})\}_{i=1}^T$*, that is,* $\mathbf{X}^{(1)}$ *is weakly separable in* $U()$.

Two major drawbacks are clearly associated with the above test. First, Condition 3 is an only sufficient one. Thus, if it does not hold, one cannot be sure that for an other set $\{(U_i, \lambda_i)\}_{i=1}^T$, GARP will not be violated. Moreover the power of the test dramatically depends on the way the indices are computed[2] [see Fleissig and Whitney (2003)]. Second, the whole procedure is clearly nonstochastic. A single

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violation of GARP leads to rejection of the null, even if caused by purely stochastic factors as measurement error. We first introduce a necessary and sufficient alternate Condition 3 and then suggest a stochastic extension.

3. AN ALTERNATE CONDITION 3

To introduce the new test for weak separability, assume that Conditions 1 and 2 hold, that is, the two sets $\{(\mathbf{x}_i, \mathbf{p}_i)\}_{i=1}^T$ and $\{(\mathbf{x}_i^{(1)}, \mathbf{p}_i^{(1)})\}_{i=1}^T$ are consistent with GARP. What we want is to replace the above Condition 3 by a necessary and sufficient one. The one we want to consider is based on (5), that is, on the independence between the marginal rate of substitution between any two goods of the separable group and the goods outside the group.

The difficulty with such an approach is that the form of the subutility, and then of course the associated gradient vector, are unknown. Nevertheless, even if unknown, it is possible to directly compute the output of the ratio of the first-order derivatives at first-order conditions. Indeed, if GARP holds for the set $\{(\mathbf{x}_i^{(1)}, \mathbf{p}_i^{(1)})\}_{i=1}^T$ then, given Theorem 1 there exists a subutility $f(\mathbf{x}_i^{(1)})$ rationalizing the data. Therefore the following lemma can be applied:

LEMMA 1 (Konyus and Byushgens). *Suppose that $f()$ is differentiable and each $\mathbf{x}_i^{(1)}$ is a solution of the maximization program for $f()$, i.e., GARP holds for $\{(\mathbf{x}_i^{(1)}, \mathbf{p}_i^{(1)})\}_{i=1}^T$, then,*

$$\frac{\mathbf{p}_i^{(1)'}}{\mathbf{p}_i^{(1)} \cdot \mathbf{x}_i^{(1)}} = \frac{\nabla f\left(\mathbf{x}_i^{(1)}\right)}{\mathbf{x}_i^{(1)} \cdot \nabla f\left(\mathbf{x}_i^{(1)}\right)}, \quad i = 1, \ldots, T, \tag{7}$$

where $\nabla f(\mathbf{x}_i^{(1)})$ is the $(k \times 1)$ gradient vector evaluated at $\mathbf{x}_i^{(1)}$.

Dividing the jth row of the system (7) by the lth row, $j, l = 1, \ldots, a$, $j \neq l$, returns the well-known condition

$$\frac{p_{ij}^{(1)}}{p_{il}^{(1)}} = \frac{\partial f\left(\mathbf{x}_i^{(1)}\right) / \partial x_{ij}^{(1)}}{\partial f\left(\mathbf{x}_i^{(1)}\right) / \partial x_{il}^{(1)}}. \tag{8}$$

At first-order conditions, the marginal rate of substitution between two goods equals the corresponding price ratio. Thus, in order to test for weak separability, knowing the form of the first-order derivatives of the subutility is not necessary, because we are able to directly compute the output of the marginal rates of substitution. Therefore, testing for weak separability amounts to checking the independence between all the unique price ratios of the goods inside the possibly separable group and the quantities outside the group.

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Let $\mathbf{Y}$ be a $(T \cdot \Sigma_{i=1}^{a-1}(a-i) \times 1)$ vector of the corresponding price ratios of the separable group, defined as

$$\mathbf{Y} = \begin{bmatrix} \log\left(\frac{\mathbf{p}_{\cdot 1}}{\mathbf{p}_{\cdot 2}}\right) \\ \vdots \\ \log\left(\frac{\mathbf{p}_{\cdot 1}}{\mathbf{p}_{\cdot a}}\right) \\ \log\left(\frac{\mathbf{p}_{\cdot 2}}{\mathbf{p}_{\cdot 3}}\right) \\ \vdots \\ \log\left(\frac{\mathbf{p}_{\cdot 2}}{\mathbf{p}_{\cdot a}}\right) \\ \vdots \\ \log\left(\frac{\mathbf{p}_{\cdot (a-1)}}{\mathbf{p}_{\cdot a}}\right) \end{bmatrix},$$

where $\mathbf{p}_{\cdot j} = (p_{1j}, p_{2j}, \ldots, p_{Tj})$ is the jth column of $\mathbf{P}^{(1)}$, $j = 1, \ldots, a$. Then, define the model

$$\mathbf{Y} = \begin{bmatrix} \mathbf{X}^{(3)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{X}^{(3)} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{X}^{(3)} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \\ \vdots \\ \boldsymbol{\beta}_{\Sigma_{i=1}^{a-1}(a-i)} \end{bmatrix} + \boldsymbol{\varepsilon}, \tag{9}$$

where $\mathbf{X}^{(3)}$ is a $(T \times (k+1))$ matrix defined as $\mathbf{X}^{(3)} = [\mathbf{1}\ \log(\mathbf{X})]$; $\boldsymbol{\beta}_i$ is a $((k+1) \times 1)$ vector of parameters, $i = 1, \ldots, \Sigma_{i=1}^{a-1}(a-i)$; and $\boldsymbol{\varepsilon}$ is a $(T \cdot \Sigma_{i=1}^{a-1}(a-i) \times 1)$ vector of residuals. Because $\mathbf{X}^{(3)} = [\mathbf{1}\ \log(\mathbf{X})]$ can be rewritten as $\mathbf{X}^{(3)} = [\mathbf{1}\ \log(\mathbf{X}^{(1)})\ \log(\mathbf{X}^{(2)})]$, then $\boldsymbol{\beta}_i = [\boldsymbol{\beta}_i^{(1)}, \boldsymbol{\beta}_i^{(2)}, \boldsymbol{\beta}_i^{(3)}]$. Weak separability of the first a columns of $\mathbf{X}$ therefore implies

$$\boldsymbol{\beta}_1^{(3)} = \boldsymbol{\beta}_2^{(3)} = \ldots = \boldsymbol{\beta}_{\Sigma_{i=1}^{a-1}(a-i)}^{(3)} = 0. \tag{10}$$

Therefore, testing for the weak separability of $\mathbf{X}^{(1)}$ amounts to checking whether the three following conditions hold:

CONDITION 1. *GARP holds for* $\{(\mathbf{x}_i, \mathbf{p}_i)\}_{i=1}^T$, *that is*, $U(\,)$ *exists*.

