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A Tutte Polynomial for Maps

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We follow the example of Tutte in his construction of the dichromate of a graph (*i.e.* the Tutte polynomial) as a unification of the chromatic polynomial and the flow polynomial in order to construct a new polynomial invariant of maps (graphs embedded in orientable surfaces). We call this the surface Tutte polynomial. The surface Tutte polynomial of a map contains the Las Vergnas polynomial, the Bollobás–Riordan polynomial and the Krushkal polynomial as specializations. By construction, the surface Tutte polynomial includes among its evaluations the number of local tensions and local flows taking values in any given finite group. Other evaluations include the number of quasi-forests.

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1. Introduction

Inspired by Tutte's construction of the dichromate of a graph, we construct a similarly defined polynomial invariant of maps (graphs embedded in an orientable surface), which we call the *surface Tutte polynomial*. For a plane map the surface Tutte polynomial is essentially the Tutte polynomial of the underlying planar graph; for non-plane maps it includes the Tutte polynomial as a specialization. Moreover the surface Tutte polynomial includes as a specialization the Las Vergnas polynomial, the Bollobás–Riordan polynomial and the Krushkal polynomial of a graph embedded in an orientable surface. The surface Tutte polynomial has evaluations that count local flows and tensions of a map taking values in a finite non-abelian group, comparable in this way to the Tutte polynomial, which has specializations counting abelian flows and tensions of a graph. We also give some other topologically significant evaluations of the surface Tutte polynomial, such as the number of quasi-forests.

1.1. Colourings, tensions and flows of graphs

Tutte [33, 35] defined the *dichromate* of a graph Γ (later to become known as the Tutte polynomial) as a bivariate generalization of the chromatic polynomial of Γ and the flow polynomial of Γ. The Tutte polynomial can be defined more generally for matrices (not just adjacency matrices of graphs), and in greater generality for matroids. The reader is referred to [2, 3, 12, 37, 38] for more on the Tutte polynomial.

The chromatic polynomial of a graph Γ evaluated at a positive integer n counts the number of proper colourings of Γ using at most n colours. The flow polynomial of Γ evaluated at a positive integer n counts the number of nowhere-zero $\mathbb{Z}_n$-flows of Γ (assignments of non-zero elements of $\mathbb{Z}_n$ to the edges of Γ with a fixed arbitrary orientation so that Kirchhoff's law is satisfied at each vertex). Proper colourings of the vertices of Γ using colours from $\mathbb{Z}_n$ may be described in terms of nowhere-zero $\mathbb{Z}_n$-tensions of Γ, which are assignments of non-zero elements of $\mathbb{Z}_n$ to the edges of Γ, with the property that for every closed walk in the oriented graph Γ the sum of values assigned to the forward edges equals the sum of values on the backward edges. A proper colouring of vertices of Γ with elements of $\mathbb{Z}_n$ gives a nowhere-zero $\mathbb{Z}_n$-tension of Γ by taking the difference between two endpoint colours of an edge (in the order specified by the edge orientation). Conversely, given a nowhere-zero $\mathbb{Z}_n$-tension of a connected graph Γ, there are exactly n proper colourings of the vertices of Γ using colours from $\mathbb{Z}_n$ which yield the given nowhere-zero $\mathbb{Z}_n$-tension upon taking differences along edges in this manner.

Tutte [33] showed that flows and tensions of a graph Γ using non-zero values from a finite additive abelian group G of order n are in bijective correspondence with nowhere-zero $\mathbb{Z}_n$-flows and nowhere-zero $\mathbb{Z}_n$-tensions of Γ, counted by $|T(\Gamma;0,1-n)|$ and $|T(\Gamma;1-n,0)|$, respectively. While colourings and flows are dual notions for planar graphs (the G-tensions of a plane graph correspond to the G-flows of the dual plane graph), for general graphs duality resides at the level of the cycle and cocycle matroids of the graph: the module of G-tensions of a non-planar graph does not correspond to the module of G-flows of a graph. (This follows from Whitney's matroid characterization of planar graphs [39].)

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1.2. Non-abelian flows and tensions of maps

In the combinatorial literature non-abelian flows seem only to have been considered by DeVos [10]. There are two significant differences between the abelian and the non-abelian case for defining flows and tensions for a graph Γ, which are most easily illustrated by considering flows.

The first is that for an abelian group the Kirchhoff condition for a flow requires that the sum of values on incoming edges is equal to the sum of values on outgoing edges where elements can be added together in any order. However, for a non-abelian group the order in which the group elements are composed matters: the edges incident with a common vertex need to be ordered so as to ensure the flow condition is well-defined. This order on edges around a vertex only matters up to cyclic permutation. Therefore the first change going from abelian to non-abelian flows is to attach a cyclic order of edges incident with a common vertex. This is equivalent to specifying an embedding of the graph in an orientable surface. Here we are moving from graphs to maps.

The second difference is that, for an abelian group, if the Kirchhoff condition for a flow is satisfied at each vertex then, for any cutset of edges (not necessarily defined by a single vertex), the flow values in one direction have the same sum as the flow values in the reverse direction. The same is not necessarily true for an assignment of values to edges from a non-abelian group: to begin with, it is not clear how to define a cyclic order of edges in an arbitrary cutset using just the cyclic orders of edges around vertices given by an orientable embedding, and, furthermore, Kirchhoff's condition may be satisfied at each vertex but there may be edge cutsets for which the product of values on outgoing edges does not equal that on incoming edges (no matter in which order the edges are taken). Flows on maps are thus defined locally, in the sense that the Kirchhoff condition is satisfied at each vertex, but does not necessarily extend to other edge cutsets.

The dual case to flows is that of tensions taking values in a non-abelian group. For a local tension, rather than requiring for every closed walk that the product of values taken in order around the walk, in which values on backward edges are inverted, is equal to the identity, we just require this condition for facial walks. A cyclic order of edges around a face is given by taking the edges in the order they are encountered when walking around the face (whether the walk is clockwise or anticlockwise is immaterial to the definition of a tension). Indeed, a cyclic order can be produced for any connected Eulerian subgraph (subgraph with vertices all of even degree) by traversing an Eulerian tour; for cycles (connected and vertices all of degree 2) this gives a unique cyclic order of edges up to reversal. Using the fact that an Eulerian subgraph is a union of cycles and isolated vertices, we may define a global tension of a graph taking values in a non-abelian group, in the sense that the product of values around any cycle must make the identity (values on backward edges are inverted). Just as for abelian groups, such global tensions correspond to vertex colourings of the graph (this time using elements of a non-abelian group as vertex colours). Local tensions do not have this correspondence, except in the case of planar graphs. However, very recently Litjens and Sevenster [25] showed that there is a correspondence between local tensions of a map and vertex colourings of a certain covering graph of the underlying graph of the map. We also note that the dual of a global non-abelian tension is not defined for graphs except in the case of planar graphs, whereas the dual of a local non-abelian tension of a map is a local non-abelian flow of a map.

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1.3. The surface Tutte polynomial

We now turn to the definition of the surface Tutte polynomial of a map. For a map M that is a 2-cell embedding of a graph Γ in an orientable surface, we let $v(M)$, $e(M)$, $k(M)$ equal respectively the number of vertices, edges and connected components of Γ (each connected component is embedded in its own surface), $f(M)$ the sum of the number of faces in the embeddings of each component of Γ, and $g(M)$ the sum of the genera of the surfaces in which the connected components of Γ are embedded. We define the *surface Tutte polynomial* of a map M (see Definition 3.12 below) to be the multivariate polynomial in variables $x, y, \mathbf{x} = (x_0, \dots, x_{g(M)}), \mathbf{y} = (y_0, \dots, y_{g(M)})$ given by

$$\mathcal{T}(M;\mathbf{x},\mathbf{y}) = \sum_{A \subseteq E} x^{e(M/A)-f(M/A)+k(M/A)} y^{e(M\backslash A^c)-v(M\backslash A^c)+k(M\backslash A^c)} \prod_{\substack{\text{conn. cpts} \\ M_i \text{ of } M/A}} x_{g(M_i)} \prod_{\substack{\text{conn. cpts} \\ M_j \text{ of } M\backslash A^c}} y_{g(M_j)},$$

where $A^c = E \setminus A$ for $A \subseteq E$, M/A is the map obtained by contracting the edges in A and $M\backslash A^c$ is the map restricted to edges in A. (Contraction and deletion of edges in maps are defined in Section 3.1 below.)

We show in Section 5 that the surface Tutte polynomial has evaluations counting nowhere-identity non-abelian local flows and tensions. It moreover contains as a specialization other polynomial invariants that have been defined for maps, notably the Las Vergnas polynomial [14, 22, 23], the Bollobás–Riordan polynomial [4, 5] and the Krushkal polynomial [20] (see Section 3.3 below). There are maps M with different surface Tutte polynomials but equal Krushkal polynomials (see Section 6.1 below). The fact that the surface Tutte polynomial has evaluations counting nowhere-identity non-abelian local flows and tensions recommends it as a natural translation of the Tutte polynomial for graphs (in its guise as the dichromate) to a Tutte polynomial for maps.

Setting $x_g = a^g$ and $y_g = b^g$ in $\mathcal{T}(M;\mathbf{x},\mathbf{y})$ for $g = 0, 1, \dots, g(M)$ gives the following quadrivariate polynomial specialization of the surface Tutte polynomial (Definition 4.10):

$$\mathcal{Q}(M;x,y,a,b) = \sum_{A \subseteq E} x^{v(M/A)-k(M/A)+2g(M/A)} y^{e(M\backslash A^c)-v(M\backslash A^c)+k(M\backslash A^c)} a^{g(M/A)} b^{g(M\backslash A^c)}.$$

Remark 1.1. Δ-matroids are to maps as matroids are to graphs [6, 9]. The surface Tutte polynomial $\mathcal{T}(M;\mathbf{x},\mathbf{y})$ cannot be extended to Δ-matroids because its definition involves the genera of the connected components of M and cannot be made independent of these; this is similar to how the U-polynomial [28] multivariate generalization of the Tutte polynomial of a graph Γ does not lift to matroids more generally as its definition ineluctably involves the ranks of the connected components of Γ. The polynomial $Q(M;x,y,a,b)$ can, however, be defined for Δ-matroids (see Remark 4.11 below).

Remark 1.2. The surface Tutte polynomial of a map is in an unbounded number of variables, just as is the case for Tutte's V-function [31, 33] of a graph $\Gamma = (V,E)$. Tutte's universal V-function is a polynomial in a sequence of commuting indeterminates $\mathbf{y} = (y_0, y_1, \dots)$ defined by

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the subgraph expansion

$$V(\Gamma;\mathbf{y}) = \sum_{A\subseteq E} \prod_{\substack{\text{conn. cpts } C_i \\ \text{of } \Gamma\setminus A}} y_{n(C_i)},$$

where $n(C_i) = |E(C_i)| - |V(C_i)| + 1$ is the nullity of the ith connected component C_i of the subgraph $\Gamma\setminus A$ (in some arbitrary ordering of connected components). Up to a prefactor, the Tutte polynomial is obtained from the universal V-function by the specialization $y_n = (x-1)(y-1)^n$. Tutte showed that the V-function is universal for graph invariants satisfying a deletion–contraction recurrence for non-loop edges, multiplicative over disjoint unions, and specified by boundary values on graphs all the edges of which are loops. Examples of V-functions that are not a specialization of the Tutte polynomial have not been so widely studied, but see for example [36].

1.4. Organization of the paper

We begin in Section 2 by recalling the definition of the Tutte polynomial of a graph and its specializations to the chromatic polynomial and the flow polynomial. In Section 3 we give the relevant background to orientably embedded graphs (maps), deferring some of the proofs to the Appendix. We then formally define the surface Tutte polynomial of a map and state our main results. We derive some basic properties of the surface Tutte polynomial and show how it specializes to the Bollobás–Riordan polynomial, the Krushkal polynomial and the Las Vergnas polynomial of a map, as well as to the Tutte polynomial of the underlying graph. In Section 4 we state the evaluations of the surface Tutte polynomial that give the number of nowhere-identity local flows (or tensions) taking values in any given finite group, leaving proofs to Section 5. We then derive further specializations of the surface Tutte polynomial, such as the number of quasi-forests. In Section 5 we enumerate non-abelian local flows and tensions. Finally, in Section 6 we review our results in the context of other work and identify some directions for future research.

2. The Tutte polynomial for graphs

2.1. Graphs

A graph $\Gamma = (V,E)$ is given by a set of vertices V and a set of edges E, together with an incidence relation between vertices and edges such that any edge $e \in E$ is either incident to two different vertices $u,v \in V$ or is incident 'twice' to the same vertex $v \in V$. In the latter case e is called a *loop*. If several edges are incident with the same pair of vertices u,v then they are called *multiple edges*. The *degree* $\deg(v)$ of a vertex v is the number of edges incident with it (any loop incident with the vertex is counted twice).

