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Degenerate random perturbations of Anosov diffeomorphisms

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Abstract. This paper deals with random perturbations of diffeomorphisms on n-dimensional Riemannian manifolds with distributions supported on k-dimensional disks, where $k < n$. First we demonstrate general but not very intuitive conditions which guarantee that all invariant measures for rank-k random perturbations of C^2 diffeomorphisms are absolutely continuous with respect to the Riemannian measure on M. For two subclasses of Anosov diffeomorphisms, hyperbolic toral automorphisms and Anosov diffeomorphisms with codimension 1 stable manifolds, the above conditions are modified in order to relate k-dimensional disks that support the distributions to certain foliations that arise from Anosov diffeomorphisms. We conclude that generic rank-k random perturbations have absolutely continuous invariant measures.

1. *Introduction*

This paper aims to address the following question. *Given a diffeomorphism f on a Riemannian manifold M subject to a small degenerate random perturbation, under what conditions can we guarantee that all the invariant measures for such a system are absolutely continuous with respect to the Riemannian measure on M and whether such conditions are satisfied for generic random perturbations of f?*

More precisely, let M be a compact Riemannian manifold and f be a diffeomorphism on M. A random perturbation of f is defined as a Markov chain on M such that for every x, the transition probability distribution $P(\cdot|x)$ is given by Q_{fx} where Q_{fx} is a probability distribution that depends only on $f(x)$ and is not far from the point mass at $f(x)$. Intuitively it means that a particle jumps from x to $f(x)$ and then disperses randomly near $f(x)$ with the distribution Q_{fx}.

Assuming Q_{fx} depends continuously on x, such Markov chains admit stationary probability measures. We will refer to these stationary measures as *invariant measures*. If we denote the Riemannian measure on M by m, an important question to ask is whether the system admits invariant measures absolutely continuous with respect to m. The answer to this question is straightforward if we assume that $\{Q_{fx}\}$ are absolutely continuous with

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respect to m (i.e. have a density) for every x. Given the absolute continuity of transitional probabilities, every invariant measure of the perturbed dynamics is absolutely continuous with respect to m.

Interesting problems arise for degenerate random perturbations in which the probability distributions Q_{fx} do not necessarily have densities. In many real systems, perturbations do not occur everywhere or uniformly in all directions. Frequently, irregular patches on the domain may introduce random patterns or perturbations can occur on the boundary. For discrete time systems, it is also natural for the Q_{fx} to have bounded or even small support, making the problem of uniqueness of invariant densities rather delicate.

The analogous problem for continuous-time systems, i.e. for stochastic differential equations with degenerate stochastic noise, has been extensively studied. Hormander's theorem [**5**] states that if a certain combination of Lie brackets of the vector fields involved in the differential operator generate the whole space, then the system supports an absolutely continuous stationary measure. Even though the formalism is well-established, a good understanding of what types of dynamics give rise to hypoellipticity in the stochastic differential equation (SDE) setting is not well understood. On the other hand, no comparable result exists for discrete-time dynamical systems so far.

The purpose of this investigation is to study the invariant measures for random perturbations of Anosov diffeomorphisms when Q_{fx} is *localized*, i.e. Q_{fx} is supported on a bounded set for all x, and *degenerate*, i.e. Q_{fx} is not absolutely continuous with respect to m. The localization prevents reaching 'everywhere' in one step and therefore makes it possible for invariant measures to be supported on small sets (compared to the scale of the phase space). The degeneracy may prevent invariant measures from being absolutely continuous with respect to the Riemannian measure m and potentially create 'too many' of them.

For definiteness, this paper focuses on uniform rank-k random perturbations, i.e. when Q_{fx} are uniform distributions on k-dimensional disks obtained from k vector fields. Such a restriction is not essential for many general properties of the perturbed system, but allows us to describe things more precisely.

The paper is organized in the following way: in §2 we provide background information on the subject of random perturbations; in §§3, 4 and 5 we study rank-one random perturbations; and in §6 we generalize the results obtained in §§3, 4 and 5 for rank-k random perturbations.