CONDITION 2. *GARP holds for* $\{(\mathbf{x}_i^{(1)}, \mathbf{p}_i^{(1)})\}_{i=1}^T$, *that is*, $f(\,)$ *exists*.

CONDITION 3′. $\boldsymbol{\beta}_1^{(3)} = \boldsymbol{\beta}_2^{(3)} = \ldots \boldsymbol{\beta}_{\Sigma_{i=1}^{a-1}(a-i)}^{(3)} = 0$, *that is*, $\mathbf{X}^{(1)}$ *is weakly separable in* $U(\,)$.

We now turn to a stochastic extension of the procedure.[3]

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4. A STOCHASTIC EXTENSION

We now relax the unrealistic assumption that data are perfectly observed.

Assumption 1. Under the null, $\mathbf{X}^*$ is generated by a weakly separable utility function but is unobservable. Only $\mathbf{X}$ is observed. It relates to $\mathbf{X}^*$ by the additive relation

$$x_{ij} = x_{ij}^* + \psi_{ij}. \tag{11}$$

Assumption 2. For one good j, $j = 1, \ldots, k$, the terms ψ_{ij} are iid with zero mean and variance $\sigma_{\psi_j}^2$, with distribution function $F_j(x)$. The distribution $F_j(x)$ is max-stable and min-stable.

Assumption 1 has two major implications. First, it leads us to take into account that some violations of GARP when testing for Conditions 1 and 2 might be caused by purely stochastic factors. Hence the need for discriminating between significant and nonsignificant violations of the axiom. Second, if the violations are nonsignificant, and thus if the data are measured with errors, model (9) is to be estimated by using particular estimators. We first focus on the way to test the significance of the violations of GARP when testing for weak separability.

The procedure we use is an extension of de Peretti (2005, 2007). It therefore inherits a similar logical structure, consisting of the following:

(i) Finding the minimal adjustment in order for the data to be consistent with both Conditions 1 and 2,
(ii) Testing the significance of this adjustment.

Concerning the former, computing the minimal adjustment is achieved by solving over z_{ij} the quadratic program:

$$\text{obj} = \min_{z_{ij}} \sum_{i=1}^{T} \sum_{j=1}^{k} (x_{ij} - z_{ij})^2, \tag{12}$$

subject to

$$\forall i \epsilon \{1, \ldots, T\} \forall j \epsilon \{1, \ldots, T\} : \mathbf{z}_i R \mathbf{z}_j \Rightarrow \mathbf{p}_j \cdot \mathbf{z}_j \leq \mathbf{p}_j \cdot \mathbf{z}_i, \tag{C.1}$$

$$\forall i \epsilon \{1, \ldots, T\} \forall j \epsilon \{1, \ldots, T\} : \mathbf{z}_i^{(1)} R \mathbf{z}_j^{(1)} \Rightarrow \mathbf{p}_j^{(1)} \cdot \mathbf{z}_j^{(1)} \leq \mathbf{p}_j^{(1)} \cdot \mathbf{z}_i^{(1)}, \tag{C.2}$$

where $\mathbf{z}_i^{(1)} = (z_{i1}, z_{i2}, \ldots, z_{ia})'$. Let $\widehat{\mathbf{Z}}$ be the matrix solution of the above program and define $\widehat{\boldsymbol{\Omega}} = \mathbf{X} - \widehat{\mathbf{Z}}$ as the matrix of residuals. Because $\widehat{\boldsymbol{\Omega}}$ is the minimal adjustment in order for $\widehat{\mathbf{Z}}$ to satisfy both Condition 1 (constraint C.1) and Condition 2 (constraint C.2), we will refer to these residuals as theoretical residuals.[4]

To test the significance of the adjustment, that is, to test whether violations are due to measurement error, we compare the theoretical residuals with the true measurement error $\boldsymbol{\Psi} = \mathbf{X} - \mathbf{X}^*$. For convenience, we will first assume that $\boldsymbol{\Psi}$ is known as well its distribution function. Because in (12) only a few bundles are

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adjusted, the idea is to build a test tracking excess adjustments in some goods and in particular to test whether extreme theoretical residuals are consistent with the perturbations induced by measurement error.[5] Statistically, we proceed as follows. For one good j, $j = 1, \ldots, k$, define $\widehat{\text{Max}}_j = \max(\widehat{\omega}_{1j}, \widehat{\omega}_{2j}, \ldots, \widehat{\omega}_{Tj})$ and $\widehat{\text{Min}}_j = \min(\widehat{\omega}_{1j}, \widehat{\omega}_{2j}, \ldots, \widehat{\omega}_{Tj})$, where $(\widehat{\omega}_{1j}, \widehat{\omega}_{2j}, \ldots, \widehat{\omega}_{Tj})$ is the jth column of the $(T \times k)$ matrix $\widehat{\boldsymbol{\Omega}}$. Similarly define $\text{Max}_j = \max(\boldsymbol{\Psi}_{.j})$ and $\text{Min}_j = \min(\boldsymbol{\Psi}_{.j})$, where $\boldsymbol{\Psi}_{.j} = (\psi_{1j}, \psi_{2j}, \ldots, \psi_{Tj})$ is the jth column of the $(T \times k)$ matrix $\boldsymbol{\Psi}$. Because $\text{Min}_j = \min(\psi_{1j}, \psi_{2j}, \ldots, \psi_{Tj}) = -\max(-\psi_{1j}, -\psi_{2j}, \ldots, -\psi_{Tj})$ and because reversal of the sign of $(\psi_{1j}, \psi_{2j}, \ldots, \psi_{Tj})$ will produce results for the smallest extreme, we will focus only on theoretical results concerning the largest extreme. Under Assumption 2, the following theorem holds:

THEOREM 2 (Fisher-Tippett). *For the iid sequence* $\boldsymbol{\Psi}_{.j} = (\psi_{1j}, \psi_{2j}, \ldots, \psi_{Tj})$, *if there exist norming constants* $a_j \in R$ *and* $b_j > 0$ *and some nondegenerate distribution function* G *such that*

$$(\text{Max}_j - a_j)b_j^{-1} \xrightarrow{\mathcal{L}} G, \tag{13}$$

then G *belongs to the type of one of the following three laws:*

$$\textit{Fréchet (type III)} : G_3(x) = \begin{cases} 0, & x \leq 0 \\ \exp(-x^{-\alpha_j}), & x > 0 \end{cases}, \quad \alpha_j > 0. \tag{14}$$

$$\textit{Weibull (type II)} : G_2(x) = \begin{cases} \exp[-(-x^{\alpha_j})], & x \leq 0 \\ 1, & x > 0 \end{cases}, \quad \alpha_j > 0. \tag{15}$$

$$\textit{Gumbel (type I)} : G_1(x) = \exp[-\exp(-x)], \quad x \in \mathbf{R}. \tag{16}$$

With a prior knowledge of the true distribution of the errors, it is possible to know the domain of attraction of the law of the extremes and then choose (14), (15), or (16). Note that a postvalidation is always possible and desirable using for instance probability plots or goodness-of-fit measures [see D'Agostino and Stephens (1986)]. For instance, under Gaussian assumptions for the true measurement error, it can be showed that the two extremes belong to the domain of attraction of the type I Gumbel law. Therefore, under this assumption, testing the significance of the adjustments for good j, is achieved by computing the two p-values

$$1 - \exp[-\exp(-y_{1j})] \tag{17}$$

for the maxima and

$$1 - \exp[-\exp(y_{2j})] \tag{18}$$

for the minima, where $y_{1j} = (\widehat{\text{Max}}_j - a_{1j})b_{1j}^{-1}$ and $y_{2j} = (\widehat{\text{Min}}_j - a_{2j})b_{2j}^{-1}$, where as before, a_{1j} and a_{2j} are the location parameters and b_{1j} and b_{2j} are the scale parameters, corresponding to the distribution of $\boldsymbol{\Psi}_{.j}$.

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The two tests simply return the probability for the two extremes of the theoretical residuals to belong to the distribution of the extremes of the true measurement error. Hence, excess adjustments will lead to a rejection of the null.

With no prior knowledge, the three laws can be expressed as the Generalized Extreme Value distribution (GEV), also known as the Von Mises-Jenkinson–type distribution (19). See Hosking, Wallis, and Wood (1985) and Hosking (1985) for estimation methods for the parameters.

$$G(x) = \begin{cases} \exp\left\{-\left[1+\lambda\dfrac{(x-a_j)}{b_j}\right]^{-\frac{1}{\lambda}}\right\}, & 1+\lambda\dfrac{(x-a_j)}{b_j} > 0, \quad \lambda > 0, \\ \exp\left\{-\exp\left[-\dfrac{(x-a_j)}{b_j}\right]\right\}, & \lambda = 0. \end{cases} \tag{19}$$

From an empirical point of view, the realizations $\mathbf{\Psi}_{.j}$, $j = 1, \ldots, k$ are seldom observable (except when there is a data revision). To compute the tests statistics, we therefore need some estimates of $\mathbf{\Psi}_{.j}$, say, $\widehat{\mathbf{\Psi}}_{.j}$. We now introduce several possible methods, all based on state-space models, assuming a normal distribution for the true measurement error.

Interestingly, (11) is the measurement equation of a time-invariant state-space model. Assuming a first-order Markov transition process for the unobservable state variable returns the following model for good j:

$$\begin{aligned} x_{ij} &= x^*_{ij} + \psi_{ij}, \\ x^*_{(i+1)j} &= F x^*_{ij} + c_j + \xi_{ij}, \end{aligned} \tag{20}$$

where x^*_{ij} is the unobserved quantity; ψ_{ij} and ξ_{ij} are two uncorrelated residuals, respectively, with variance $\sigma^2_{\psi_j}$ and $\sigma^2_{\xi_j}$; and F, c_j, $\sigma^2_{\psi_j}$ and $\sigma^2_{\xi_j}$ are the hyperparameters of the model. In the following, to simplify the discussion, we will consider only the so-called linear trend model (20) with $F = 1$ and $c_j = 0$. Nevertheless, much more complicated structural time series models can also be considered [see Harvey (1989) or Durbin and Koopman (2001)]. Under Gaussian assumptions, the innovations of the model are normal and so are the residuals ψ_{ij} and ξ_{ij}. Let $\widehat{\sigma}^2_{\psi_j}$ be the maximum likelihood estimate of $\sigma^2_{\psi_j}$. Following Guégan (2003), it can be shown that the maximum of a series of T variates drawn in a centered and reduced normal law has (17) as limiting distribution with location and scale parameters defined by

$$a_j = (2\ln T)^{1/2} - \frac{\ln\ln T + \ln 4\pi}{2(2\ln T)^{1/2}}, \tag{21}$$

$$b_j = (2\ln T)^{-1/2}. \tag{22}$$

One can then compute the p-value (17) with $a_{1j} = a_j$ and $b_{1j} = b_j$ as defined above, replacing $\widehat{\text{Max}}_j$ by $\widehat{\text{Max}}R_j$ defined as $\widehat{\text{Max}}R_j = \max(\widehat{\omega}_{1j}\widehat{\sigma}^{-1}_{\psi_j},$

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$\widehat{\omega}_{2j}\widehat{\sigma}_{\psi_j}^{-1}, \ldots, \widehat{\omega}_{Tj}\widehat{\sigma}_{\psi_j}^{-1}$). A similar result holds for the minimum using $\widehat{\text{Min}}R_j = \min(\widehat{\omega}_{1j}\widehat{\sigma}_{\psi_j}^{-1}, \widehat{\omega}_{2j}\widehat{\sigma}_{\psi_j}^{-1}, \ldots, \widehat{\omega}_{Tj}\widehat{\sigma}_{\psi_j}^{-1})$ and $a_{2j} = -a_j$. We will refer to this method as method 1.