The graph $\Gamma\setminus e$ obtained from Γ by *deletion* of e is the graph $(V, E\setminus\{e\})$. The graph Γ/e obtained from Γ by *contraction* of e is defined by first deleting e and then identifying its endpoints. Contracting a loop of a graph coincides with deleting it.

Definition 2.1. For a graph Γ we let $v(\Gamma)$, $e(\Gamma)$, $k(\Gamma)$ denote the number of vertices, edges and connected components of Γ. The *rank* of Γ is defined by

$$r(\Gamma) = v(\Gamma) - k(\Gamma),$$

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and the *nullity* of Γ by

$$n(\Gamma) = e(\Gamma) - r(\Gamma) = e(\Gamma) - v(\Gamma) + k(\Gamma).$$

An edge e is a loop of Γ precisely when $n(\Gamma/e) = n(\Gamma) - 1$; an edge e is a *bridge* of Γ (deleting e disconnects the connected component of Γ to which it belongs) precisely when $r(\Gamma\backslash e) = r(\Gamma) - 1$.

2.2. The Tutte polynomial

The Tutte polynomial of a graph $\Gamma = (V, E)$ is defined by the subgraph expansion

$$T(\Gamma; x, y) = \sum_{A \subseteq E} (x-1)^{r(\Gamma) - r(\Gamma\backslash A^c)} (y-1)^{n(\Gamma\backslash A^c)}, \tag{2.1}$$

where $A^c = E\backslash A$ is the complement of $A \subseteq E$.

The Tutte polynomial contains the chromatic polynomial as a specialization. In particular, the number of nowhere-zero $\mathbb{Z}_n$ tensions of Γ is given by $(-1)^{r(\Gamma)} T(\Gamma; 1-n, 0)$.

Dually, the flow polynomial $\varphi(\Gamma; z)$ evaluated at $n \in \mathbb{N}$ is equal to the number of nowhere-zero $\mathbb{Z}_n$-flows of Γ and is given by $\varphi(\Gamma; z) = (-1)^{n(\Gamma)} T(\Gamma; 0, 1-z)$.

3. A Tutte polynomial for maps

3.1. Graph embeddings and maps

For embeddings of graphs in surfaces we follow [21]. See also [13].

A *surface* in this paper is a compact oriented two-dimensional topological manifold. Such orientable surfaces are classified by a non-negative integer parameter, called the *genus* g of the surface (the number of 'handles', or 'doughnut holes'); thus the sphere has genus 0 and the torus has genus 1. Surfaces are not only orientable but have been given a fixed orientation, which in particular allows one to distinguish left and right relative to a directed line.

Definition 3.1. A *connected map* M is a graph Γ embedded into a connected surface Σ (*i.e.* considered as a subset $\Gamma \subset \Sigma$) in such a way that

(1) vertices are represented as distinct points in the surface,
(2) edges are represented as continuous curves in the surface that intersect only at vertices,
(3) cutting the surface along the graph thus drawn, what remains (*i.e.* the set $\Sigma \setminus \Gamma$) is a disjoint union of connected components, called *faces*. Each face is homeomorphic to an open disk.

The graph Γ is said to be the *underlying graph* of M.

A map is also known as an *orientably embedded graph*, an *orientable ribbon graph* or *cyclic graph*, with the attendant variations in the diagrammatic representation of maps. (See [13] and the references therein.) Below we shall use Tutte's permutation approach.

A contractible closed curve in a surface Σ is one that can be continuously deformed (or contracted) in Σ to a single point. A cycle C of a graph Γ embedded in Σ is contractible in Σ if the subgraph $(V(C), E(C))$ of Γ forms a contractible closed curve in Σ. A region of a surface is a 2-cell if its boundary is a contractible cycle. An embedding of a graph Γ into a surface

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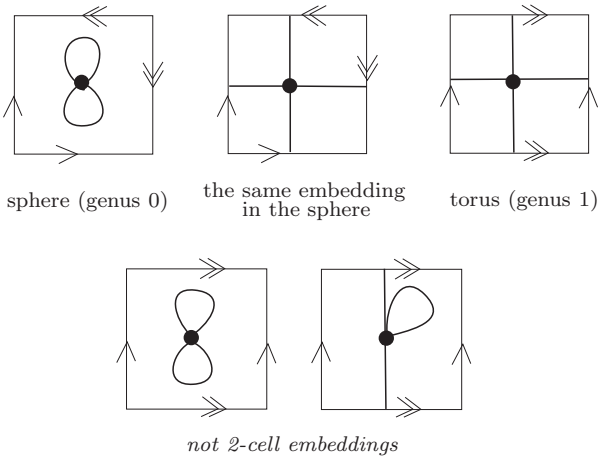


Figure 1. Embeddings of the graph consisting of two loops on a single vertex, in the sphere and the torus. (Edges of the square with matching arrows are glued together.) The lower pair of embeddings are not 2-cell embeddings as in each of them one of the faces is not homeomorphic to an open disk.

Σ subject to condition (3) in Definition 3.1 (that each of the connected components of $\Sigma \setminus \Gamma$ is homeomorphic to an open disk) is called a *2-cell embedding* of Γ and has the property that each face is a 2-cell. See Figure 1. A graph must be connected in order for it to have a 2-cell embedding.

To a map M embedding a graph $\Gamma = (V,E)$ in a surface Σ we identify the vertices and edges of Γ with their representations in Σ and let F be the set of faces of M. A face is identified with the subset of edges forming its boundary. We will denote M by the triple (V,E,F).

We now introduce Tutte's permutation axiomatization for maps, which allows us to rigorously define deletion and contraction of edges in maps, and which also gives the most convenient way of defining local flows and local tensions of maps for our purposes.

3.1.1. Tutte's permutation axiomatization for connected maps. We follow Chapter X of Tutte's monograph [34] with some changes in notation and terminology. Tutte uses the term *oriented map* for our term *map*; we shall silently use his Theorem X.13 and the discussion before it in order to translate statements for his more generally defined maps (orientable or non-orientable) to our case of oriented maps.

Let D be a set of even size, elements of which will be called *darts*. We write permutations of D as functions, composing them from right to left; as parentheses will feature in the cycle notation for orbits of a permutation, however, we drop the usual parentheses around the argument of a function and write πd for $\pi(d)$. The identity permutation is denoted by ι.

Definition 3.2. A *connected map* M is an ordered pair (α,τ) of permutations acting on a set D of $2m$ elements, such that

(1) $\alpha^2 = \iota$,
(2) for any $d \in D$ the elements d and αd are distinct,
(3) the permutation group $\langle \alpha,\tau \rangle$ acts transitively on D.

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Conditions (1) and (2) stipulate that α is a fixed-point-free involution, and condition (3) that the map is connected. A *map* is a disjoint union of connected maps.

The orbit of a permutation π of D with $d \in D$ is denoted by $\begin{pmatrix} d & \pi d & \cdots & \pi^{-1}d \end{pmatrix}$, and will be regarded interchangeably as the set of elements $\{d, \pi d, \ldots, \pi^{-1}d\}$ permuted transitively by π and as the cyclic permutation equal to π restricted to this set of elements.

An orbit $\begin{pmatrix} d & \alpha d \end{pmatrix}$ of α is an *edge*, and an orbit $\begin{pmatrix} d & \tau d & \cdots & \tau^{-1}d \end{pmatrix}$ of τ a *vertex* of M. An edge and a vertex are *incident* if some dart belongs to both. An edge $\begin{pmatrix} d & \alpha d \end{pmatrix}$ is a *loop* of M if d and αd belong to the same vertex, and a *link* of M otherwise. With this definition of incidence, the edges, vertices, loops and links of M are the edges, vertices, loops and links, respectively, of a graph $\Gamma(M)$, which we shall call the *underlying graph* of M. By Edmonds' construction [11] the pair (α, τ) induces a 2-cell embedding of $\Gamma(M)$ in a compact orientable surface. Conversely, any connected graph Γ that is 2-cell embedded in a compact orientable surface gives rise to a map $M = (\alpha, \tau)$ with underlying graph isomorphic to Γ: the orbits of τ are formed by the vertex rotations on Γ, which are cyclic orders of half-edges around vertices, and the orbits of α correspond to gluing the two half-edges comprising an edge together. A half-edge of a graph Γ corresponds to a dart in the permutation definition of the map with underlying graph Γ.

Tutte's permutation axiomatization of connected maps gives a representation of graphs embedded in orientable surfaces. It allows for a particularly simple representation of the surface dual of a map (formed by putting a vertex in each face and connecting two vertices by an edge if their faces share an edge).

Theorems X.18, X.19 and X.22 of [34] show that if $M = (\alpha, \tau)$ is a connected map then $M^* = (\alpha, \tau\alpha)$ is also a connected map. The vertices of M^*, equal to the orbits of the permutation $\varphi = \tau\alpha$, are called the *faces* of M. The connected map M^* is the *dual* of M. Since $\varphi\alpha = \tau$, the dual of M^* is M. See [34, Sections X.3 and X.5] for how this corresponds to the geometric surface dual of M; the face rotations of M have the opposite sense to the vertex rotations of M.

3.1.2. Maps and map parameters.

Definition 3.3. A *map* M is a disjoint union of connected maps, one for each connected component of the underlying graph Γ: each connected component of Γ is embedded as a connected map into its own surface. If $M_i = (\alpha_i, \tau_i)$ is a connected map with set of darts D_i, for $i = 1, \ldots, k$, we let $D = D_1 \sqcup \ldots \sqcup D_k$ and define the disjoint union $M = M_1 \sqcup \ldots \sqcup M_k$ to be the map (α, τ), where α (resp. τ) is the permutation of D that restricted to D_i is equal to α_i (τ_i).

From now on we will interchangeably use the two representations $M = (V, E, F)$ and $M = (\alpha, \tau)$ when we speak about a map M. The *dual* of a map M is the union of the dual of its components and is denoted by M^*. It follows that if $M = (\alpha, \tau)$ then $M^* = (\alpha, \tau\alpha)$.

A map $M_1 = (\alpha_1, \tau_1)$, with α_1, τ_1 permutations of D_1, and a map $M_2 = (\alpha_2, \tau_2)$, with α_2, τ_2 permutations of D_2, are *isomorphic* if there exists a bijection $\beta : D_1 \to D_2$ such that $\beta\alpha_1 = \alpha_2$ and $\beta\tau_1 = \tau_2$. A map parameter $P = P(M)$ is said to be a *map invariant* if $P(M_1) = P(M_2)$ whenever M_1 and M_2 are isomorphic. We introduce notation for a number of map invariants that we require below.

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Definition 3.4. The *genus* of a connected map M is the genus of the connected surface Σ in which it is embedded and is denoted by $g(M)$. The genus of a graph is the minimum genus of an embedding of Γ as a map.

In particular, graphs of genus 0 are called *planar*, the maps witnessing this being *plane graphs* (or plane maps). Plane maps are viewed rather as embeddings of planar graphs in the sphere (the unbounded outer face in the plane becomes bounded once the plane has a point at infinity added to make it a sphere). For a given graph Γ there in general exist non-isomorphic maps with underlying graph Γ, and these maps may be of various genera. (The two maps in Figure 1 provide an example.)

For a connected map $M = (V,E,F)$, let $v(M) = |V|$, $e(M) = |E|$, and $f(M) = |F|$, and let $\chi(M) = |V| - |E| + |F|$ be its Euler characteristic. The well-known formula of Euler is that $\chi(M) = 2 - 2g(M)$ for a connected map M.

Definition 3.5. Let $M = M_1 \sqcup \cdots \sqcup M_k$ be a map, equal to the disjoint union of connected maps $M_1, \ldots, M_k$. We define $k(M) = k$ (the number of connected components of the underlying graph of M) and extend the map parameters v, e, f, g and χ additively over the disjoint union.

The *rank* and *nullity* of M are defined by

$$r(M) = v(M) - k(M), \quad n(M) = e(M) - v(M) + k(M),$$

and the *dual rank* and *dual nullity* by

$$r^*(M) = f(M) - k(M), \quad n^*(M) = e(M) - f(M) + k(M).$$

The number of vertices, edges and connected components and the rank and nullity of a map are parameters shared with those of its underlying graph (Definition 2.1) and we use the same notation for them.

Remark 3.6. The number of faces $f(M)$ in a map is the sum of the number of faces in the embeddings of components of M in disjoint surfaces: embedding a disconnected graph in one surface does not give a map (as there is a face not homeomorphic to an open disk).

By applying Euler's formula for connected maps to each connected component of a disconnected map M and using the additivity of the parameters v, e, f and g over disjoint unions, we have

$$\chi(M) = v(M) - e(M) + f(M) = 2k(M) - 2g(M). \tag{3.1}$$

This in turn implies

$$n^*(M) = r(M) + 2g(M). \tag{3.2}$$

Definition 3.7. A map M is a *quasi-tree* if $f(M) = 1$ and a *bouquet* if $v(M) = 1$.

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Having defined maps via permutations of darts we can now give the formal definitions of deletion and contraction that we need.