More precisely, in §3 we discuss very general conditions that apply for rank-one random perturbations of any C^2 diffeomorphism f and guarantee absolute continuity of all invariant measures. However, these conditions do not provide an intuitive description of the relation between f and the vector field along which the perturbation occurs.

In §4 we restrict f to the class of C^2 Anosov diffeomorphisms and aim to provide a different set of conditions that relate the vector field along which the perturbation occurs to certain f-invariant foliations. The conditions that guarantee absolute continuity of all invariant measures are obtained for two important subclasses of Anosov diffeomorphisms: hyperbolic toral automorphisms and Anosov diffeomorphisms with codimension 1 stable manifolds. In §5 we prove genericity of this these conditions.

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2. *Background and preliminaries*

Let M be a Riemannian manifold and let $f : M \to M$ be a continuous or a piecewise-continuous function. Denote by $P(M)$ the space of Borel probability measures on M with the topology of weak convergence. We consider a family $Q_x \in P(M)$ such that $Q : M \to P(M)$ is Borel. Given Q_x, by a random perturbation of f we will mean a Markov chain X_n, $n = 0, 1, 2, \ldots$ with transition probabilities $P(A|x) = P\{X_{n+1} \in A : X_n = x\} = Q_{fx}(A)$ defined for any $x \in M$, Borel set $A \subset M$, and $n \in \mathbb{Z}^+$. Intuitively it means that a particle jumps from x to fx and then disperses randomly near fx with the distribution Q_{fx}. A standard reference for this material is [**4**]. Given $f : M \to M$ and a family $\{Q_x\}$, we will denote the randomly perturbed dynamics $P(\cdot|x) = Q_{fx}$ by $\mathcal{F}$. Given a measure ν, the push forward of ν under f will be given by $(\mathcal{F}_*\nu)(A) = \int_M P(A|x)\, d\nu$.

Definition. Invariant measure: A probability measure μ on M is called an invariant measure of the Markov chain X_n if for any Borel set $A \subset M$, $\int_M P(A|x)\, d\mu(x) = \mu(A)$, i.e. $\mathcal{F}_*\mu = \mu$. By the invariant measure of a random perturbation $\mathcal{F}$ of f we will mean the invariant probability measure of the Markov chain with the corresponding transition probabilities.

If M is compact, there is a simple condition that ensures the existence of an invariant measure.

LEMMA 2.1. [**4**, Proposition 1.4] *Let X_n be a Markov chain on a compact space M with transition probabilities $P(A|x) = P\{X_1 \in A : X_0 = x\}$. Suppose that the measures $P(\cdot|x) \in P(M)$ depend continuously on x in the topology of weak convergence in $P(M)$. Then the Markov chain X_n has at least one invariant probability measure.*

In the context of a random perturbation, we have $P(A|x) = Q_{fx}(A)$, so continuity of f plus continuous dependence of Q_x on x ensures the existence of an invariant measure.

Definition. We will say that a measure μ has a density if it is absolutely continuous with respect to the Riemannian measure m on the manifold.

When a measure μ is absolutely continuous with respect to m, we will denote it by $\mu \ll m$. If μ is singular with respect to m, we will denote it $\mu \perp m$.

LEMMA 2.2. *If $\{P(\cdot|x)\}$ have densities with respect to the Riemannian measure m, i.e. $P(\cdot|x) \ll m$, then all invariant measures μ also have densities, i.e. $\mu \ll m$.*

Proof. Let $g_x = dP(\cdot|x)/dm$, then for every continuous function φ

$$\int_M \varphi(x)\, d\mu(x) = \int_M \int_M \varphi(y) P(dy|x)\, d\mu(x)$$
$$= \int_M \int_M \varphi(y) g_x(y)\, dm(y)\, d\mu(x) = \int_M \varphi(y) g_x(y)\, dm(y).$$

Therefore, $\mu \ll m$. □

LEMMA 2.3. *Let $\mathcal{F}$ be a random dynamics on M defined by a Markov chain X_n with transition probabilities $P(A|x)$. Assume that for any finite measure ν, if $\nu \ll m$, then*

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$\mathcal{F}_*\nu \ll m$. Suppose μ is invariant under $\mathcal{F}$. Let $\mu = \mu_\perp + \mu_\ll$, where $\mu_\ll \ll m$ and $\mu_\perp \perp m$. Then both $\mu_\perp$ and $\mu_\ll$ are invariant under $\mathcal{F}$.