Alternatively, one may also consider retabulating the empirical distribution functions for the two extremes using simulations methods. Let for good j, $j \in \{1, \ldots, k\}$, $\widehat{\mathbf{x}^*_{.j}} = (\widehat{x^*}_{1j}, \ldots, \widehat{x^*}_{Tj})$ be the smoothed estimate of the unobserved variable $\mathbf{x}^*_{.j} = (x^*_{1j}, \ldots, x^*_{Tj})$ computed thanks to the Kalman filter, and then let $\widehat{\mathbf{\Psi}}_{.j} = (x_{1j} - \widehat{x^*}_{1j}, \ldots, x_{Tj} - \widehat{x^*}_{Tj})$ be the so-called smoothed residuals,[6] also defined as $\widehat{\psi}_{ij} = x_{ij} - \widehat{x^*}_{ij} = E(\psi_{ij} \mid (x_{1j}, \ldots, x_{Tj}))$. The idea is to draw, a large number of times, samples of the smoothed residuals and for each draw to collect the minimum and maximum of $\widehat{\psi}_{.j}$. Testing the significance of the theoretical adjustment is then achieved by using a simple quantile approach. We will refer to this approach as method 2. To draw samples, we use the simulation smoother as introduced by de Jong and Shephard (1995). It is defined as follows. First define the updating relations of the Kalman filter, from i to $i+1$, as

$$v_{ij} = x_{ij} - x^*_{ij},\ F_i = P_i + \sigma^2_{\psi_j},\ K_i = P_i F_i^{-1},$$

$$x^*_{(i+1)j} = x^*_{ij} + K_i v_{ij},\ P_{i+1} = P_i(1 - K_i) + \sigma^2_{\xi_j}, \quad \text{for } j \epsilon \{1, \ldots, k\},$$

where v_{ij} is the innovation, F_i is the variance of the innovation, K_i is the Kalman gain, and P_{i+1} is the variance of $x^*_{(i+1)j}$ given the information in period i. For simplicity, the subscript j for F, P and K are omitted. A simulated sample for the measurement error is then drawn recursively using the following set of equations:

$$\widetilde{\psi}_{ij} = d_i + \sigma^2_{\psi_j}\left(v_{ij} F_i^{-1} - K_i \widetilde{r}_i\right),$$

$$\widetilde{r}_{i-1} = v_{ij} F_i^{-1} - \widetilde{W}_i d_i C_i^{-1} + (1 - K_i)\widetilde{r}_i,$$

where $d_i \sim N(0, C_i)$, $C_i = \sigma^2_{\psi_j} - \sigma^4_{\psi_j}(F_i^{-1} + K_i^2 \widetilde{N}_i)$, $\widetilde{W}_i = \sigma^2_{\psi_j}(F_i^{-1} - K_i \widetilde{N}_i(1 - K_i))$, and $\widetilde{N}_{i-1} = F_i^{-1} + \widetilde{W}_i^2 C_i^{-1} + \widetilde{N}_i(1 - K_i)^2$, $\widetilde{r}_T = \widetilde{N}_T = 0$.

Note that this approach is not strictly equivalent to the one using the estimated second moment of the disturbance error, because the smoothed residuals are likely to underestimate the variance of the true residuals[7] and because the smoothed residuals will have a stationary ARMA structure rather than an iid one [Bell and Hillmer (1984), Harvey and Koopman (1992)]. Note that even with an ARMA structure, the distribution of the extremes will still be the Gumbel one [Rootzén (1986)]. Note also that a less time-consuming solution consists of using the simulation smoother, say, only several hundred times, collecting for each draw the maximum and minimum, and then estimating the location and scale parameters of the law of interest by using a suitable method such as maximum likelihood, moments, or probability weighted moments [see Johnson, Kotz, and Balakrishnan (1994)].

If the data pass one of the above tests, i.e., if the violations are caused by measurement errors, testing for weak separability is achieved by estimating the auxiliary model (9) with observed data by using an instrumental variables (IV)

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estimator,[8] a natural choice for the instruments being therefore given by the smoothed estimates of the state vectors. This allows account to be taken of the amount of measurement error in the data. We next turn to a simulation study.

5. A SIMULATION STUDY

We now implement some simulations to investigate the magnitude of the type I and II errors under the stochastic case, i.e., when data are measured with errors. Recall that the type I error is the probability of incorrectly rejecting separability and the type II error is the probability of incorrectly accepting separability. Following Blackorby, Primont, and Russell (1998), let for period $i, i = 1, \ldots, T$, and the preferences are defined by the utility function

$$U_i = U(x_{i1}^*, x_{i2}^*, x_{i3}^*, x_{i4}^*) = x_{i1}^{*(1/3)} x_{i3}^{*(1/3)} x_{i4}^{*(1/3)} + x_{i2}^{*(1/2)} x_{i3}^{*(1/4)} x_{i4}^{*(1/4)}. \quad (23)$$

Clearly, the utility is weakly separable over (x_{i3}^*, x_{i4}^*), but not over (x_{i1}^*, x_{i2}^*). It thus admits the following rewriting:

$$U_i = V(x_{i1}^*, x_{i2}^*, f(x_{i3}^*, x_{i4}^*)) = x_{i1}^{*(1/3)} \nu_i + x_{i2i}^{*(1/2)} \nu_i^{(3/4)}, \quad (24)$$

where $\nu_i = x_{i3}^{*(1/3)} x_{i4}^{*(1/3)}$. We maximize (23) for four observed prices, i.e., services (x_{i1}^*), food (x_{i2}^*), other nondurable goods (x_{i3}^*), and durable goods (x_{i4}^*), for the United States over the period 1989:01-2004:10, subject to an observed per capita income. Given model (11) we add to the quantities solution of the above program normally distributed iid terms with zero mean and variance defined according to the signal-to-noise ratio (SNR_j). The signal-to-noise ratio is defined as $\text{SNR}_j = \sigma_{\xi_j}^2 / \sigma_{\psi_j}^2$, where $\sigma_{\xi_j}^2$ is the variance of the residuals of the transition equation and $\sigma_{\psi_j}^2$ is the variance of the measurement error for good $j, j = 1, \ldots, k$. It therefore measures the degree of corruption of the data. Here, we use $\text{SNR}_j = 0.25, 0.5, 1$, and 1.5. The smallest value corresponding to the largest measurement error (comparative to $\sigma_{\xi_j}^2$). Let $\mathbf{X}$ be the quantities with measurement error.