3.1.3. Deletion and contraction of edges in maps. It will be convenient to define the 'vertex-map' $M_\bullet$ with one vertex and no edges, which has an empty set of darts. (The underlying graph of $M_\bullet$ is an isolated vertex.) In order to simplify subsequent definitions in the case of $M_\bullet$ we consider α and τ to each be the unique permutation of the empty set, denoted in cycle notation by $()$. Tutte avoids consideration of the vertex-map $M_\bullet$, and in particular does not define deletion of an edge of a map when it produces a null cycle [34, Chapter X.7]. However, we shall need to extend Tutte's definition of deletion and contraction of edges in maps to such cases.

A linear sequence of darts in a permutation cycle beginning with dart a and ending with dart a' will be denoted by $a \cdots a'$. When this linear sequence forms part of the cyclic permutation

$$\gamma = (b \quad a \quad \cdots \quad a' \quad b' \quad \cdots)$$

we have $\gamma b = a$, $a' = \gamma^{-1}b'$. The sequence $a \cdots a'$ may be empty (in which case $\gamma b = b'$), or consist of just a single dart (in which case $\gamma b = a = a' = \gamma^{-1}b'$).

Definition 3.8 (deletion). Let $M = (\alpha, \tau)$ be a map with set of darts D, $e = (a \quad \alpha a)$ an edge of M, and $D' = D \setminus \{a, \alpha a\}$. Then the map $M\backslash e$ obtained by *deleting* e is defined as the map $M' = (\alpha', \tau')$ in which

(1) $\alpha' d = \alpha d$ for each $d \in D'$, and

(2) τ' is obtained from τ

(i) if e is a link by replacing the disjoint pair of cycles

$$(a \quad b \quad \cdots \quad b'), \quad (\alpha a \quad c \quad \cdots \quad c')$$

with the (possibly empty) cycles

$$(b \quad \cdots \quad b')(c \quad \cdots \quad c');$$

(ii) if e is a loop by replacing the cycle

$$(a \quad b \quad \cdots \quad b' \quad \alpha a \quad c \quad \cdots \quad c')$$

with the (possibly empty) cycle

$$(b \quad \cdots \quad b' \quad c \quad \cdots \quad c').$$

Deletion of an edge e in a map M corresponds to deleting the edge in its underlying graph Γ, although deletion of e may reduce the genus of the map, in which case the underlying graph $\Gamma\backslash e$ of $M\backslash e$ is embedded in a different surface to Γ.

We now turn to the definition of contracting an edge.

Definition 3.9 (contraction). Let $M = (\alpha, \tau)$ be a map with set of darts D, $e = (a \quad \alpha a)$ an edge of M, and $D' = D \setminus \{a, \alpha a\}$. Then the map M/e obtained by *contracting* e is the map $M'' = (\alpha', \tau'')$ in which

(1) $\alpha' d = \alpha d$ for each $d \in D'$, and

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(2) τ'' is obtained from τ

(i) if e is a link by replacing the disjoint pair of cycles

$$\left(a \quad b \quad \cdots \quad b'\right), \quad \left(\alpha a \quad c \quad \cdots \quad c'\right)$$

with the (possibly empty) cycle

$$\left(b \quad \cdots \quad b' \quad c \quad \cdots \quad c'\right);$$

(ii) if e is a loop by replacing the cycle

$$\left(a \quad b \quad \cdots \quad b' \quad \alpha a \quad c \quad \cdots \quad c'\right)$$

with the product of the pair of (possibly empty) cycles

$$\left(b \quad \cdots \quad b'\right)\left(c \quad \cdots \quad c'\right).$$

Contraction of an edge in a map does not always correspond to contraction of the edge in the underlying graph. In particular, contracting a loop in a map has the effect of splitting in two the vertex with which it is incident. Contraction of an edge can be related to deletion of the edge in the dual map.

Proposition 3.10. *Let M be a map with edge e. Then the map M/e is isomorphic to $(M^*\backslash e)^*$.*

We postpone the proof of Proposition 3.10 to the Appendix.

Using the above definitions of deletion and contraction it is immediate that if $e \neq f$ are edges of M then $(M\backslash e)\backslash f \cong (M\backslash f)\backslash e$, $(M/e)/f \cong (M/f)/e$, and $M/e\backslash f \cong M\backslash f/e$. Consequently, if A and B are disjoint subsets of edges of M then we can unambiguously write $M\backslash A$, M/B and $M\backslash A/B$ for the maps obtained from M by respectively deleting the edges in A, contracting the edges in B, and both deleting edges from A and contracting edges in B, taking the edges in any order.

A *submap* of M is a map $M\backslash A$ obtained from M by deleting a subset of edges. A submap of M shares the same vertex set as M.

We finish this section with a technical lemma, the proof of which is given in the Appendix.

Lemma 3.11. *If M is a map and A a subset of edges then*

$$g(M\backslash A^c) + g(M/A) \leqslant g(M),$$

with equality if and only if

$$\begin{aligned} k(M\backslash A^c) - k(M) - f(M \setminus A^c) + k(M/A) &= 0 \quad \textit{and} \\ k(M/A) - k(M) - v(M/A) + k(M\backslash A^c) &= 0. \end{aligned}$$

A simple but useful corollary of Lemma 3.11 is that neither deletion nor contraction of edges increase the genus: $g(M\backslash A^c) \leqslant g(M)$ and $g(M/A) \leqslant g(M)$.

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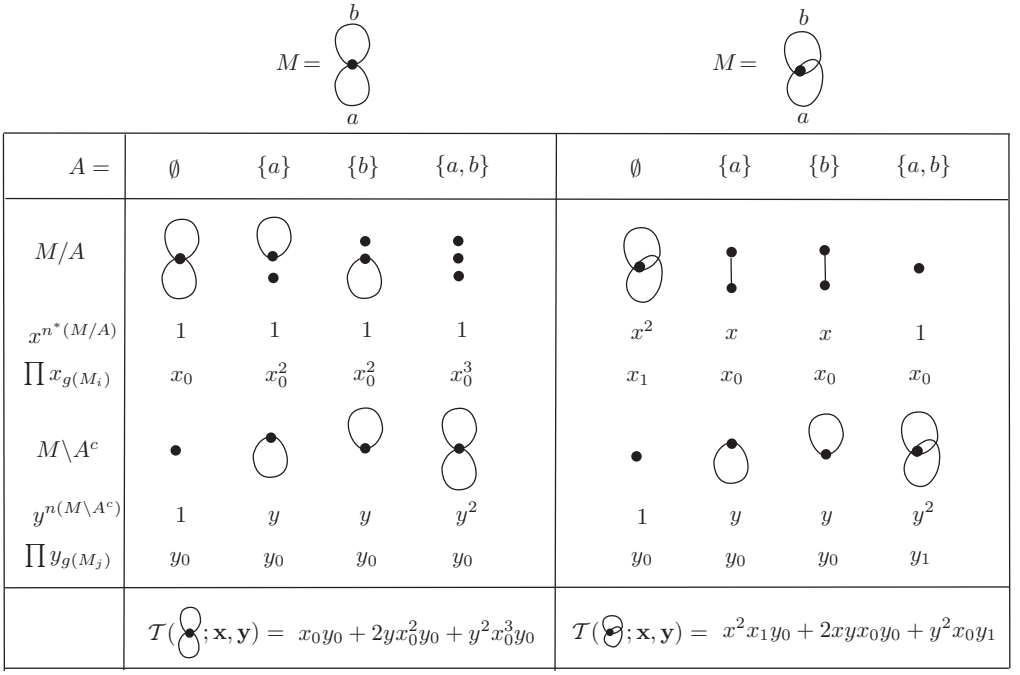


Figure 2. Calculating the surface Tutte polynomial of two loops on a single vertex, on the left embedded in the plane and on the right embedded in the torus.

3.2. A Tutte polynomial for maps

Definition 3.12. Let $\mathbf{x} = (x, x_0, x_1, x_2, \dots)$, $\mathbf{y} = (y, y_0, y_1, \dots)$ be two infinite sequences of commuting indeterminates.

Given a map $M = (V, E, F)$, the *surface Tutte polynomial* of M is the multivariate polynomial

$$\mathcal{T}(M; \mathbf{x}, \mathbf{y}) = \sum_{A \subseteq E} x^{n^*(M/A)} y^{n(M \setminus A^c)} \prod_{\substack{\text{conn. cpts } M_i \\ \text{of } M/A}} x_{g(M_i)} \prod_{\substack{\text{conn. cpts } M_j \\ \text{of } M \setminus A^c}} y_{g(M_j)}, \tag{3.3}$$

where $A^c = E \setminus A$ for $A \subseteq E$.

Remark 3.13. By Lemma 3.11, for a given map M, $\mathcal{T}(M; \mathbf{x}, \mathbf{y})$ is a polynomial in indeterminates $x, x_0, \dots, x_{g(M)}$ and $y, y_0, \dots, y_{g(M)}$.

Remark 3.14. In the summation (3.3) defining $\mathcal{T}(M; \mathbf{x}, \mathbf{y})$ the exponent of x is $n^*(M/A) = n((M/A)^*) = n(M^* \setminus A)$. (For convenience, and where no confusion can arise, we make the usual identification of the edges of the surface dual M^* with the edges of M.)

See Figure 2 for a small example of the calculations involved in computing $\mathcal{T}(M; \mathbf{x}, \mathbf{y})$ from its subset expansion (3.3).

The surface Tutte polynomial is multiplicative over the connected components of a map.

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Proposition 3.15. *For maps M_1 and M_2,*

$$\mathcal{T}(M_1 \sqcup M_2; \mathbf{x}, \mathbf{y}) = \mathcal{T}(M_1; \mathbf{x}, \mathbf{y})\mathcal{T}(M_2; \mathbf{x}, \mathbf{y}).$$

Proof. The nullity parameter $n(M)$ is additive over disjoint unions. The set of connected components of $(M_1 \sqcup M_2)/A$ is the disjoint union of the connected components of M_1/A_1 and those of M_2/A_2, where $A = A_1 \sqcup A_2$ with $A_1 \subseteq E(M_1)$ and $A_2 \subseteq E(M_2)$. Likewise for $(M_1 \sqcup M_2)\backslash A^c$. □

The surface Tutte polynomial behaves with respect to geometric duality in the same way as the Tutte polynomial of a matroid does with respect to matroid duality.

Proposition 3.16. *If M is a map and M^* its surface dual then*

$$\mathcal{T}(M^*; \mathbf{x}, \mathbf{y}) = \mathcal{T}(M; \mathbf{y}, \mathbf{x}).$$

Proof. This is immediate from Definition 3.12, the identities

$$n^*(M^*/A) = n(M\backslash A), \quad n(M^*\backslash A^c) = n^*(M/A^c) \quad \text{and} \quad g(M^*) = g(M),$$

and the fact that the connected components of M^*/A (resp. $M^*\backslash A^c$) are in one-one correspondence with and have the same genus as the connected components of its dual $M\backslash A$ (resp. M/A^c). □

The surface Tutte polynomial coincides with the Tutte polynomial for plane embeddings of planar graphs.

Proposition 3.17. *If M is a plane map embedding of a planar graph Γ then $\mathcal{T}(M; \mathbf{x}, \mathbf{y})$ is a polynomial in x, y, x_0, y_0 and*

$$\mathcal{T}(M; \mathbf{x}, \mathbf{y}) = (x_0 y_0)^{k(\Gamma)} T(\Gamma; y_0 x + 1, x_0 y + 1).$$

Proof. For a plane map M with underlying graph Γ we have $g(M) = 0$ and $g(M\backslash A^c) = 0 = g(M/A)$ for every $A \subseteq E$ (a consequence of Lemma 3.11). Hence $\mathcal{T}(M; \mathbf{x}, \mathbf{y})$ is a polynomial in x, y, x_0, y_0.

Using the defining subset expansion (3.3) for $\mathcal{T}(M; \mathbf{x}, \mathbf{y})$ and equation (3.2), we have

$$\mathcal{T}(M; \mathbf{x}, \mathbf{y}) = \sum_{A \subseteq E} x^{r(M/A)} y^{n(M\backslash A^c)} x_0^{k(M/A)} y_0^{k(M\backslash A^c)}. \tag{3.4}$$

We check that the exponents of x, y, x_0 and y_0 in equation (3.4) are, comparing with the defining expansion of the Tutte polynomial (2.1), respectively equal to $k(\Gamma\backslash A^c) - k(\Gamma), n(\Gamma\backslash A^c), k(\Gamma\backslash A^c)$ and $n(\Gamma\backslash A^c) + k(\Gamma)$.