Proof. By definition of the push-forward measure,

$$\mathcal{F}_*\mu = \int_M P(\cdot|x)\, d\mu(x) = \int_M P(\cdot|x)\, d\mu_\ll(x) + \int_M P(\cdot|x)\, d\mu_\perp(x) = \mathcal{F}_*\mu_\ll + \mathcal{F}_*\mu_\perp.$$

Let A, B be such that $A \cup B = M$, $A \cap B = \emptyset$, $m(A) = 1 = \mu_\perp(B)$, $m(B) = 0 = \mu_\perp(A)$. So for any Borel set E, $\mu(E) = \mu_\ll(E \cap A) + \mu_\perp(E \cap B)$. Since

$$(\mathcal{F}_*\mu_\ll) \ll m,\ (\mathcal{F}_*\mu_\ll)(E \cap B) = 0$$

and

$$(\mathcal{F}_*\mu_\ll)(E) = (\mathcal{F}_*\mu_\ll)(E \cap A) + (\mathcal{F}_*\mu_\ll)(E \cap B) = (\mathcal{F}_*\mu_\ll)(E \cap A).$$

By the invariance of μ we conclude that

$$\begin{aligned}\mu_\ll(E \cap A) + \mu_\perp(E \cap B) &= \mu(E) = (\mathcal{F}_*\mu)(E)\\ &= (\mathcal{F}_*\mu_\ll)(E \cap A) + (\mathcal{F}_*\mu_\perp)(E \cap A) + (\mathcal{F}_*\mu_\perp)(E \cap B).\end{aligned}$$

The above relation holds for any Borel set E, thus

$$\mu_\ll(E \cap A) = (\mathcal{F}_*\mu_\ll)(E \cap A) + (\mathcal{F}_*\mu_\perp)(E \cap A)$$

and

$$\mu_\perp(E \cap B) = (\mathcal{F}_*\mu_\perp)(E \cap B).$$

$\{P(\cdot|x)\}$ are probability distributions with $P(M|x) \equiv 1$. Thus $(\mathcal{F}_*\mu_\perp)(M) = \mu_\perp(M) = \mu_\perp(M \cap B) = (\mathcal{F}_*\mu_\perp)(M \cap B)$, implying that $(\mathcal{F}_*\mu_\perp)(M \cap A) = (\mathcal{F}_*\mu_\perp)(M \setminus B) = 0$. Therefore

$$\begin{aligned}\mu_\ll(E \cap A) &= (\mathcal{F}_*\mu_\ll)(E \cap A),\\ \mu_\perp(E \cap B) &= (\mathcal{F}_*\mu_\perp)(E \cap B)\end{aligned}$$

for any Borel set E implying that both $\mu_\ll$ and $\mu_\perp$ are invariant under $\mathcal{F}$. □

3. *General conditions for absolutely continuous invariant density (for rank-one random perturbations)*

Let M be a compact n-dimensional C^2 Riemannian manifold, f be a C^2 diffeomorphism (not necessarily Anosov for this section), V be a C^2 unit vector field on M (assume M supports such vector fields) and $\epsilon > 0$ be fixed. By m we will denote the Riemannian measure on M.