To estimate the whole procedure type I error, we run GARP on $\{(\mathbf{x}_i, \mathbf{p}_i)\}_{i=1}^T$ and $\{(\mathbf{x}_i^{(1)}, \mathbf{p}_i^{(1)})\}_{i=1}^T$, where $\mathbf{x}_i^{(1)} = (x_{i3}, x_{i4})'$. If violations appear, we first solve (12) and then estimate the model (20) by maximizing a diffuse (Gaussian) log-likelihood, because the local trend model is nonstationary. To test the significance of the theoretical adjustment, we use the two different methods mentioned above: (i) By using the maximum likelihood estimate of $\sigma_{\psi_j}^2$ and then by computing the two p-values (17) and (18) using $\widehat{\text{Max}} R_j$ and $\widehat{\text{Min}} R_j$, the norming constants being computed as (21) and (22) for the maximum and as $a_{2j} = -a_j$ and $b_{2j} = b_j$ for the minimum; (ii) by using the simulation smoother (10,000 replications are used), that is, by retabulating the two laws of extremes and by comparing the extremes of the theoretical adjustment with four quantiles of the estimated densities (see below). If the data pass the test(s), we estimate (25) by

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TABLE 1. Initial estimates

	Diffuse log-likelihood	$\hat{\sigma}^2_{\psi_j}$	$\hat{\sigma}^2_{\xi_j}$	Max[abs(gradient)][a]
Good 1	−263.67	5.53×10^{-6}	149.24	1.90×10^{-6}
Good 2	−299.34	7.45×10^{-6}	432.85	3.77×10^{-6}
Good 3	−341.70	5.36×10^{-13}	1,532.98	0.00
Good 4	−334.01	1.62×10^{-5}	1,218.23	4.23×10^{-6}

[a]Estimated at $\hat{\sigma}^2_{\psi_j}$ and $\hat{\sigma}^2_{\xi_j}$.

using IV estimators, thus following Hsiao (1997):

$$\log\left(\frac{\mathbf{p}_{\cdot 3}}{\mathbf{p}_{\cdot 4}}\right) = \beta_1 \log(\mathbf{x}_{\cdot 1}) + \beta_2 \log(\mathbf{x}_{\cdot 2}) + \beta_3 \log(\mathbf{x}_{\cdot 3}) + \beta_4 \log(\mathbf{x}_{\cdot 4}) + \beta_5 + \varepsilon, \quad \textbf{(25)}$$

where $\mathbf{p}_{\cdot j} = (p_{1j}, p_{2j}, \ldots, p_{Tj})$ is the jth column of $\mathbf{P}$ and $\mathbf{x}_{\cdot j} = (x_{1j}, x_{2j}, \ldots, x_{Tj})$ is the jth column of $\mathbf{X}$. Clearly, in (25) the separability of $\mathbf{x}_{\cdot 3}$ and $\mathbf{x}_{\cdot 4}$ implies $\beta_1 = \beta_2 = 0$. Let p_w be the p-value of the Wald-type restriction test. The type I error is therefore defined as the probability of having $\{\min[(17), (18)] < \alpha,\ j = 1, \ldots, k\}$ or $\{\min[(17), (18)] \geq \alpha,\ j = 1, \ldots, k$ and $p_w < \alpha\}$ using $\widehat{\text{Max}} R_j$ and $\widehat{\text{Min}} R_j$ for the first method and $\{\widehat{\text{Max}}_j > Q^{\max}_{1-\alpha}$ and/or $\widehat{\text{Min}}_j < Q^{\min}_{\alpha},\ j = 1, \ldots, k\}$ or $\{\widehat{\text{Max}}_j \leq Q^{\max}_{1-\alpha}$ and $\widehat{\text{Min}}_j \geq Q^{\min}_{\alpha},\ j = 1, \ldots, k$ and $p_w < \alpha\}$ for the second method, where $Q^{\max}_{1-\alpha}$ is the quantile of the density of maximums at $1 - \alpha$ and $Q^{\min}_{\alpha}$ of the density of minimums at α, computed using the simulation smoother, $\alpha = 1\%, 0.05\%, 10\%$, and 15%.

To estimate the whole procedure type II error, we proceed in a similar way, except that (i) we run GARP on $\{(\mathbf{x}_i, \mathbf{p}_i)\}_{i=1}^T$ and $\{(\mathbf{x}_i^{(1)}, \mathbf{p}_i^{(1)})\}_{i=1}^T$, where now $\mathbf{x}_i^{(1)} = (x_{i1}, x_{i2})'$, and (ii) if violations are seen as nonsignificant using the two above tests, we estimate (26) by using IV estimators:

$$\log\left(\frac{\mathbf{p}_{\cdot 1}}{\mathbf{p}_{\cdot 2}}\right) = \beta_1 \log(\mathbf{x}_{\cdot 1}) + \beta_2 \log(\mathbf{x}_{\cdot 2}) + \beta_3 \log(\mathbf{x}_{\cdot 3}) + \beta_4 \log(\mathbf{x}_{\cdot 4}) + \beta_5 + \varepsilon. \quad \textbf{(26)}$$

Nonseparability of $\mathbf{x}_{\cdot 2}$ and $\mathbf{x}_{\cdot 1}$ implies $\beta_3 \neq 0$ and $\beta_4 \neq 0$ in the above equation. Let p_w be the p-value of the Wald-type restriction test. The type II error is defined as the probability of having $\{\min[(17), (18)] \geq \alpha,\ j = 1, \ldots, k$ and $p_w \geq \alpha\}$ using $\widehat{\text{Max}} R_j$ and $\widehat{\text{Min}} R_j$ for the first method and $\{\widehat{\text{Max}}_j \leq Q^{\max}_{1-\alpha}$ and $\widehat{\text{Min}}_j \geq Q^{\min}_{\alpha},\ j = 1, \ldots, k$ and $p_w \geq \alpha\}$ for the second method.

Table 1 presents the estimated hyperparameters of (20) for the perfectly observed data, that is, for $\mathbf{X}^*$, as well as the log likelihood and the maximum of the gradient (in absolute value) evaluated at $\widehat{\sigma}^2_{\psi_j}$ and $\widehat{\sigma}^2_{\xi_j}$. The model is estimated by maximizing a diffuse (Gaussian) log likelihood. For filtering and smoothing, the exact Kalman filter is used [Koopman (1997)]. Note that because $\widehat{\sigma}^2_{\psi_j}$ is very small for all goods, the Kalman filter will return a perfect fit, thus not biasing the estimation of the type I and II errors.

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TABLE 2. Type I and II errors for various signal-to-noise (SNR_j) ratios

Method 1[a]								
	Type I error: SNR_j				Type II error: SNR_j			
$\alpha(\%)$	0.25	0.50	1.00	1.50	0.25	0.50	1.00	1.50
1	0.027	0.032	0.063	0.056	0.000	0.000	0.000	0.000
5	0.115	0.112	0.128	0.110	0.000	0.000	0.000	0.000
10	0.185	0.165	0.191	0.176	0.000	0.000	0.000	0.000
15	0.251	0.222	0.244	0.232	0.000	0.000	0.000	0.000
Method 2[b]								
	Type I error: SNR_j				Type II error: SNR_j			
$\alpha(\%)$	0.25	0.50	1.00	1.50	0.25	0.50	1.00	1.50
1	0.027	0.032	0.063	0.056	0.000	0.000	0.000	0.000
5	0.115	0.112	0.128	0.110	0.000	0.000	0.000	0.000
10	0.185	0.165	0.191	0.176	0.000	0.000	0.000	0.000
15	0.251	0.222	0.244	0.232	0.000	0.000	0.000	0.000

[a] Statistics (17) and (18) are used to test the significance of the violations of GARP, using reduced theoretical extremes.