The exponent of y is $n(M\backslash A^c) = n(\Gamma\backslash A^c)$. For the exponent of x, since $g(M) = 0$ and $g(M/A) + g(M\backslash A^c) \leqslant g(M)$ we have $g(M/A) + g(M\backslash A^c) = g(M)$, whence $v(M/A) - k(M/A) = k(M\backslash A^c) -$

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$k(M)$ by Lemma 3.11. This implies

$$\begin{aligned} r(M/A) &= k(M\backslash A^c) - k(M) \\ &= k(\Gamma\backslash A^c) - k(\Gamma). \end{aligned}$$

The exponent of y_0 in equation (3.4) is equal to

$$k(M \setminus A^c) = k(\Gamma\backslash A^c) = [k(\Gamma\backslash A^c) - k(\Gamma)] + k(\Gamma).$$

The identity $k(M \setminus A^c) - k(M) = r(M/A)$ in dual form states that $k(M/A) - k(M) = n(M \setminus A^c)$ for each $A \subseteq E$, from which we find that the exponent of x_0 is equal to

$$k(M/A) = n(M\backslash A^c) + k(M) = n(\Gamma\backslash A^c) + k(\Gamma). \qquad \square$$

3.3. Relation to other map polynomials

The Krushkal polynomial [20] of a graph Γ embedded in an orientable surface Σ as a map M is, using the definition in [7, 9], Lemma 4.1 of [1], and our notation, given by

$$\mathcal{K}(M;x,y,a,b) = \sum_{A\subseteq E} (x-1)^{k(M\backslash A^c)-k(M)} y^{n(M\backslash A^c)} a^{g(M/A)} b^{g(M\backslash A^c)}.$$

The Bollobás–Riordan polynomial [4] of a graph Γ orientably embedded in Σ as a map M is defined by

$$\mathcal{R}(M;x,y,z) = \sum_{A\subseteq E} (x-1)^{k(M\backslash A^c)-k(M)} y^{n(M\backslash A^c)} z^{2g(M\backslash A^c)},$$

and is obtained from the Krushkal polynomial [20] by setting $a = 1, b = z^2$ in $\mathcal{K}(M;x,y,a,b)$. (Bollobás and Riordan defined their polynomial more generally for 2-cell embeddings of graphs in non-orientable surfaces [5], and Butler extended the Krushkal polynomial to 2-cell embeddings of graphs in non-orientable surfaces [7].)

The Las Vergnas polynomial [23] of a map M is shown in [1] and Proposition 3.3 of [14] to be given by, in our notation,

$$\mathcal{L}(M;x,y,z) = \sum_{A\subseteq E} (x-1)^{k(M\backslash A^c)-k(M)} (y-1)^{n(M\backslash A^c)-g(M)-g(M\backslash A^c)+g(M/A)} z^{g(M)-g(M\backslash A^c)+g(M/A)}.$$

The surface Tutte polynomial $\mathcal{T}(M;\mathbf{x},\mathbf{y})$ specializes to the Krushkal polynomial (and hence the Bollobás–Riordan polynomial of M, the Las Vergnas polynomial of M and the Tutte polynomial of the underlying graph of M), as may be verified by making the requisite substitutions.

Proposition 3.18. *The surface Tutte polynomial $T(M;\mathbf{x},\mathbf{y})$ in $\mathbf{x} = (x, x_0, x_1, x_2, \dots)$ and $\mathbf{y} = (y, y_0, y_1, y_2, \dots)$ has the following specializations:*

- *the Krushkal polynomial of a map M is given by*

$$\mathcal{K}(M;X,Y,A,B) = (X-1)^{-k(M)} \mathcal{T}(M;\mathbf{x},\mathbf{y}),$$

  in which $x = 1, x_g = A^g, y = Y, y_g = (X-1)B^g$, for $g = 0, 1, 2, \dots$,
- *the Bollobás–Riordan polynomial of a map M is given by*

$$\mathcal{R}(M;X,Y,Z) = (X-1)^{-k(M)} \mathcal{T}(M;\mathbf{x},\mathbf{y}),$$

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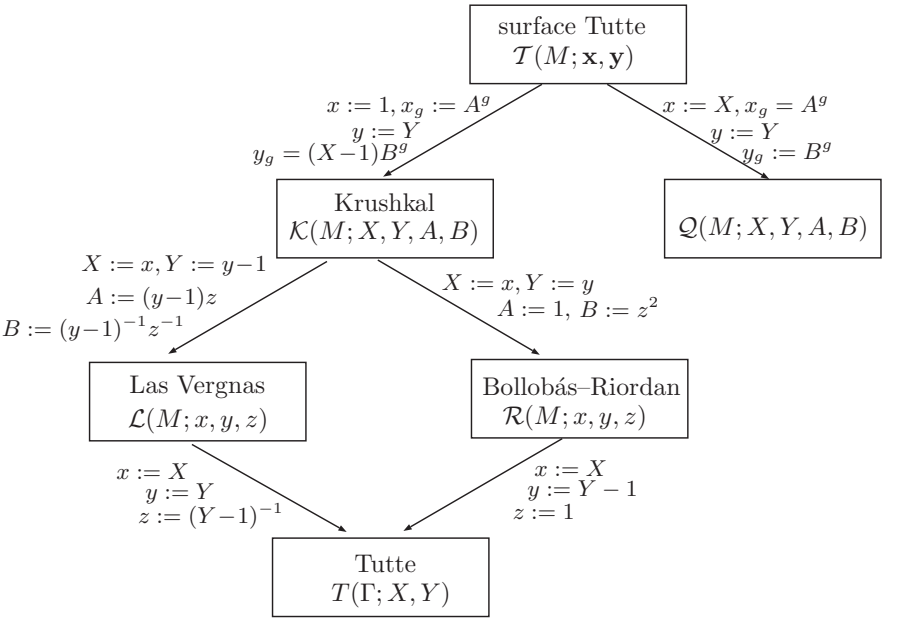


Figure 3. Specializations of the surface Tutte polynomial for a graph Γ embedded in an orientable surface Σ as a map M (prefactors, dependent only on $k(M)$ and $g(M)$, have been omitted: see Proposition 3.18.) When M is plane (Σ is the sphere) the hierarchy collapses to the Tutte polynomial of the underlying planar graph Γ.

in which $x = 1 = x_g$, $y = Y$ and $y_g = (X-1)Z^{2g}$, for $g = 0,1,2,\ldots$,

- *the Las Vergnas polynomial of a map M is given by*

$$\mathcal{L}(M;X,Y,Z) = (X-1)^{-k(M)}(Y-1)^{-g(M)}Z^{g(M)}\mathcal{T}(M;\mathbf{x},\mathbf{y}),$$

with $x = 1$, $x_g = (Y-1)^g Z^g$, $y = Y-1$, $y_g = (X-1)(Y-1)^{-g}Z^{-g}$, for $g = 0,1,2,\ldots$, and

- *the Tutte polynomial of a graph Γ is given by*

$$T(\Gamma;X,Y) = (X-1)^{-k(M)}\mathcal{T}(M;\mathbf{x},\mathbf{y}),$$

with $x = 1 = x_g$, $y = Y-1$, $y_g = X-1$, for $g = 0,1,2,\ldots$, and in which M is an arbitrary embedding of Γ as a map.

Figure 3 displays the relationship between the various map polynomials in Proposition 3.18 together with the polynomial $\mathcal{Q}(M;X,Y,A,B)$, defined at the end of Section 1.3 and which is considered in more detail in Section 4.3. The relationship between $\mathcal{Q}(M;X,Y,A,B)$ and the Krushkal polynomial is as yet unclear. In Section 6.1 we ask whether $\mathcal{Q}(M;X,Y,A,B)$ and the Krushkal polynomial are equivalent as map invariants (Problem 6.1), even though neither one appears to be a specialization of the other.

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4. Specializations

4.1. Flows and tensions

We begin by giving analogues for maps of the specializations of the Tutte polynomial of a graph to the chromatic polynomial and the flow polynomial described at the end of Section 2.2.

4.1.1. Local flows and local tensions defined for maps.

Definition 4.1 (local flow). Let $M = (\alpha, \tau)$ be a map on set of darts D, and let G be a finite group. A function $f : D \to G$ is a *local G-flow* of M if, for each $d \in D$,

(i) $f(\alpha d) = f(d)^{-1}$,
(ii) $f(d)f(\tau d)\cdots f(\tau^{-1}d) = 1$.

In other words, a local G-flow of M is an assignment of elements of G to the darts of M with the property that the product of values on each cycle $(d \quad \alpha d)$ of α and on each cycle $(d \quad \tau d \quad \cdots \quad \tau^{-1}d)$ of τ is equal to the identity.

Let x_d, $d \in D$, be variables taking values in G supplemented by the relations $x_{\alpha d} = x_d^{-1}$ for each $d \in D$. A local G-flow of M is a solution to a set of equations in variables x_d in which $x_d x_{\tau d} \cdots x_{\tau^{-1}d} = 1$ for each $d \in D$. If $v = (d \quad \tau d \quad \cdots \quad \tau^{-1}d)$ is a vertex of M, then the equations

$$x_d x_{\tau d} \cdots x_{\tau^{-1}d} = 1, \quad x_{\tau d} \cdots x_{\tau^{-1}d} x_d = 1, \quad \ldots, \quad x_{\tau^{-1}d} x_d \cdots x_{\tau d} = 1$$

obtained by cyclically permuting the variables are equivalent to each other (by conjugation).

If $e = \{a, \alpha a\}$ is a link of M, the darts a and αa appearing in τ in disjoint cycles

$$(a \quad b \quad \cdots \quad b'), \quad (\alpha a \quad c \quad \cdots \quad c'),$$

the operation of contracting e corresponds to eliminating the variable x_a between the equations

$$x_a x_b \cdots x_{b'} = 1 \quad \text{and} \quad x_a^{-1} x_c \cdots x_{c'} = 1$$

to obtain the single equation

$$x_b \cdots x_{b'} x_c \cdots x_{c'} = 1.$$

This observation proves the following proposition.

Proposition 4.2. *If e is a link of M then there is a bijection between local G-flows of M and local G-flows of M/e.*

Definition 4.3 (local tension). Let $M = (\alpha, \tau)$ be a map on set of darts D, $\varphi = \tau\alpha$, and let G be a finite group. A function $f : D \to G$ is a *local G-tension* of M if, for each $d \in D$,

(i) $f(\alpha d) = f(d)^{-1}$,
(ii) $f(d)f(\varphi d)\cdots f(\varphi^{-1}d) = 1$.

A local G-tension of M is an assignment of elements of G to darts of M with the property that the product of values on each cycle $(d \quad \alpha d)$ of α and on each cycle $(d \quad \varphi d \quad \cdots \quad \varphi^{-1}d)$ of φ is equal to the identity.

Since $M^* = (\alpha, \tau\alpha)$, a local G-tension of M is a local G-flow of M^* and conversely.

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4.1.2. Counting local flows and local tensions. Let G be a finite group of order n. There is a finite number, ℓ say, of pairwise non-isomorphic irreducible representations of G over $\mathbb{C}$. The dimensions of these representations, $n_1,\dots,n_\ell$, satisfy $\sum_{i=1}^{\ell} n_i^2 = n$. If G is abelian, there are exactly n such representations, each of dimension 1. In Section 5.2 we prove the following result.

Theorem 4.4. *Let M be a map, let $\mathcal{T}(M;\mathbf{x},\mathbf{y})$ be the surface Tutte polynomial of M, and let G be a finite group, the irreducible representations of which have dimensions $n_1,\dots,n_\ell$. Then the number of nowhere-identity local G-tensions of M is given by*

$$(-1)^{e(M)-f(M)}\mathcal{T}(M;\mathbf{x},\mathbf{y}), \quad \text{with } x=-|G|, y=1, \quad x_g=-\frac{1}{|G|}\sum_{i=1}^{\ell} n_i^{2-2g},\ y_g=1,$$

and the number of nowhere-identity local G-flows by

$$(-1)^{e(M)-v(M)}\mathcal{T}(M;\mathbf{x},\mathbf{y}), \quad \text{with } x=1, y=-|G|, \quad x_g=1,\ y_g=-\frac{1}{|G|}\sum_{i=1}^{\ell} n_i^{2-2g},$$

for $g=0,1,2,\dots$.

Remark 4.5. As noted in Remark 3.13, the polynomial $\mathcal{T}(M;\mathbf{x},\mathbf{y})$ is a polynomial in finitely many indeterminates: its specializations for counting flows and tensions involve only the variables x_g, y_g for $g=0,1,\dots,g(M)$. In Theorem 4.4 we can set $x_g, y_g = 0$ for $g > g(M)$.

4.2. Quasi-trees of given genus

Replacing x_g with $x^{-2g}x_g$ and y_g with $y^{-2g}y_g$ for $g=0,1,2,\dots$ in $\mathcal{T}(M;\mathbf{x},\mathbf{y})$, we obtain the following renormalization of the surface Tutte polynomial.

Definition 4.6. Let $\mathbf{x}=(x,x_0,x_1,x_2,\dots)$, $\mathbf{y}=(y,y_0,y_1,\dots)$ be two infinite sequences of commuting indeterminates.