Throughout this section, the $\{Q_x\}$ will be defined as follows. Given x and ϵ, let $I_\epsilon(x)$ be a curve passing through x along the flow of the vector field V such that for any $y \in I_\epsilon(x)$ the Riemannian distance from x to y is less than or equal to ϵ. Since M is differentiable and V is C^2, $I_\epsilon(x)$ is well defined. Let Q_x be the uniform distribution on $I_\epsilon(x)$. Let $\mathcal{F}$ denote the perturbed dynamics given by a Markov chain with transition probabilities $P(\cdot|x) = Q_{fx}$.

Remark. By Lemma 2.1, our perturbed system has at least one invariant measure.

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3.1. *A 'bracket-like' condition.* The following argument suggests that, provided the dynamics are rich enough, it is natural to expect that there should exist an absolutely continuous invariant measure for the system subject to a rank-one random perturbation. If we start with the point measure at x and push it forward by the perturbed dynamics, after the first step the new measure is equal to Q_{fx} and is supported on a one-dimensional curve. If the image of this curve under f is not tangent to the vector field, then the 'smearing' will produce a transition probability $P^2(\cdot|x)$ that has density along a two-dimensional surface, i.e. it acquires an extra dimension from the perturbation. Now consider the f-image of the support of $P^2(\cdot|x)$. If this image is never tangent to the vector field, then in the next step yet another dimension is acquired, i.e. $P^3(\cdot|x)$ now has a three-dimensional density. This process may be continued as long as the non-tangency condition is satisfied. We will show that if f is a C^2 diffeomorphism and, for $n = \dim M$, $P^n(\cdot|x)$ is absolutely continuous with respect to m for every x, then every invariant measure of the perturbed system has a density.

The heuristic argument in the previous paragraph requires the vector field to be not tangent to the support of $P^k(\cdot|x)$ everywhere and for every x. The condition of this kind is rather too strong and many important examples that possess absolutely continuous invariant measures do not satisfy this property. Acquiring a non-zero absolutely continuous component even with a tiny support for every x might often be enough for absolutely continuous invariant measures to exist.

We are going to start with a very simple generalization of the non-tangency condition that works for a larger class of systems. Later, in §3.2, we are going to extend this condition further to ensure that we obtain certain properties we will use in §4.

Let $Df_x^k : T_xM \to T_{f^kx}M$ be the derivative of f^k at x. Then $Df_x^k(V(x))$ is a tangent vector at $f^k(x)$ to $f^k(I_\epsilon(x))$. Note that $Df_x^k(V(x))$ varies C^1 with x since f and V are C^2.

Definition. $n_0(x)$. For every $x \in M$, $\dim M = n$, let $n_0(x)$ be the minimum k such that

$$\mathrm{Span}\{Df_{fx}^{k-1}(V(fx)),\, Df_{f^2x}^{k-2}(V(f^2x)),\, \ldots,\, Df_{f^{k-1}x}(V(f^{k-1}x)),\, V(f^kx)\} = \mathbb{R}^n.$$

If $\mathrm{Span}\{Df_{fx}^{k-1}(V(fx)), Df_{f^2x}^{k-2}(V(f^2x)), \ldots, Df_{f^{k-1}x}(V(f^{k-1}x)), V(f^kx)\}$ is a proper subspace of $\mathbb{R}^n$ for all k, define $n_0(x) = \infty$.

Given f, V and ϵ, let $\mathcal{F}$ be a rank-one perturbation of f. Clearly, if $n_0(x) < \infty$, $\mathcal{F}_*^{n_0}\delta_x$ has a non-zero absolutely continuous component. We will say that a singular measure ν acquires density in k steps if $\mathcal{F}_*^k\nu$ has a non-zero absolutely continuous component.