[b] The significance of the violations of GARP is tested by comparing the theoretical extremes adjustments with quantiles of the retabulated density, computed using the simulation smoother.

Table 2 returns for 1,000 replications, four different signal-to-noise ratios, and the two methods to test the significance of the violations of GARP, the estimated type I and II errors. Clearly, methods 1 and 2, return exactly the same overall results. Nevertheless, it turns out that using the simulation smoother is slightly more efficient. Indeed, when violations appear when running GARP on $\{(\mathbf{x}_i^{(1)}, \mathbf{p}_i^{(1)})\}_{i=1}^T$, where $\mathbf{x}_i^{(1)} = (x_{i1}, x_{i2})'$, it allows rejection more often than method 1 (data not showed here). Concerning the type I and II errors, note that the test is slightly biased toward rejection because at 5% the type I error is 0.115 for both methods ($SNR_j = 0.25$) and 0.185 at 10% ($SNR_j = 0.50$). It nevertheless remains within a very acceptable range. Note at last that the type II error is set to zero, whatever the considered threshold and signal-to-noise ratio. This suggests that the test perfectly recognizes nonseparability, either by rejecting the nonsignificance of the violations of GARP or by rejecting the restriction $\beta_3 \neq 0$ and $\beta_4 \neq 0$ in (26). On simulated data, the test thus appears to be quite powerful.

6. CONCLUSION AND DISCUSSION

As already mentioned, weak separability is the key existence condition for an aggregate to exist [Barnett (1980)] and is also the condition needed to separate out sectors of the economy. It is therefore of central importance in economics and econometrics. Starting from the well-known lack of power of existing tests, we

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have introduced a new class of weak separability tests that is semi-nonparametric. With regard to the standard Varian (1983) test, the one we have considered differs in two points. First, the only sufficient separability condition based on the Afriat inequalities is replaced by a necessary and sufficient one. This one uses the well-known independence condition between the marginal rates of substitution between the goods inside the group and the goods outside. Second, the whole procedure allows us to deal with the common situation in which the data are not perfectly observed. On simulated data, on the basis of the framework of Blackorby et al. (1998), we have shown that the procedure performed quite well, even for large random measurement errors.

Two points are worth noting. First, in simulations, we have used classical IV estimators. Nevertheless, one may also consider using fully modified–generalized instrumental variable estimation (FM–GIVE) or FM–IV estimators, resulting in a possible gain of efficiency. Second, throughout the paper we have used results based on normality assumptions for the unobserved measurement error. Nevertheless, non-Gaussian models, especially with fatter tails, can easily be considered still within the Kalman filter framework. This is left for future research.

NOTES

1. It could be nevertheless possible to combine the results of Fleissig and Whitney (2003, 2005).

2. Note that the indices U_i produce an ordinal measurement, that is, a coherent ranking consistent with the information returned by the transitive closure matrix. They thus provide no information about a cardinal measurement. That is why using the test produces an only sufficient solution. In their paper, Fleissig and Whitney (2003) suggest solving the Afriat inequalities by finding utility and price indices satisfying the Afriat inequalities such that the distance with an observed superlative Divisia index and a dual price index is minimal. Because under homotheticity, the Divisia tracks the variations of the subutility function, if the deviations between the index and the indices are small enough, the indices can be seen as a cardinal measurement. Note that the result can be extended to the nonhomothetic case if the variations of prices and revenues are small from one period to another [see Barnett (1979)]. This may partly explain why their test overperforms the Varian one. Nevertheless, it is apparent that in their minimization program λ_i will act as a buffer variable and in some cases will be very small, thus leading to a very large "price" λ_i^{-1} and therefore to a very large value for $U_i \cdot \lambda_i^{-1}$. Because the power of GARP dramatically depends on the number of budget line intersections, introducing some very large values for $U_i \cdot \lambda_i^{-1}$ with regard to the other goods is likely to introduce a strong bias in the analysis. In other words, the bias arrises because $U_i \cdot \lambda_i^{-1}$ does not reflect the budget in $i, i = 1, \ldots, T$. This may be a major limitation of such an approach.

3. To avoid any confusion, note that equation (9) is only an independence test. Thus the estimated parameters will generally not be estimates of the parameters of the marginal rate of substitution. Exceptions are for instance for the Cobb–Douglas function. Note also that squared and cross terms can be taken into account to take account of complex relations.

4. See Appendix 1 for computational details.

5. On extreme values, see Embrecht, Klüppelberg, and Mikosch (2003).

6. Note that such residuals, also known as auxiliary residuals, return information about outliers in the series.

7. For another approach, see Stoffer and Wall (1991).

8. Interestingly, if the data are nonstationary, a result of Hsiao (1997) states that classical IV estimators, as well as classical associated Wald tests, are still valid, without modifications. Kitamura and Phillips (1997) also suggest using FM–GIVE or FM–IV estimators. In empirical work, it may be useful to look at the two different estimators.

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APPENDIX: SOLVING THE ADJUSTMENT PROCEDURE

Here, we focus on the way the quadratic program (12) is solved, i.e., how the objective function is minimized subject to a set of transitive constraints. Recall that we want to produce a set consistent with both

(i) the overall utility maximization program and
(ii) the subutility maximization program,

such that the distance with the observed data is minimal.

Following de Peretti (2005, 2007), define the binary relation $\mathbf{x}_i V R \mathbf{x}_j$ if $\mathbf{x}_i R \mathbf{x}_j$ and $\mathbf{x}_j P^0 \mathbf{x}_i$ or if there exists a sequence between $\mathbf{x}_i$ and $\mathbf{x}_j$ such that $\mathbf{x}_i R \mathbf{x}_k$ and $\mathbf{x}_k P^0 \mathbf{x}_i$, $\mathbf{x}_k R \mathbf{x}_l$ and $\mathbf{x}_l P^0 \mathbf{x}_k, \ldots, \mathbf{x}_m R \mathbf{x}_j$ and $\mathbf{x}_j P^0 \mathbf{x}_m$. Also define $\mathbf{x}_i S R \mathbf{x}_j$ if $S(i) = S(j)$, where $S(i) = (\sum_{j=1}^{T} r_{ij}) - 1$, r_{ij} being the element at the ith row and jth column of the transitive closure matrix. The sequence we want to consider is as follows:

(i) Adjust the quantities to produce data consistent with the subutility,
(ii) adjust the quantities to produce data consistent with the overall utility, such that the data must still be consistent with the subutility.