Given a map $M=(V,E,F)$, define

$$\widetilde{\mathcal{T}}(M;\mathbf{x},\mathbf{y}) = \sum_{A\subseteq E} x^{r(M/A)}y^{r^*(M\setminus A^c)} \prod_{\substack{\text{conn. cpts } M_i \\ \text{of } M/A}} x_{g(M_i)} \prod_{\substack{\text{conn. cpts } M_j \\ \text{of } M\setminus A^c}} y_{g(M_j)}, \tag{4.1}$$

where $r(M)=v(M)-k(M)$, $r^*(M)=f(M)-k(M)$ and $A^c=E\setminus A$ for $A\subseteq E$.

Proposition 4.7. *Let M be a connected map with $g(M)=g$. Let h be an integer with $0\leqslant h\leqslant g$. Then the evaluation of $\widetilde{\mathcal{T}}(M;\mathbf{x},\mathbf{y})$ at $x=y=0$, $x_i=0$ for $i\neq g-h$, $x_{g-h}=1$, $y_j=0$ for $j\neq h$, and $y_h=1$ is equal to the number of quasi-trees of M of genus h (which is also equal to the number of quasi-trees of M^* of genus $g-h$).*

Proof. Let $A\subseteq E$ be such that it gives a non-zero contribution to the sum (4.1) with the given values assigned to the indeterminates $\mathbf{x},\mathbf{y}$. Then $r(M/A)+r^*(M\setminus A^c)=0$ (from the fact that $x=0=y$), and each component of M/A has genus $g-h$ and each component of $M\setminus A^c$ has genus h (from the fact that $x_{g-h}=1=y_h$ while $x_i=0$ for $i\neq g-h$ and $y_j=0$ for $j\neq h$). By additivity of the genus over connected components, this immediately implies that $g(M/A)\geqslant g-h$ and

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$g(M \setminus A^c) \geqslant h$. Then by Lemma 3.11 we know that equality must hold, that is, $g(M/A) = g - h$ and $g(M \setminus A^c) = h$.

Since rank and dual rank take non-negative values, we have $r(M/A) = 0 = r^*(M\backslash A^c)$, whence

$$v(M/A) = k(M/A) \quad \text{and} \quad f(M\backslash A^c) = k(M\backslash A^c). \tag{4.2}$$

As $g(M) = g(M/A) + g(M \setminus A^c)$, we know by Lemma 3.11 that

$$k(M/A) + k(M\backslash A^c) = k(M) + f(M\backslash A^c), \tag{4.3}$$

and, dually,

$$k(M/A) + k(M\backslash A^c) = k(M) + v(M/A). \tag{4.4}$$

From equations (4.2) and (4.4) we have $k(M\backslash A^c) = k(M) = 1$, and from equations (4.2) and (4.3) we have $k(M/A) = k(M) = 1$. Hence $M\backslash A^c$ is a quasi-tree and $h = g(M\backslash A^c)$, while $M^*\backslash A \cong (M/A)^*$ is a quasi-tree and $g - h = g(M/A) = g(M^*\backslash A)$.

Conversely, if $M\backslash A^c$ is a quasi-tree of genus h (or $M^*\backslash A$ a quasi-tree of genus $g - h$), then A contributes 1 to the sum (4.1) with the given values assigned to the indeterminates. Hence for

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$0 \leqslant h \leqslant g$ the given evaluation is equal to

$$\#\{A \subseteq E : f(M\backslash A^c) = k(M\backslash A^c) = 1, g(M\backslash A^c) = h\},$$

that is, the number of quasi-trees of M of genus h. □

Remark 4.8. The quasi-trees of maximum genus (genus zero) in a map M form the bases of the matroid whose bases are the feasible sets of maximum (minimum) size in the Δ-matroid associated with M (see [6]). Thus the evaluations of Proposition 4.7 for $h \in \{0, g(M)\}$ are evaluations of the Tutte polynomial of the upper and lower matroids lying within the Δ-matroid of M.

Remark 4.9. As shown in [8], the ordinary generating function for quasi-trees of a connected map M according to genus is given by evaluating the specialization

$$q(M;t,Y) = \mathcal{R}(M;1,Y,tY^{-2}) = \sum_{\substack{A\subseteq E \\ k(M\backslash A^c)=1}} t^{g(M\backslash A^c)} Y^{n(M\backslash A^c)-2g(M\backslash A^c)}$$

of the Bollobás–Riordan polynomial at $Y = 0$ (we have $n(M\backslash A^c) - 2g(M\backslash A^c) = f(M\backslash A^c) - k(M\backslash A^c)$). The coefficients of $q(M;t,0)$ are the evaluations of $\widetilde{T}(M;\mathbf{x},\mathbf{y})$ given in Proposition 4.7. Let ζ be a primitive $(g(M)+1)$th root of unity. Then the number of quasi-trees of genus h, evaluated in Proposition 4.7, is also given by

$$\frac{1}{g(M)+1} \sum_{j=0}^{g(M)} q(M;\zeta^j,0)\zeta^{-jh}.$$

4.3. Quasi-forests

A *quasi-forest* of M is a submap of M each of whose connected components is a quasi-tree. A *maximal quasi-forest* of a map M is a quasi-forest each of whose components is a quasi-tree of a connected component of M. When M is connected a maximal quasi-forest is a quasi-tree. When $M\backslash A^c$ is a maximal quasi-forest we have $k(M\backslash A^c) = k(M)$.

For the remaining specializations of the surface Tutte polynomial that follow it will be convenient to first specialize $\mathcal{T}(M;\mathbf{x},\mathbf{y})$ and its renormalization given in Definition 4.6 to polynomials in four variables.

Definition 4.10. We set $x_g = a^g$ and $y_g = b^g$ in $\mathcal{T}(M;\mathbf{x},\mathbf{y})$ (in which $\mathbf{x} = (x, x_0, x_1, \dots)$ and $\mathbf{y} = (y, y_0, y_1, \dots)$) to give the quadrivariate polynomial

$$\mathcal{Q}(M;x,y,a,b) = \sum_{A\subseteq E} x^{n^*(M/A)} y^{n(M\backslash A^c)} a^{g(M/A)} b^{g(M\backslash A^c)}. \tag{4.5}$$

Likewise, setting $x_g = a^g$ and $y_g = b^g$ in $\widetilde{\mathcal{T}}(M;\mathbf{x},\mathbf{y})$,

$$\widetilde{\mathcal{Q}}(M;x,y,a,b) = \sum_{A\subseteq E} x^{r(M/A)} y^{r^*(M\backslash A^c)} a^{g(M/A)} b^{g(M\backslash A^c)}. \tag{4.6}$$

The polynomials of Definition 4.10 are simply related by

$$\mathcal{Q}(M;x,y,a,b) = \widetilde{\mathcal{Q}}(M;x,y,ax^2,by^2),$$

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using the fact that $n(M) = r^*(M) + 2g(M)$ (the dual version of equation (3.2)). However, it is useful to have notation for them both separately, since we shall be making evaluations where some of the variables x, y, a, b are set to zero.

By Proposition 3.17, if $M = (V, E, F)$ is a plane embedding of $\Gamma = (V, E)$ then

$$\mathcal{Q}(M; x, y, a, b) = T(\Gamma; x+1, y+1) = \widetilde{\mathcal{Q}}(M; x, y, ax^2, by^2).$$

Remark 4.11. Just as the Las Vergnas, Bollobás–Riordan and Krushkal polynomials of a ribbon graph can be extended to Δ-matroids (see [9, Section 6]), this is also true for the polynomials $\mathcal{Q}(M; x, y, a, b)$ and $\widetilde{\mathcal{Q}}(M; x, y, a, b)$. A short explanation for $\widetilde{\mathcal{Q}}(M; x, y, a, b)$ is as follows. For a subset A of the edges of M the coefficient of x is given by $r(M/A)$, which in Δ-matroid terminology (see [9]) is simply the rank of the lower matroid of the Δ-matroid underlying M/A. The coefficient of a is the genus of M/A, which in Δ-matroid terminology is equal to half the *width* of the Δ-matroid underlying M/A. The coefficients of y and b are similarly expressed in terms of parameters of the Δ-matroid underlying M^*/A^c.

The specializations and evaluations of $\mathcal{Q}(M; x, y, a, b)$ and $\widetilde{\mathcal{Q}}(M; x, y, a, b)$ (and of $\mathcal{T}(M; \mathbf{x}, \mathbf{y})$) that follow are related to the Tutte polynomial specializations

$$T(\Gamma; x+1, 1) = \sum_{\substack{A \subseteq E \\ r(A) = |A|}} x^{r(\Gamma) - |A|},$$

$$T(\Gamma; 1, y+1) = \sum_{\substack{A \subseteq E \\ r(A) = r(E)}} y^{|A| - r(\Gamma)},$$

giving respectively generating functions for spanning forests of Γ according to their number of edges and for connected spanning subgraphs. We need two more definitions, though. A *bridge* of a map $M = (V, E, F)$ is a an edge e such that $M \setminus e$ has more components than M. By definition a bridge is incident to only one face. A *dual bridge* of a map $M = (V, E, F)$ is a loop e of M such that e is a bridge of the dual of M.

Proposition 4.12. *For a map M,*

$$\mathcal{Q}(M; x, 0, 1, 1) = x^{2g(M)} \sum_{\substack{A \subseteq E\text{: conn. cpts of} \\ M \setminus A^c \text{ plane quasi-trees}}} x^{r(M) - |A|},$$

and

$$\mathcal{Q}(M; 0, y, 1, 1) = y^{2g(M)} \sum_{\substack{A \subseteq E\text{: conn. cpts of} \\ M/A \text{ plane bouquets}}} y^{r^*(M) - |A|}.$$

(A plane quasi-tree consists solely of bridges, and so corresponds to an embedding of a tree. A plane bouquet consists solely of dual bridges.)

Proof. From equation (4.5),

$$\mathcal{Q}(M; x, 0, 1, 1) = \sum_{\substack{A \subseteq E: \\ n(M \setminus A^c) = 0}} x^{n^*(M/A)},$$

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and

$$\mathcal{Q}(M;0,y,1,1) = \sum_{\substack{A\subseteq E:\\ n^*(M/A)=0}} y^{n(M\backslash A^c)}.$$

We prove the first identity in the proposition statement; the second follows by duality, with $\mathcal{Q}(M;0,y,1,1) = \mathcal{Q}(M^*;y,0,1,1)$.

By Euler's relation,

$$n(M\backslash A^c) = e(M\backslash A^c) - v(M\backslash A^c) + k(M\backslash A^c) = f(M\backslash A^c) - k(M\backslash A^c) + 2g(M\backslash A^c)$$

and $f(M\backslash A^c) \geqslant k(M\backslash A^c)$ with equality if and only if each component of $M\backslash A^c$ has just one face. Thus, for $n(M\backslash A^c) = 0$ to hold, $M\backslash A^c$ must be a disjoint union of plane quasi-trees (*i.e.* trees). Given that the edges of A form a disjoint union of plane quasi-trees, we have, beginning with Euler's relation,

$$\begin{aligned} n^*(M/A) &= e(M/A) - f(M/A) + k(M/A) \\ &= v(M/A) - k(M/A) + 2g(M/A) \\ &= v(M) - |A| - k(M) + 2g(M) \\ &= 2g(M) + r(M) - |A|, \end{aligned}$$

since contracting a bridge of a map changes neither connectivity nor genus. This establishes the result. □

Corollary 4.13. *For a map $M = (V,E,F)$ with underlying graph $\Gamma = (V,E)$, the constant term of the specialization $\mathcal{Q}(M;x,y,1,1)$ of $\mathcal{T}(M;\mathbf{x},\mathbf{y})$ is given by*

$$\mathcal{Q}(M;0,0,1,1) = \begin{cases} T(\Gamma;1,1) & \text{if } M \text{ is a plane embedding of planar graph } \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Further, we have the following evaluations for any map $M = (V,E,F)$:

$$\mathcal{Q}(M;1,0,1,1) = \#\{A \subseteq E : \text{connected components of } M\backslash A^c \text{ are plane quasi-trees}\},$$
$$\mathcal{Q}(M;0,1,1,1) = \#\{A \subseteq E : \text{connected components of } M/A \text{ are plane bouquets}\}.$$

Remark 4.14. The last two evaluations are analogous (and for plane maps, identical) to the following Tutte polynomial evaluations for a graph $\Gamma = (V,E)$ giving for connected Γ the number of spanning forests and number of connected spanning subgraphs:

$$\begin{aligned} T(\Gamma;2,1) &= \#\{A \subseteq E : n(\Gamma\backslash A^c) = 0\} \\ &= \#\{A \subseteq E : \text{connected components of } \Gamma\backslash A^c \text{ are trees}\}, \end{aligned}$$

and

$$\begin{aligned} T(\Gamma;1,2) &= \#\{A \subseteq E : r(\Gamma\backslash A^c) = r(\Gamma)\} \\ &= \#\{A \subseteq E : \text{connected components of } \Gamma/A \text{ are single vertices with loops}\}. \end{aligned}$$

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Proof of Corollary 4.13. This is immediate from Proposition 4.12 with $x = 1$. By Proposition 3.17, when M is a plane embedding of planar Γ, $\mathcal{Q}(M;0,0,1,1) = T(\Gamma;1,1)$. □

Proposition 4.15. *For a map M,*

$$\widetilde{\mathcal{Q}}(M;x,0,1,1) = \sum_{\substack{A\subseteq E\text{: conn. cpts of} \\ M\backslash A^c \text{ quasi-trees}}} x^{r(M/A)},$$

$$\widetilde{\mathcal{Q}}(M;0,y,1,1) = \sum_{\substack{A\subseteq E\text{: conn. cpts of} \\ M/A \text{ bouquets}}} y^{r^*(M\backslash A^c)}.$$

Proof. This follows from the defining equation (4.6) for $\widetilde{\mathcal{Q}}(M;x,y,a,b)$ given in Definition 4.10 and the fact that $r^*(M\backslash A^c) = f(M\backslash A^c) - k(M\backslash A^c) = 0$ if and only if each connected component of $M\backslash A^c$ has just one face, that is, they are quasi-trees, and dually $r(M/A) = v(M/A) - k(M/A) = 0$ if and only if the connected components of M/A each have one vertex, that is, they are bouquets. □

Corollary 4.16. *For a map $M = (V,E,F)$,*

$$\widetilde{\mathcal{Q}}(M;1,0,1,1) = \#\{A \subseteq E : M\backslash A^c \text{ is a quasi-forest}\},$$

$$\widetilde{\mathcal{Q}}(M;0,1,1,1) = \#\{A \subseteq E : \text{connected components of } M/A \text{ are bouquets}\}.$$

The evaluations of Corollary 4.16 are analogous (and for plane maps identical) to the evaluations $T(\Gamma;2,1)$ and $T(\Gamma;1,2)$ of the Tutte polynomial as the number of spanning forests and connected spanning subgraphs of Γ. (Compare Remark 4.14 concerning the evaluations of Corollary 4.13.)