PROPOSITION 3.1. *If, for all $x \in M$, $n_0(x) < \infty$ then there exists an absolutely continuous invariant measure. In fact all invariant measures are absolutely continuous with respect to m.*

LEMMA 3.1. *If $n_0(x) < \infty$, then there exists an open neighborhood U of x such that for every $y \in U$, $n(y) \le n(x)$.*

Proof. If $n_0(x) < \infty$,

$$\mathrm{Span}\{Df_{fx}^{n_0(x)-1}(V(fx)), \ldots, Df_{f^{n_0(x)-1}x}(V(f^{n_0(x)-1}x)), V(f^{n_0}x)\} = \mathbb{R}^n.$$

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Each of the vectors in the Span varies C^1 with x, so there exist an open neighborhood U of x such that for any $y \in U$,

$$\mathrm{Span}\{Df_{fy}^{n_0(x)-1}(V(fy)), \ldots, Df_{f^{n_0(y)-1}y}(V(f^{n_0(y)-1}y)), V(f^{n_0}y)\} = \mathbb{R}^n.$$

Thus $n_0(y) \le n_0(x)$. □

LEMMA 3.2. *Let $\mathcal{F}$ be a rank-one perturbation, given f, V, and ϵ. If μ is an invariant measure under $\mathcal{F}$ and $\mu = \mu_\perp + \mu_{\ll}$, where $\mu_{\ll} \ll m$ and $\mu_\perp \perp m$, then both $\mu_\perp$ and $\mu_{\ll}$ are invariant under $\mathcal{F}$.*

Proof. To apply Lemma 2.3 we need to show that if $\nu \ll m$, then $\mathcal{F}_*\nu \ll m$.

$\mathcal{F}_*\nu = (\mathcal{F}_*\nu)_{\ll} + (\mathcal{F}_*\nu)_\perp$. Let A, B be such that $A \cup B = M$, $A \cap B = \emptyset$ and for every Borel set E, $(\mathcal{F}_*\nu)(E) = (\mathcal{F}_*\nu)_{\ll}(E \cap A) + (\mathcal{F}_*\nu)_{\ll}(E \cap B)$.

We want to show that the singular component is zero, i.e. $(\mathcal{F}_*\mu)_\perp(B) = 0$. Fix some small neighborhood U and define Γ to be a smooth transversal to V in U. For every $x \in U$, denote the local segment in U along the vector field flow through x by $I(x)$, the one-dimensional Riemannian measure on $I(x)$ by m_x and the Riemannian measure on Γ by m_Γ. Since $m(B \cap U) = 0$ and V is C^2, by the Fubini theorem for m_Γ-almost every y, $m_y(B \cap U) = 0$. $\{Q_x\}$ are supported on the curves $I_\epsilon(x)$ along V, thus for m_Γ-almost every y and any $x \in I(y)$, $Q_x(B \cap U) = 0$, i.e. the set $\{x : Q_x(B \cap U) > 0\}$ lies on a m_Γ-zero measure of $I(y)$. Applying the Fubini theorem again we conclude that $m\{x : Q_x(B \cap U) > 0\} = 0$. Since M is compact, we can cover M by finitely many such U and conclude that $m\{x : Q_x(B) > 0\} = 0$.

Since f is a C^2 diffeomorphism, f and f^{-1} map Riemannian zero-measure sets to Riemannian zero-measure sets. Thus $m\{x : P(B|x) > 0\} = m\{x : Q_{fx}(B) > 0\} = 0$ implies $\nu\{x : P(B|x) > 0\} = 0$ by the absolute continuity of ν. Therefore for any Borel set E, $(\mathcal{F}_*\nu)_\perp = \int_M P(E \cap B|x)\, d\nu = 0$ and $\mathcal{F}_*\nu \ll m$. Applying Lemma 2.3 proves the result. □

Proof of Proposition 3.1. First let us show that there exists N such that for all x, $n(x) \le N$. Suppose this is not the case. Then there exists x_k such that $n(x_k) > k$. Since M is compact, we have a convergent subsequence $x_{k(j)}$ and a limit $x = \lim_{k(j)\to\infty} x_{k(j)}$. By assumption, we must have $n_0(x) < \infty$. Then by Lemma 3.1 there exists an open neighborhood U of x such that $n(y) \le n(x)$ for any $y \in U$. This contradicts the supposition.