Concerning the subutility adjustment program for the set $D^{(1)} = \{(\mathbf{x}_i^{(1)}, \mathbf{p}_i^{(1)})\}_{i=1}^{T}$, the procedure is as follows:

Step 1.1. Test $D^{(1)}$ for consistency with GARP, let $\mathbf{R}^{(1)}$ be the transitive closure matrix, and let $r_{ij}^{(1)}$ be an element at the ith row and jth column. Define nvio as the number of violations:

$$\text{if } \begin{cases} \text{nvio} = 0 \text{ then stop the iterative procedure,} \\ \text{otherwise go to step 1.2.} \end{cases}$$

Step 1.2. Because $\mathbf{x}_i^{(1)} V R \mathbf{x}_j^{(1)}$ implies $\mathbf{x}_i^{(1)} S R \mathbf{x}_j^{(1)}$, build B_l set(s), $l = 1, \ldots, n$ such that every couple $(\mathbf{x}_i^{(1)}, \mathbf{p}_i^{(1)})$ and $(\mathbf{x}_j^{(1)}, \mathbf{p}_j^{(1)})$ belonging to $B_l, \forall l \in \{1, \ldots, n\}$ satisfies $\mathbf{x}_i^{(1)} S R \mathbf{x}_j^{(1)}$. Search for the one, B_1, containing bundles violating GARP being at the same highest place

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in the preference chain such that for $(\mathbf{x}_i^{(1)}, \mathbf{p}_i^{(1)}) \in B_1$ and $(\mathbf{x}_j^{(1)}, \mathbf{p}_j^{(1)}) \notin B_1 : S(i) > S(j)$. Go to step 1.3.

Step 1.3. In the set B_1, solve (A.1) subject to (C.1) and (C.2), for each bundle related by $\mathbf{x}_i^{(1)} R \mathbf{x}_j^{(1)}$ and $\mathbf{x}_j^{(1)} P^0 \mathbf{x}_i^{(1)}$. Select the bundle for which the objective function is minimal. Let $(\widehat{\mathbf{z}}_i^{(1)}, \mathbf{p}_i^{(1)})$ be the bundle solution, where $\widehat{\mathbf{z}}_i^{(1)} = (\widehat{z}_{i1}^{(1)}, \widehat{z}_{i2}^{(1)}, \ldots, \widehat{z}_{ia}^{(1)})'$. Replace in $D^{(1)}(\mathbf{x}_i^{(1)}, \mathbf{p}_i^{(1)})$ by $(\widehat{\mathbf{z}}_i^{(1)}, \mathbf{p}_i^{(1)})$. Go to step 1.1.

$$\text{obj}_i = \min_{z_{ij}^{(1)}} \sum_{j=1}^{a} \left(x_{ij}^{(1)} - z_{ij}^{(1)} \right)^2, \tag{A.1}$$

subject to

$$\begin{cases} \mathbf{p}_i^{(1)} \cdot \mathbf{x}_i^{(1)} = \mathbf{p}_i^{(1)} \cdot \mathbf{z}_i^{(1)} \text{ and } \mathbf{p}_j^{(1)} \cdot \mathbf{x}_j^{(1)} \leq \mathbf{p}_j^{(1)} \cdot \mathbf{z}_i^{(1)}, \\ \mathbf{p}_m^{(1)} \cdot \mathbf{x}_m^{(1)} \leq \mathbf{p}_m^{(1)} \cdot \mathbf{z}_i^{(1)} \text{ for all } \mathbf{x}_m^{(1)} \text{ such that } \mathbf{x}_i^{(1)} V R \mathbf{x}_m^{(1)}, m \neq j, \end{cases} \tag{C.1}$$

$$\left| \mathbf{p}_q^{(1)} \cdot \mathbf{x}_q^{(1)} \leq \mathbf{p}_q^{(1)} \cdot \mathbf{z}_i^{(1)} \text{ for all } (\mathbf{x}_q^{(1)}, \mathbf{p}_q^{(1)}) \notin B_1 \text{ such that } r_{iq}^{(1)} = 1 \right. \tag{C.2}$$

The above sequence returns a set $\widehat{D}^{(1)}$ consistent with the subutility maximization program and a coherent transitive closure matrix $\widehat{\mathbf{R}}^{(1)}$. To simplify, redefine $\widehat{D}^{(1)} = \{(\boldsymbol{\xi}_i, \mathbf{p}_i^{(1)})\}_{i=1}^T$, where $\xi_i = (\xi_{i1}, \xi_{i2}, \ldots, \xi_{ia})' = \widehat{\mathbf{z}}_i^{(1)}$ for an adjusted bundle or $\xi_i = (\xi_{i1}, \xi_{i2}, \ldots, \xi_{ia})' = \mathbf{x}_i^{(1)}$ otherwise. Also, define the set $D = \{(\boldsymbol{\xi}_i, \mathbf{x}_i^{(2)}), (\mathbf{p}_i^{(1)}, \mathbf{p}_i^{(2)})\}_{i=1}^T$.

The next step is to produce a set consistent with the overall utility maximization program, such that the data remain consistent with the subutility maximization program. This is done by using a similar iterative procedure, simply adding an additional constraint based on the transitive closure matrix $\widehat{\mathbf{R}}^{(1)}$. The procedure is as follows:

Step 2.1. Test $D = \{(\boldsymbol{\xi}_i, \mathbf{x}_i^{(2)}), (\mathbf{p}_i^{(1)}, \mathbf{p}_i^{(2)})\}_{i=1}^T$ for consistency with GARP; let $\mathbf{R}$ be the transitive closure matrix and r_{ij} be an element at the ith row and jth column. Define nvio as the number of violations:

$$\text{if } \begin{cases} \text{nvio} = 0 \text{ then stop the iterative procedure,} \\ \text{otherwise go to step 2.2.} \end{cases}$$