5. Enumerating flows and tensions

Definition 5.1. For a map $M = (V,E,F)$, let $p_G^1(M)$ denote the number of local G-tensions of G (allowing the identity 1) and $p_G(M)$ the number of nowhere-identity local G-tensions of M (see Definitions 4.1 and 4.3). Dually, let $q_G^1(M)$ denote the number of local G-flows of M and $q_G(M)$ the number of nowhere-identity local G-flows of M.

By definition of the dual map we have

$$q_G^1(M) = p_G^1(M^*)$$

and

$$q_G(M) = p_G(M^*).$$

The main goal of this section is to establish formulas for $q_G(M)$ and $p_G(M)$ in terms of the size of the group G and the dimensions of its irreducible representations.

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By partitioning local G-flows according to the set of edges A on which the flow value is equal to the identity,

$$q_G^1(M) = \sum_{A \subseteq E} q_G(M \backslash A).$$

Then by the inclusion–exclusion principle

$$q_G(M) = \sum_{A \subseteq E} (-1)^{|A^c|} q_G^1(M \backslash A^c). \tag{5.1}$$

So we see that it suffices to find formulas for $q_G^1(M)$, which is the main focus of the following subsection. In Section 5.2 we give formulas for $q_G(M)$ and $p_G(M)$ and discuss some special cases.

5.1. Enumerating local G-flows

Here we establish the following result.

Theorem 5.2. *Let G be a finite group with irreducible representation of dimensions n_ℓ. Let $M = (V, E, F)$ be a connected map. Then the number of local G-flows of M is given by*

$$q_G^1(M) = |G|^{|E|-|V|} \sum_\ell n_\ell^{\chi(M)}. \tag{5.2}$$

Remark 5.3. This result has already implicitly appeared in [27], but we give a proof below so as to make our paper as self-contained as possible.

Before giving a proof of Theorem 5.2, let us further remark that it includes a remarkable result due to Mednyh [26] (Theorem 5.4 below), which we now explain. Let M_g be the map given by a single vertex with $2g$ loops attached to it with permutation representation (α, τ) where $\alpha = (1 \quad 3)(2 \quad 4)\ldots(4g-3 \quad 4g-1)(4g-2 \quad 4g)$ and where $\tau = (1 \quad 2 \quad \ldots \quad 4g-1 \quad 4g)$. (The genus of M_g is easily seen to be equal to g.) A local G-flow of M_g is a solution in G to the equation

$$[a_1, b_1] \cdots [a_g, b_g] = 1, \tag{5.3}$$

where $[a, b] = aba^{-1}b^{-1}$ is the commutator of a and b in G.

If Σ is an orientable compact surface of genus g, its fundamental group $\pi_1(\Sigma)$ has the presentation

$$\pi_1(\Sigma) \cong \langle a_1, b_1, \ldots, a_g, b_g \,:\, [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle.$$

(See for example [17, p. 51].) This implies that solutions in G to equation (5.3) are exactly the homomorphisms from $\pi_1(\Sigma)$ to G. Let us denote the set all homomorphisms from $\pi_1(\Sigma)$ to G by $\mathrm{Hom}(\pi_1(\Sigma), G)$. Thus we have $q_G^1(M_g) = |\mathrm{Hom}(\pi_1(\Sigma), G)|$, and so Theorem 5.2 implies the following result of Frobenius [15] (for $g = 1$) and Mednyh [26] (for $g > 1$); see *e.g.* the survey [18, Section 7] for more details.

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Theorem 5.4. *Let Σ be a surface of genus $g > 0$ and G a finite group with dimensions of irreducible representations n_ℓ. Then*

$$\frac{|\mathrm{Hom}(\pi_1(\Sigma), G)|}{|G|} = \sum_\ell \left(\frac{|G|}{n_\ell}\right)^{2g-2}.$$

Remark 5.5. The numbers $|G|/n_\ell$ in Theorem 5.4 are positive integers.

When $g = 0$ the formula still holds: when Σ is the sphere,

$$|\mathrm{Hom}(\pi_1(\Sigma), G)| = 1 = \frac{1}{|G|}\sum_\ell n_\ell^2,$$

which is a well-known result from representation theory [30, p. 18, Corollary 2].

Remark 5.6. Although Mednyh's theorem is a consequence of our Theorem 5.2, it also possible to derive Theorem 5.2 from Mednyh's theorem (see [27]).

We now turn to the proof of Theorem 5.2. As mentioned above we prove it using representation theory. We refer the reader to the book by Serre [30] for the necessary definitions from representation theory and its background. (In fact the first twenty pages of [30] contain everything that we need.) We shall use the following three facts. The regular character χ_{reg} of a finite group G satisfies ([30, p. 18, Proposition 2])

$$\chi_{\mathrm{reg}}(g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{5.4}$$

for all $g \in G$. Let $C(G)$ denote the set of all irreducible representations of G (up to isomorphism). Then ([30, p. 18, Corollary 1])

$$\chi_{\mathrm{reg}} = \sum_{\rho \in C(G)} \dim(\rho)\chi_\rho. \tag{5.5}$$

Let $\rho, \rho' \in C(G)$, which we assume to be in matrix form. Then, for $i, j \in [\dim(\rho)]$ and $i', j' \in [\dim(\rho')]$, we have ([30, p. 14, Corollary 3])

$$\sum_{g\in G} \rho(g)_{i,j}\rho'(g^{-1})_{j',i'} = \begin{cases} |G|/\dim(\rho) & \text{if } \rho = \rho',\ i = i' \text{ and } j = j', \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

For a connected map $M = (V, E, F)$ with permutation representation (α, τ) and set of darts D, a dart d' occurring in the orbit of a vertex $v = \begin{pmatrix} d & \tau d & \dots & \tau^{-1}d \end{pmatrix} \in V$ is said to be contained in v and, abusing notation, we write $d' \in v$.

Proof of Theorem 5.2. Let us denote the vertices, edges and darts of the underlying graph of M by V, E and D respectively. Let us also denote by S the set of all functions $f : D \to G$ that satisfy $f(d) = f(\alpha d)^{-1}$ for all $d \in D$. Then we can write

$$q_G^1(M) = \sum_{f\in S} \prod_{\substack{v\in V \\ v=(d\ \tau d\cdots \tau^{-1}d)}} \mathbf{1}(f(d)f(\tau d)\cdots f(\tau^{-1}d)),$$

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where $\mathbf{1}(g)$ is equal to 1 if and only if $g = 1$ and zero otherwise. By (5.4) we can rewrite this as

$$q_G^1(M) = \sum_{f\in S} |G|^{-|V|} \prod_{\substack{v\in V \\ v=(d\,\tau d\cdots\tau^{-1}d)}} \chi_{\mathrm{reg}}(f(d)f(\tau d)\cdots f(\tau^{-1}d)).$$

Using (5.5) we obtain that

$$\begin{aligned}|G|^{|V|}q_G^1(M) &= \sum_{f\in S} \prod_{\substack{v\in V \\ v=(d\,\tau d\cdots\tau^{-1}d)}} \sum_{\rho\in C(G)} \dim(\rho)\chi_\rho(f(d)f(\tau d)\cdots f(\tau^{-1}d)) \\ &= \sum_{f\in S} \sum_{\kappa:V\to C(G)} \prod_{\substack{v\in V \\ v=(d\,\tau d\cdots\tau^{-1}d)}} \dim(\kappa(v))\chi_{\kappa(v)}(f(d)f(\tau d)\cdots f(\tau^{-1}d)).\end{aligned}$$

Let us now fix an assignment $\kappa : V \to C(G)$ and look at its contribution to the sum above. For notational convenience we will write κ_v instead of $\kappa(v)$. This contribution is given by (using that representations are multiplicative and where tr denotes the trace)

$$\begin{aligned}&\sum_{f\in S} \prod_{\substack{v\in V \\ v=(d\,\tau d\cdots\tau^{-1}d)}} \dim(\kappa_v)\cdot \mathrm{tr}(\kappa_v(f(d))\kappa_v(f(\tau d))\cdots\kappa_v(f(\tau^{-1}d))) \qquad (5.7)\\ &= \sum_{f\in S} \prod_{\substack{v\in V \\ v=(d\,\tau d\cdots\tau^{-1}d)}} \dim(\kappa_v)\cdot \sum_{i_d,i_{\tau(d)},\ldots,i_{\tau^{-1}(d)}=1}^{\dim(\kappa_v)} (\kappa_v(f(d))_{i_d,i_{\tau d}}\cdots\kappa_v(f(\tau^{-1}d))_{i_{\tau^{-1}d},i_d}) \\ &= \sum_{f\in S}\sum_{v\in V}\sum_{d\in v}\sum_{i_d=1}^{\dim(\kappa_v)}\prod_{u\in V}\dim(\kappa_u)\cdot\prod_{d\in u}(\kappa_u(f(d)))_{i_d,i_{\tau d}} \\ &= \sum_{v\in V}\sum_{d\in v}\sum_{i_d=1}^{\dim(\kappa_v)}\prod_{u\in V}\dim(\kappa_u)\cdot\prod_{e=\{d,\alpha d\}\in E}\cdot\sum_{g\in G}(\kappa_d(g)_{i_d,i_{\tau d}}(\kappa_{\alpha d}(g^{-1}))_{i_{\alpha d},i_{\tau\alpha d}},\end{aligned}$$

where we slightly abuse notation by writing κ_d for the representation κ_v for which $d \in v$. By (5.6), the last sum on the last line of (5.7) is zero unless $\kappa_v = \kappa \in C(G)$ for all $v \in V$, and additionally $i_d = i_{\tau\alpha d}$ and $i_{\tau d} = i_{\alpha d}$ for all $d \in D$. So to get a non-zero contribution we need that, for each $d \in D$, i_d only depends on the face of M to which d belongs and that κ is constant on V. Each such assignment then gives a contribution of $\dim(\kappa)^{|V|-|E|}|G|^{|E|}$, and so in total for constant κ we get that (5.7) is equal to $\dim(\kappa)^{|V|+|F|-|E|}|G|^{|E|}$. Summing this over all irreducible representations of G and dividing by $|G|^{|V|}$, we obtain the desired expression for $q_G^1(M)$. □

5.2. Enumerating nowhere identity G-flows and tensions

In the previous subsection we enumerated local G-flows. We shall now use this to prove the following result, which immediately implies Theorem 4.4.

Theorem 5.7. *Let G be a group with irreducible representations of dimensions n_ℓ.*

(i) *The number of nowhere-identity local G-flows of a map $M = (V,E,F)$ is given by*

$$q_G(M) = \sum_{A\subseteq E}(-1)^{|A^c|}|G|^{n(M\setminus A^c)}\prod_{\substack{\text{conn. cpts} \\ M_j \text{ of } M\setminus A^c}}\left(\frac{1}{|G|}\sum_\ell n_\ell^{\chi(M_j)}\right), \qquad (5.8)$$

where $A^c = E\setminus A$ and $\chi(M_j) = 2-2g(M_j)$.