Now suppose ν is an invariant measure that is not absolutely continuous with respect to m. Then $\nu = \nu_\perp + \nu_{\ll}$ and both are invariant by Lemma 3.2. $\nu_\perp$ acquires density in N steps, which contradicts its invariance and singularity. Therefore all invariant measures must be absolutely continuous with respect to m. □

3.2. *'SDE' condition.* We defined $n_0(x)$ to represent the time when the density is acquired locally around the orbit of x. However, the dynamics of $\mathcal{F}$ are rich and many random orbits deviate from the orbit of x under f. It seems plausible that acquiring density along some random orbit of x would suffice to get the same conclusion as in

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Proposition 3.1. So we would like to extend the definition of n_0 of x to incorporate a broader class of situations.

Yet there might still be points that do not 'acquire density' under the extended definition. We would like to define and study the properties of the set S of such 'deficient' points. That, for example, will enable us to conclude in §4 that under certain conditions on Anosov diffeomorphisms, $S = \emptyset$.

Given $x \in M$ with dim $M = n$, we can consider a nested sequence of sets constructed in the following manner:

$$H_0(x) = \{x\}, \quad H_1(x) = \bigcup_{y \in H_0(x)} I_\epsilon(fy),$$
$$\ldots, \quad H_k(x) = \bigcup_{y \in H_{k-1}(x)} I_\epsilon(fy), \quad \ldots.$$

Suppose $n_0(x_k) < \infty$ for some $x \in H_k(x)$. Then by Lemma 3.1, there exists an open neighborhood $U \ni x_k$ such that $n_0(y) \leq n_0(x_k)$ for all $y \in U$. In particular, this is true for any $y \in U \cap H_k(x)$. Thus any measure supported on $U \cap H_k(x)$ acquires density in $n_0(x_k)$ steps. Since Q_x are defined to be uniform on $I_\epsilon(x)$, which lie along the flow of the C^2 vector field V, $\mathcal{F}_*^k \delta_x(U \cap H_k(x)) > 0$. Therefore $\mathcal{F}_*^{n_0(x_k)+k} \delta_x$ has a non-zero absolutely continuous component.

Definition. $n(x)$. Let $n(x)$ be the minimum of $n_0(y) + k$ such that $y \in H_k(x)$ and $k \in \mathbb{Z}^+ \cup \{0\}$. Define $n(x) = \infty$ if $n_0(y) = \infty$ for all $y \in \bigcup_k H_k$.

LEMMA 3.3. *If $x \in M$ is such that $n(x) < \infty$, then there exists an open set $U \ni x$ such that for every $y \in U$, $n(y) \leq n(x)$.*

Proof. If $n(x) < \infty$, there exists k and $y \in H_k(y)$ such that $n_0(y) + k = n(x)$. We are going to use induction to show the following statement.

(*) For all $y \in H_i(x)$, if U_y is a neighborhood of y, there exists a neighborhood W of x such that for any $z \in W$, $H_i(z) \cap U_y \neq \emptyset$.

For $i = 0$ the statement is trivial. For $i = 1$, $H_1(x) = I_\epsilon(x)$ and $H_1(z) = I_\epsilon(z)$. Since both curves are defined along the flow of C^2 vector field V, the statement follows.

Suppose the statement (*) holds for $i = j$. For $i = j + 1$,

$$H_{j+1}(x) = \bigcup_{y \in H_j(x)} I_\epsilon(fy) \quad \text{and} \quad H_{j+1}(w) = \bigcup_{z \in H_j(w)} I_\epsilon(fz).$$

Pick any $q \in H_{j+1}(x)$ and $U_q \ni q$. $q \in I_\epsilon(fy)$ for some $y \in H_j(x)$. Since $I_\epsilon(\cdot)$-curves are defined along the flow of C^2 vector field V, there exists a neighborhood U_y of y, such that for all $p \in U_y$, $I_\epsilon(p) \cap U_q \neq \emptyset$. By induction assumption, there exists a neighborhood W of x, such that for all $z \in W$, $H_j(z) \cap U_y \neq \emptyset$. Thus for all $z \in W$, $H_{j+1}(z) \cap U_q \neq \emptyset$. This proves (*).