Step 2.2. Because $(\boldsymbol{\xi}_i, \mathbf{x}_i^{(2)}) V R (\boldsymbol{\xi}_j, \mathbf{x}_j^{(2)})$ implies $(\boldsymbol{\xi}_i, \mathbf{x}_i^{(2)}) S R (\boldsymbol{\xi}_j, \mathbf{x}_j^{(2)})$, build B_l set(s), $l = 1, \ldots, n$ such that every couple $\{(\boldsymbol{\xi}_i, \mathbf{x}_i^{(2)}), (\mathbf{p}_i^{(1)}, \mathbf{p}_i^{(2)})\}$ and $\{(\boldsymbol{\xi}_j, \mathbf{x}_j^{(2)}), (\mathbf{p}_j^{(1)}, \mathbf{p}_j^{(2)})\}$ belonging to B_l, $\forall l \in \{1, \ldots, n\}$ satisfies $\{(\boldsymbol{\xi}_i, \mathbf{x}_i^{(2)}), (\mathbf{p}_i^{(1)}, \mathbf{p}_i^{(2)})\} S R \{(\boldsymbol{\xi}_j, \mathbf{x}_j^{(2)}), (\mathbf{p}_j^{(1)}, \mathbf{p}_j^{(2)})\}$. Search for the one, B_1, containing bundles violating GARP being at the same highest place in the preference chain such that for $\{(\boldsymbol{\xi}_i, \mathbf{x}_i^{(2)}), (\mathbf{p}_i^{(1)}, \mathbf{p}_i^{(2)})\} \in B_1$ and $\{(\boldsymbol{\xi}_j, \mathbf{x}_j^{(2)}), (\mathbf{p}_j^{(1)}, \mathbf{p}_j^{(2)})\} \notin B_1 : S(i) > S(j)$. Go to step 2.3.

Step 2.3. In the set B_1, solve (A.2) subject to (C.1), (C.2), (C.3), and (C.4) for each bundle related by $(\boldsymbol{\xi}_i, \mathbf{x}_i^{(2)}) R (\boldsymbol{\xi}_j, \mathbf{x}_j^{(2)})$ and $(\boldsymbol{\xi}_j, \mathbf{x}_j^{(2)}) P^0 (\boldsymbol{\xi}_i, \mathbf{x}_i^{(2)})$. Select the bundle for which the objective function is minimal. Let $\{(\widehat{\mathbf{z}}_i^{(1)}, \widehat{\mathbf{z}}_i^{(2)}), (\mathbf{p}_i^{(1)}, \mathbf{p}_i^{(2)})\}$ be the bundle solution. Replace in D $\{(\boldsymbol{\xi}_i, \mathbf{x}_i^{(2)}), (\mathbf{p}_i^{(1)}, \mathbf{p}_i^{(2)})\}$ by $\{(\widehat{\mathbf{z}}_i^{(1)}, \widehat{\mathbf{z}}_i^{(2)}), (\mathbf{p}_i^{(1)}, \mathbf{p}_i^{(2)})\}$. Go to step 2.1.

$$\text{obj}_i = \min_{z_{ij}^{(1)}, z_{ij}^{(2)}} \left[\sum_{j=1}^{a} \left(x_{ij} - z_{ij}^{(1)} \right)^2 + \sum_{j=a+1}^{k} \left(x_{ij} - z_{i(j-a)}^{(2)} \right)^2 \right], \tag{A.2}$$

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subject to

$$\left|\begin{array}{l}\mathbf{p}_i^{(1)}\cdot\boldsymbol{\xi}_i+\mathbf{p}_i^{(2)}\cdot\mathbf{x}_i^{(2)}=\mathbf{p}_i^{(1)}\cdot\mathbf{z}_i^{(1)}+\mathbf{p}_i^{(2)}\cdot\mathbf{z}_i^{(2)},\\ \mathbf{p}_j^{(1)}\cdot\boldsymbol{\xi}_j+\mathbf{p}_j^{(2)}\cdot\mathbf{x}_j^{(2)}\le\mathbf{p}_j^{(1)}\cdot\mathbf{z}_i^{(1)}+\mathbf{p}_j^{(2)}\cdot\mathbf{z}_i^{(2)},\\ \mathbf{p}_m^{(1)}\cdot\boldsymbol{\xi}_m+\mathbf{p}_m^{(2)}\cdot\mathbf{x}_m^{(2)}\le\mathbf{p}_m^{(1)}\cdot\mathbf{z}_i^{(1)}+\mathbf{p}_m^{(2)}\cdot\mathbf{z}_i^{(2)}\\ \text{for all observations: } (\boldsymbol{\xi}_m,\mathbf{x}_m^{(2)}) \text{ satisfying } (\boldsymbol{\xi}_i,\mathbf{x}_i^{(2)})VR(\boldsymbol{\xi}_m,\mathbf{x}_m^{(2)}),\ m\neq j,\end{array}\right. \tag{C.1}$$

$$\left|\begin{array}{l}\mathbf{p}_q^{(1)}\cdot\boldsymbol{\xi}_q+\mathbf{p}_q^{(2)}\cdot\mathbf{x}_q^{(2)}\le\mathbf{p}_q^{(1)}\cdot\mathbf{z}_i^{(1)}+\mathbf{p}_q^{(2)}\cdot\mathbf{z}_i^{(2)}\quad\text{for all }\left(\boldsymbol{\xi}_q^{(1)},\mathbf{x}_q^{(2)}\right)\\ \text{such that } r_{iq}=1;\end{array}\right. \tag{C.2}$$

$$\left|\mathbf{p}_i^{(1)}\cdot\boldsymbol{\xi}_i=\mathbf{p}_i^{(1)}\cdot\mathbf{z}_i^{(1)};\right. \tag{C.3}$$

$$\left|\mathbf{p}_r^{(1)}\cdot\boldsymbol{\xi}_r\le\mathbf{p}_r^{(1)}\cdot\mathbf{z}_i^{(1)}\quad\text{for all }\left(\boldsymbol{\xi}_r,\mathbf{p}_r^{(1)}\right)\text{ such that }\widehat{r}_{ir}^{(1)}=1,\right. \tag{C.4}$$

where $z_i^{(1)}=(z_{i1}^{(1)},z_{i2}^{(1)},\ldots,z_{ia}^{(1)})'$ and $z_i^{(2)}=(z_{i1}^{(2)},z_{i2}^{(2)},\ldots,z_{i(k-a)}^{(2)})'$.

The above sequence returns a set $\widehat{D}$ consistent with both the subutility maximization program and the overall utility maximization program, such that the L2 norm with the observed quantities is minimal.

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