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(ii) *The number of nowhere-identity local G-tensions of a map $M=(V,E,F)$ is given by*

$$p_G(M)=\sum_{A\subseteq E}(-1)^{|A|}|G|^{n^*(M/A)}\prod_{\substack{\text{conn. cpts}\\ M_i \text{ of } M/A}}\left(\frac{1}{|G|}\sum_{\ell}n_\ell^{\chi(M_i)}\right).$$

Remark 5.8. The subset A is complemented in the summation (5.8) for later convenience. The submap $M\backslash A^c$ is the restriction of M to edges in A.

Proof. The statement for $q_G(M)$ follows from Theorem 5.2 by (5.1). To prove the statement for $p_G(M)$, by noting that

$$\begin{aligned}e(M^*\backslash A^c)&=|A|=e(M/A^c),\\ k(M^*\backslash A^c)&=k((M^*\backslash A^c)^*)=k(M/A^c),\\ v(M^*\backslash A^c)&=f((M^*\backslash A^c)^*)=f(M/A^c),\end{aligned}$$

we see that $n(M^*\backslash A^c)=n^*(M/A^c)$. Then, as the components of $M^*\setminus A^c$ are the components of $(M^*\setminus A^c)^*=M/A^c$, using that $p_G(M)=q_G(M^*)$, and replacing A^c with A, we arrive at the desired equality for $p_G(M)$. □

We now discuss some special cases of Theorem 5.7.

Example 5.9. If $M=(V,E,F)$ is a plane map, then $\chi(M_j)=2$ for all choices of A and j in equation (5.8). Combining this with the fact that $\sum n_\ell^2=|G|$, $q_G(M)$ coincides for arbitrary finite group G with the flow polynomial of the planar graph $\Gamma=(V,E)$ evaluated at $|G|$.

Similarly, $p_G(M)$ also depends only on $|G|$. This can also be seen from that for a plane map M local tensions are global, and we have the correspondence between G-tensions and vertex G-colourings of the graph underlying M: this makes it evident that the number of nowhere-identity G-tensions of a plane map depends only on $|G|$ and not on the structure of G (proper vertex G-colourings depend only on distinctness of colours, that is, on G as a set).

Example 5.10. When G is abelian, $n_\ell=1$ for $i=1,\dots,|G|$, and as $\sum_{i=1}^{|G|}1^{\chi(M_j)}=|G|$ we obtain for any map M the same expression as for plane maps with arbitrary finite group G,

$$q_G(M)=\sum_{A\subseteq E}(-1)^{|A^c|}|G|^{n(M\backslash A^c)}=\phi(\Gamma;|G|),$$

where ϕ denotes the flow polynomial. Thus when G is abelian the number of local abelian G-flows of a map M is equal to the number of global G-flows of M, evaluated by the flow polynomial of the graph underlying M.

Example 5.11 (dihedral group). Let $G=D_{2n}=\langle r,s : s^2=r^n=1, srs=r^{-1}\rangle$ be the dihedral group of order $2n$. The number of nowhere-identity D_{2n}-flows of a map M is given by a quasi-polynomial in n of period 2 as follows.

(1) *n odd.* There are two one-dimensional irreducible representations of D_{2n}, the remaining $(n-1)/2$ being two-dimensional. This corresponds to $(n+3)/2$ conjugacy classes in D_{2n}.

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Setting $n_i = 1$ for $i = 1,2$ and $n_i = 2$ for $i = 3,4\ldots,(n+3)/2$, we obtain

$$q_{D_{2n}}(M) = \sum_{A\subseteq E} (-1)^{|E\setminus A|}(2n)^{|E\setminus A|-|V|} \prod_{\substack{\text{conn. cpts}\\ M_j \text{ of } M\setminus A}} \left(2 + \frac{n-1}{2} 2^{\chi(M_j)}\right).$$

(2) *n even.* There are four one-dimensional irreducible representations of D_{2n}, the remaining $n/2-1$ being two-dimensional. This corresponds to $n/2+3$ conjugacy classes in D_{2n}. Setting $n_i = 1$ for $i = 1,2,3,4$ and $n_i = 2$ for $i = 5,6\ldots,n/2+3$, we obtain

$$q_{D_{2n}}(M) = \sum_{A\subseteq E} (-1)^{|E\setminus A|}(2n)^{|E\setminus A|-|V|} \prod_{\substack{\text{conn. cpts}\\ M_j \text{ of } M\setminus A}} \left(4 + \frac{n-2}{2} 2^{\chi(M_j)}\right).$$

Remark 5.12. The flow polynomial of a graph Γ evaluated at $n \in \mathbb{N}$ enumerates the number of nowhere-zero $\mathbb{Z}_n$-flows of Γ. A *nowhere-zero n-flow* is a $\mathbb{Z}$-flow that takes values in $\{\pm1,\pm2,\ldots,\pm(n-1)\}$. Tutte [32] showed that the existence of a nowhere-zero $\mathbb{Z}_n$-flow of Γ is equivalent to the existence of a nowhere-zero n-flow of Γ. Kochol [19] showed that the number of nowhere-zero n-flows is in general larger than the number of nowhere-zero $\mathbb{Z}_n$-flows, but still given by a polynomial in n. Example 5.11 shows that the number of nowhere-identity D_{2n}-flows is a quasi-polynomial in n of period 2. The dihedral group D_{2n} is isomorphic to its representation in $GL_2(\mathbb{Z}_n)$ as

$$\left\{\begin{pmatrix}\pm1 & x\\ 0 & 1\end{pmatrix} : x \in \mathbb{Z}_n\right\}.$$

Let us define

$$\mathbb{D} = \left\{\begin{pmatrix}\pm1 & x\\ 0 & 1\end{pmatrix} : x \in \mathbb{Z}\right\}.$$

Very recently, Litjens [24] showed that there are maps that have a nowhere-identity D_6-flow while not having a nowhere-identity $\mathbb{D}$-flow in which all values $\left(\begin{smallmatrix}\pm1 & x\\ 0 & 1\end{smallmatrix}\right)$ have $|x| \leqslant 2$. This raises the question for which maps is having a nowhere-identity D_{2n}-flow equivalent to having a nowhere-identity $\mathbb{D}$-flow with $|x| \leqslant n-1$, and furthermore whether the latter are counted by a quasi-polynomial in n of period 2 (analogous to Kochol's integer flow polynomial).

DeVos [10] appears to be the first to have considered the problem of counting non-abelian flows. In [10, Lemma 6.1.6], DeVos argues directly that the number of nowhere-identity D_8-flows is equal to the number of nowhere-identity Q_8-flows, where Q_8 is the quaternion group, with presentation $\langle r,s : r^4 = 1, s^2 = r^2 = srs^{-1}r^{-1}\rangle$. This can be seen from Theorem 5.7 by observing that D_8 and Q_8 each have irreducible representations of dimensions $1,1,1,1,2$.

Let $G' = \langle xyx^{-1}y^{-1} : x,y \in G\rangle$ denote the commutator subgroup of G. Let us call an edge e of a connected map M a *plane-sided bridge* if deleting e from M results in a disconnected map one of whose components is plane, that is, it has genus zero. The following is the main result proved by DeVos concerning non-abelian flows.

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Theorem 5.13 (Theorem 6.0.7 of [10]). *Let M be a connected map and G a non-abelian group.*

(1) *If $|G'| > 2$ then M has a nowhere-identity local G-flow if and only if M has no plane-sided bridge.*
(2) *If $|G'| = 2$ and $G \notin \{D_8, Q_8\}$ then M has a nowhere-identity local G-flow if and only if M has no odd-sized subset B of bridges of M such that each $e \in B$ is a plane-sided bridge of $M\backslash(B\backslash e)$.*
(3) *If $G \in \{D_8, Q_8\}$ then M has a nowhere-identity local G-flow if M has no bridge, but it is NP-complete to decide if M with a bridge has a nowhere-identity local G-flow.*

6. Concluding remarks

6.1. The surface Tutte polynomial and the Krushkal polynomial

Consider the following two maps M_1 and M_2 in the 2-torus (orientable surface of genus 2). The map M_2 is a 2-vertex 5-edge map constructed by adding a pendant edge to the 4-loop graph on one vertex in the 2-torus. M_1 is the unique map in the 2-torus in which each of the two vertices is attached to a non-loop edge and to two loops. Both M_1 and M_2 have a non-loop edge (e_1 and e_2 respectively) that is a bridge in their underlying graph; we have $M_1/e_1 = M_2/e_2$.

By direct computation we find that

$$\begin{aligned}\mathcal{T}(M_1;\mathbf{x},\mathbf{y}) = {} & x^5x_2{y_0}^2 + x^4x_2y_0 + 4x^4x_1y{y_0}^2 + 4x^3x_1yy_0 + 2x^3x_1y^2y_0y_1 + 2x^2x_1y^2y_1 \\ & + 4x^3x_0y^2{y_0}^2 + 4x^2x_0y^2y_0 + 4x^2x_0y^3y_1y_0 + 4xx_0y^3y_1 + xx_0y^4{y_1}^2 + x_0y^4y_2 \end{aligned} \tag{6.1}$$

and

$$\begin{aligned}\mathcal{T}(M_2;\mathbf{x},\mathbf{y}) = {} & x^5x_2{y_0}^2 + x^4x_2y_0 + 4x^4x_1y{y_0}^2 + 4x^3x_1yy_0 + 2x^3x_1y^2y_0y_1 + 2x^2x_1y^2y_1 \\ & + 4x^3x_0y^2{y_0}^2 + 4x^2x_0y^2y_0 + 4x^2x_0y^3y_1y_0 + 4xx_0y^3y_1 + xx_0y^4y_0y_2 + x_0y^4y_2, \end{aligned} \tag{6.2}$$

which are different as

$$\mathcal{T}(M_1;\mathbf{x},\mathbf{y}) - \mathcal{T}(M_2;\mathbf{x},\mathbf{y}) = xx_0y^4{y_1}^2 - xx_0y^4y_0y_2.$$

Using [20, Lemma 2.2(2)] for edges e_1 and e_2 in M_1 and M_2, respectively, together with the observation that $M_1/e_1 = M_2/e_2$, we conclude that $\mathcal{K}(M_1;x,y,a,b) = \mathcal{K}(M_2;x,y,a,b)$. Therefore, the surface Tutte polynomial $\mathcal{T}(M;\mathbf{x},\mathbf{y})$ distinguishes these two maps, whereas the Krushkal polynomial does not (and hence neither does the Las Vergnas polynomial nor the Bollobás–Riordan polynomial). Thus, by Proposition 3.18, $\mathcal{T}(M;\mathbf{x},\mathbf{y})$ strictly refines the partition on maps induced by the Krushkal polynomial.

Since $\mathcal{T}(M;\mathbf{x},\mathbf{y})$ has $4+2g(M)$ variables, this latter fact that the distinguishing power of $\mathcal{T}(M;\mathbf{x},\mathbf{y})$ (partition of maps into equivalence classes according to the value of their surface Tutte polynomial) refines that of $\mathcal{K}(M;x,y,a,b)$ is perhaps unsurprising when $g(M) > 0$. On the other hand, by Proposition 3.17, given two plane maps, the two underlying graphs have the same Tutte polynomial if and only if the two plane maps have the same surface Tutte polynomial (a polynomial in four variables).

In light of the specializations of Section 4.3, it seems natural to consider the 4-variable specialization $\mathcal{Q}(M;x,y,a,b)$ of the surface Tutte polynomial (setting $x_g = a^g$, $y_g = b^g$ for $g = 0,1,\ldots$)

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and ask for its distinguishing power relative to the Krushkal polynomial. As yet we have no counterexample that would answer the following question in the negative.

Problem 6.1. *Let M and M' be maps. Is it true that*

$$\mathcal{Q}(M;x,y,a,b) = \mathcal{Q}(M';x,y,a,b) \quad \textit{if and only if} \quad \mathcal{K}(M;x,y,a,b) = \mathcal{K}(M';x,y,a,b)\,?$$

6.2. Non-orientable surfaces

It is natural to ask whether $\mathcal{T}(M;\mathbf{x},\mathbf{y})$ can be extended from maps to graphs embedded in non-orientable surfaces in such a way that there are evaluations of this extended polynomial which enumerate nowhere-identity local G-flows and nowhere-identity local G-tensions (the definition of which extend to non-orientable embeddings). Three of the authors together with Litjens have managed to do so. An extended abstract [16] reporting this work recently appeared.

6.3. Deletion–contraction recurrence

The Tutte polynomial is universal for graph invariants satisfying a deletion–contraction recurrence for ordinary edges, loops and bridges [29]: together with the boundary condition that it takes the value 1 on edgeless graphs this recurrence determines the polynomial.

The Bollobás–Riordan polynomial and Krushkal polynomial both satisfy a similar recurrence, except reduction is only for ordinary edges and bridges, with boundary conditions determined by their values on bouquets [4, 20].

What about other specializations of the surface Tutte polynomial, such as $\mathcal{Q}(M;a,y,a,b)$? As we have already remarked, contraction of loops in maps does not behave in the same way as for graphs.