Applying the statement (*) to the previous argument, we conclude that there exists a neighborhood W of x such that for all $w \in W$, $H_k(w) \cap U \neq \emptyset$ and for any $z \in H_k(w) \cap U$, $n_0(z) \leq n_0(y)$. Pick any random orbit of w, $\{w_0 = z, \ldots, w_k\}$ such that $w_k \in H_k(w) \cap U$ to conclude that $n(x) \leq n(y)$. □

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Points that acquire density are the key for having absolutely continuous invariant measures, while those that do not acquire density require special attention.

Definition. Set S. Let $S = \{x \in X : n(x) = \infty\}$.

We will call a set A invariant under f if $f(A) = A$, forward invariant under f if $f(A) \subset A$, and forward invariant under the perturbed dynamics if for all $x \in A$, $I_\epsilon(fx) \subset A$.

THEOREM 1. *S is closed, forward invariant and forward invariant under the perturbed dynamics.*

Proof. If $n(x) < \infty$, there is an open set U_x around x such that for all $y \in U_x$, $n(y) \leq n(x)$ by Lemma 3.3. This fact plus the definition of $n(x)$ gives $n(f^{-1}x) \leq n(x) + 1$. Let $A_k = \{x : n(x) \leq k\}$; then $f^{-1}A_k \subset A_{k+1}$. If $A = \bigcup_{k=1}^{\infty} A_k$, then A is open and backward invariant, $A = S^c$, so S is closed and forward invariant.

Assume there exists $x \in S$, such that for some $y \in I_\epsilon(fx)$, $n(y) < \infty$. Then there exists a random orbit $\{y = y_0, \ldots, y_k\}$ such that $n_0(y_k) + k = n(y)$. Thus choosing the random orbit $\{x, y = y_0, y_1, \ldots, y_k\}$, we conclude that $n(x) \leq (k+1) + n_0(y_k) = n(y) + 1$, which contradicts the fact the $x \in S$. Therefore S is forward invariant under the perturbed dynamics. □

COROLLARY 3.1. *If $S \neq \emptyset$, S contains a C^2 curve along the flow of the vector field V.*

Proof. By Theorem 1 for every $x \in S$, $I_\epsilon(fx) \subset S$. □

THEOREM 2. *If $S = \emptyset$, then any invariant measure is absolutely continuous with respect to m.*

Proof. The proof is exactly the same as the proof of Proposition 3.1, applying Lemma 3.3 instead of Lemma 3.1. □

Theorem 2 describes the situation when $S = \emptyset$. When $S \neq \emptyset$, there might be invariant measure(s) that are singular with respect to m. By Theorem 1, S is forward invariant and contains curves everywhere tangent to V. Often, there are very few or no proper subsets of this kind. Thus it is important to rule out the situation $S = M$ as atypical.

LEMMA 3.4. *Let $f : M \to M$, $\dim M = n$, be such that there exists at least one $x \in M$ with $x, fx, \ldots, f^n x$ all distinct. Then $S \neq M$ for an open and dense subset $\mathcal{V}$ of vector fields on M.*

Proof. If for given V, there exists x such that $n(x) < \infty$, then there exists an open neighborhood of vector fields around V with $n(x) < \infty$ for all vector fields in this neighborhood. This is similar to the statement in Lemma 3.3. Therefore $\mathcal{V}$ is open.