In subsequent work we aim to determine what sort of deletion–contraction recurrence is satisfied by $\mathcal{T}(M;\mathbf{x},\mathbf{y})$, $\mathcal{Q}(M;a,y,a,b)$ and other polynomials derived from the surface Tutte polynomial, and to describe the boundary conditions for them. For example, for the number of nowhere-identity local G-flows of M, $q_G(M)$, it is straightforward to see that $q_G(M)$ satisfies a recurrence for non-loops and, after establishing a four-term relation for $q_G(M)$ on bouquets, that its value is determined by its value on bouquets whose chord diagram is of the form $D_{i,j}$ in the notation of [4, Lemma 5].

Appendix: Section 3.1 proofs

Here we provide proofs of Proposition 3.10 and Lemma 3.11.

Proof of Proposition 3.10. Let $M = (\alpha,\tau)$ be a map with set of darts D. The dual map is given by $M^* = (\alpha,\tau\alpha) = (\alpha,\varphi)$ with the same set of darts D. Hence, deleting an edge $e = \{a,\alpha a\}$ of M^* gives us the map $M^* \setminus e = (\alpha',\varphi')$ in which the permutation α' is the restriction of α to $D' = D\setminus\{a,\alpha a\}$, and φ' is defined by

$$\varphi' d = \begin{cases} \varphi d & \text{if } d \in D' \setminus \{\varphi^{-1}a, \varphi^{-1}\alpha a\}, \\ \varphi a & \text{if } d = \varphi^{-1}a \neq \alpha a, \varphi a \neq \alpha a, \text{ or } d = \varphi^{-1}\alpha a, \varphi\alpha a = a, \\ \varphi\alpha a & \text{if } d = \varphi^{-1}\alpha a \neq a, \varphi\alpha a \neq a, \text{ or } d = \varphi^{-1}a, \varphi a = \alpha a, \end{cases}$$

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and if $\varphi a = \alpha a$ and $\varphi\alpha a = a$, or $a = \varphi a$, or $\alpha a = \varphi\alpha a$, then φ' has an empty cycle for each of the three possible cases (where an empty cycle corresponds to a copy of $M_\bullet$). For instance, two such empty cycles occur if both a and αa are fixed points of φ (in which case M is a loop-map and contracting its edge makes two copies of $M_\bullet$), and the case where $\varphi a = \alpha a$ and $\varphi\alpha a = a$ corresponds to a component consisting of a single edge.

The various cases are more transparent in cycle notation: if

$$\begin{pmatrix} d & a & \varphi a & \cdots \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} d & \alpha a & a & \varphi a & \cdots \end{pmatrix}$$

is a cycle of φ then $\varphi' d = \varphi a$ (removing a and αa from the cycles of φ), if

$$\begin{pmatrix} d & \alpha a & \varphi\alpha a & \cdots \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} d & a & \alpha a & \varphi\alpha a & \cdots \end{pmatrix}$$

is a cycle of φ then $\varphi' d = \varphi\alpha a$ (removing a and αa from the cycles of φ), and if $\begin{pmatrix} a \end{pmatrix}$, $\begin{pmatrix} \alpha a \end{pmatrix}$ or $\begin{pmatrix} a & \alpha a \end{pmatrix}$ is an orbit of φ, then an empty cycle results in φ' upon removing a and αa (two empty cycles when both $\begin{pmatrix} a \end{pmatrix}$ and $\begin{pmatrix} \alpha a \end{pmatrix}$ are orbits of φ).

By definition $(M^* \setminus e)^* = (\alpha', \varphi'\alpha')$. Let $M/e = (\alpha', \tau'')$. We wish to show that $\varphi'\alpha'$ is equal to τ''.

Consider first the case where $d \in D'$ is a dart occurring in a cycle of τ of the form

$$\begin{pmatrix} d & b & \cdots & b' \end{pmatrix}, \quad \text{where } b \notin \{a, \alpha a\}.$$

Then $d = \tau^{-1} b = \varphi^{-1}\alpha b$, in which $\alpha b \notin \{a, \alpha a\}$, whence $d \notin \{a, \alpha a, \varphi^{-1} a, \varphi^{-1}\alpha a\}$. In this case $\varphi' d = \varphi d$. Hence $\varphi'\alpha' d = \varphi\alpha d = \tau d$ for darts d such that $\alpha d \notin \{\alpha a, a, \alpha\varphi^{-1} a, \alpha\varphi^{-1}\alpha a\}$; equivalently, for $d \notin \{a, \alpha a, \tau^{-1} a, \tau^{-1}\alpha a\}$, using $\alpha^2 = \iota$ and $\varphi = \tau\alpha$. Likewise, by definition we have $\tau'' d = \tau d$ for $d \notin \{a, \alpha a, \tau^{-1} a, \tau^{-1}\alpha a\}$. Hence $\varphi'\alpha' d = \tau'' d$ for such darts d.

It remains to consider $d \in \{\tau^{-1} a, \tau^{-1}\alpha a\}$, that is, d is a dart occurring in a cycle of τ of the form

$$\begin{pmatrix} d & a & b & \cdots \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} d & \alpha a & c & \cdots \end{pmatrix}.$$

If $d = \tau^{-1} a = \alpha\varphi^{-1} a$, then $\alpha d = \varphi^{-1} a$ and by definition $\varphi'\alpha d = \varphi'\alpha' d = \varphi a$ provided $\varphi^{-1} a \neq \alpha a \neq \varphi a$, that is, $\tau a \neq a$ (or $\alpha\varphi^{-1} a = \tau^{-1} a \neq \alpha\alpha a = a$) and $\tau\alpha a \neq \alpha a$ (or $\alpha a \neq \varphi a = \tau\alpha a$). Similarly, if $d = \tau^{-1}\alpha a = \alpha\varphi^{-1}\alpha a$ then $\alpha d = \varphi^{-1}\alpha a$ and by definition $\varphi'\alpha d = \varphi'\alpha' d = \varphi\alpha a$ provided $\varphi^{-1}\alpha a \neq a \neq \varphi\alpha a$, that is, $\tau^{-1}\alpha a \neq \alpha a$ (or $\alpha\varphi^{-1}\alpha a = \tau^{-1}\alpha a \neq \alpha a$) and $\tau a \neq a$. Since $d \notin \{a, \alpha a\}$ and $d \in \{\tau^{-1} a, \tau^{-1}\alpha a\}$, neither a nor αa are fixed points of τ (and $\begin{pmatrix} a & \alpha a \end{pmatrix}$ is not a cycle of τ either, in the case under consideration). We conclude that

$$\varphi'\alpha' d = \begin{cases} \tau\alpha a & \text{if } d = \tau^{-1} a, \\ \tau a & \text{if } d = \tau^{-1}\alpha a. \end{cases}$$

When a and αa belong to disjoint cycles of τ, one of which also contains d, say

$$\begin{pmatrix} a & b & \cdots & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha a & c & \cdots & c' \end{pmatrix},$$

or

$$\begin{pmatrix} a & b & \cdots & b' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha a & c & \cdots & d \end{pmatrix},$$

then these cycles in τ'' become the non-empty cycles

$$\begin{pmatrix} b & \cdots & d & c & \cdots & c' \end{pmatrix},$$

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respectively

$$\begin{pmatrix} b & \cdots & b' & c & \cdots & d \end{pmatrix}.$$

In the first case, when $d = \tau^{-1}a$, we have $\tau''d = c = \tau\alpha a$; in the second case, when $d = \tau^{-1}\alpha a$, we have $\tau''d = b = \tau a$. Therefore $\tau''d = \varphi'\alpha' d$ in these two cases.

Lastly, when a and αa belong to the same cycle in τ and which also contains d, say

$$\begin{pmatrix} a & b & \cdots & d & \alpha a & c & \cdots & c' \end{pmatrix},$$

or

$$\begin{pmatrix} a & b & \cdots & b' & \alpha a & c & \cdots & d \end{pmatrix},$$

then in τ'' these cycles become

$$\begin{pmatrix} b & \cdots & d \end{pmatrix}\begin{pmatrix} c & \cdots & c' \end{pmatrix},$$

respectively

$$\begin{pmatrix} b & \cdots & b' \end{pmatrix}\begin{pmatrix} c & \cdots & d \end{pmatrix}.$$

In the first case, when $d = \tau^{-1}\alpha a$, we have $\tau''d = b = \tau a$; in the second case, when $d = \tau^{-1}a$, we have $\tau''d = c = \tau\alpha a$. Again, therefore, $\tau''d = \varphi'\alpha' d$ in both cases.

Finally, if a and/or αa belong to cycles of τ that contain no other darts $d \in D'$, that is, $\begin{pmatrix} a \end{pmatrix}$, $\begin{pmatrix} \alpha a \end{pmatrix}$ or $\begin{pmatrix} a & \alpha a \end{pmatrix}$ are cycles of τ, then, respectively, $\begin{pmatrix} \alpha a \end{pmatrix}$, $\begin{pmatrix} a \end{pmatrix}$ or $\begin{pmatrix} a & \alpha a \end{pmatrix}$ are cycles of φ. Therefore the empty cycle(s) obtained when deleting an edge to obtain φ' are in correspondence with the empty cycles obtained when contracting an edge to obtain τ''.

We have completed the verification that $\varphi'\alpha' = \tau''$ on D' and that empty cycles of φ' and τ'' correspond. □

Proof of Lemma 3.11. Using Euler's formula (3.1),

$$\begin{aligned} 2g(M\backslash A^c) + 2g(M/A) &= 2k(M\backslash A^c) + 2k(M/A) - v(M\backslash A^c) - v(M/A) + e(M\backslash A^c) + e(M/A) \\ &\quad - f(M\backslash A^c) - f(M/A) \\ &= 2g(M) + [k(M\backslash A^c) - k(M) - f(M\setminus A^c) + k(M/A)] \\ &\quad + [k(M/A) - k(M) - v(M/A) + k(M\backslash A^c)]. \end{aligned}$$

We now claim that

$$k(M/A) - k(M) - v(M/A) + k(M\backslash A^c) \leqslant 0. \tag{A.1}$$

It suffices to prove that $k(M/A) - 1 \leqslant v(M/A) - k(M\backslash A^c)$ for a connected map M, since the map parameters $v(M)$ and $k(M)$ are additive over disjoint unions.

Let $F \subseteq A$ be a maximal spanning forest of $M\backslash A^c$ (*i.e.* $M\backslash F^c$ contains no cycles and for each $e \in A\backslash F$ the underlying graph of $M\backslash(F \cup \{e\})^c$ contains a cycle). We have then $k(M\backslash F^c) = k(M\backslash A^c)$ and we now need to prove that $k(M/A) - 1 \leqslant v(M/A) - k(M\backslash F^c)$.

Each edge in F is a non-loop of M. Contracting a non-loop edge in M corresponds to its contraction in the underlying graph Γ, and in particular preserves connectivity of M. We thus

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have $k(M/F)-1=1-1=0=v(M/F)-k(M\backslash F^c)$, as each connected component of $M\backslash F^c$ is reduced to a single vertex in M/F.

If $A=F$, we thus have equality in (A.1). Suppose then that there is $e\in A\backslash F$. There is a unique cycle of $M\backslash A^c$ whose edges are contained in $F\cup\{e\}$. An edge $e\in A\backslash F$ is thus a loop of M/F and when it is contracted its incident vertex v splits into two new vertices v_1 and v_2. These vertices are either in the same connected component, in which case $v(M/(F\cup\{e\}))=v(M/F)+1$ while $k(M/F)=k(M/(F\cup\{e\}))$, or not in the same connected component, in which case $v(M/(F\cup\{e\}))=v(M/F)+1$ and $k(M/(F\cup\{e\}))=k(M/F)+1$. In both cases we have $k(M/(F\cup\{e\})-1\leqslant v(M/(F\cup\{e\}))-k(M\backslash F^c)$.

It may be that, when contracting the loop $e\in A\backslash F$ of M/F on vertex v, a loop $e'\in A\backslash(F\cup\{e\})$ of M/F becomes in $M/(F\cup\{e\})$ a non-loop edge joining v_1 and v_2. Then $v(M/(F\cup\{e,e'\}))=v(M/(F\cup\{e\}))-1=v(M/F)$ while $k(M/(F\cup\{e,e'\}))=k(M/(F\cup\{e\}))=k(M/F)$. In this case $k(M/(F\cup\{e,e'\})-1\leqslant v(M/(F\cup\{e,e'\}))-k(M\backslash F^c)$, and again the desired inequality holds. Any other edges $e''\in A\backslash(F\cup\{e,e'\})$ that are loops in M/F are loops in $M/(F\cup\{e,e'\})$.

We may now repeat the argument: having contracted edges $B\subseteq A\backslash F$ to leave just loops, preserving the desired inequality $k(M/(F\cup B))-1\leqslant v(M/(F\cup B))-k(M\backslash F^c)$, choose an edge $e\in A\backslash(F\cup B)$ that is a loop of $M/(F\cup B)$. Eventually all the edges of A are contracted and the inequality (A.1) holds. By duality we immediately obtain $k(M\backslash A^c)-k(M)-f(M\setminus A^c)+k(M/A)\leqslant 0$. This proves the lemma. □

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