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Now suppose for some V, $S = M$. Pick any x such that $x, fx, \ldots, f^n x$ are distinct. A small perturbation of V can break the tangencies (if tangencies exist) between

$$Df_{fx}(V(fx)) \quad \text{and} \quad V(f^2x),$$
$$\text{Span}\{Df^2_{fx}(V(fx)), Df_{f^2x}(V(f^2x))\} \quad \text{and} \quad V(f^3x),$$
$$\ldots$$
$$\text{Span}\{Df^{n-1}_{fx}(V(fx)), Df^{n-2}_{f^2x}(V(f^2x)), \ldots, Df_{f^{n-1}x}(V(f^{n-1}x))\} \quad \text{and} \quad V(f^n x),$$

ensuring that density is acquired along the orbit $x, fx, \ldots, f^n x$ and $n_0(x) = n$. Thus $n(x) = n < \infty$. So $\mathcal{V}$ is dense. □

4. *Conditions for absolute continuity of invariant measures for rank-one random perturbations of Anosov diffeomorphisms*

Let M be an n-dimensional C^2 Riemannian manifold, $f : M \to M$ be C^2 Anosov diffeomorphism. Let V be a unit C^2 vector field on M. Given ϵ, define rank-one random perturbation of f along V, $\mathcal{F}$, as in §3.

Given an Anosov diffeomorphism $f : M \to M$, we would like to focus on the following question.

Question. Under what conditions on the vector field V are we guaranteed that all the invariant measures in the system are absolutely continuous with respect to the Riemannian measure m on M?

In §3 we have shown that if the set of 'deficient' points S is empty, then all invariant measures are absolutely continuous. However this condition does not describe the properties of the vector field V in relation to the diffeomorphism f in any simple fashion. In this section we are going to use certain facts about Anosov diffeomorphisms to provide conditions on the vector field V which guarantee that $S = \emptyset$. With this information, we would be able to conclude in §5 that for certain classes of Anosov Diffeomorphisms, $S = \emptyset$ for a residual set of C^2 vector fields.

4.1. *Hyperbolic toral automorphisms, linear vector fields.*

We are going to start with the simplest example to get an idea of what might be important for the general case.

Let f be a hyperbolic toral automorphism on $\mathbb{T}^n$ defined by some matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$ all distinct and eigenvectors $v_1, \ldots, v_n$. Let $\overline{V}$ be a constant vector field on $\mathbb{R}^n$, $V(p) = \zeta$, such that ζ is not contained in any of the proper invariant subspaces of the form $\text{Span}\{v_{n_1}, \ldots, v_{n_k}\}$, $n_j \in \{1, 2, \ldots, n\}$, and V be its projection to $\mathbb{T}^n$.

PROPOSITION 4.1. *In the setting above, $S = \emptyset$ and all the invariant measures are absolutely continuous with respect to the Riemannian measure m on $\mathbb{T}^n$.*

Proof. For any $x \in \mathbb{T}^2$, $I_\epsilon(fx)$ is a line segment along the flow of V. If we lift it to $\mathbb{R}^n$, $\overline{I_\epsilon(fx)}$ has a unit tangent vector ζ that does not belong to any of the proper invariant subspaces. Thus its coordinates, $a_1, \ldots, a_n$, with respect to the eigenbasis $\{v_1, \ldots, v_n\}$ are all non-zero, i.e. $\zeta = a_1 v_1 + \cdots + a_n v_n$ with $a_i \neq 0$, $1 \leq i \leq n$.

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Given an orbit $\{x, f(x), \dots, f^n x\}$, the vectors $\zeta, A\zeta, A^2\zeta, \dots, A^{n-1}\zeta$ correspond exactly to the vectors

$$V(f^n x),\ Df_{f^{n-1}x}V(f^{n-1}x),\ Df^2_{f^{n-2}x}V(f^{n-2}x), \dots,\ Df^{n-1}_{fx}V(fx)$$

in the definition of $n_0(x)$ (see §3). If we show that they span $\mathbb{R}^n$, then $n(x) = n_0(x) = n$, for all $x \in \mathbb{T}^n$ and $S = \emptyset$. By Proposition 3.1 the result will follow.

Indeed the coordinates of $\zeta, A\zeta, A^2\zeta, \dots, A^{n-1}\zeta$ in the eigenbasis are

$$\zeta = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix}, \quad A\zeta = \begin{pmatrix} a_1\lambda_1 \\ a_2\lambda_2 \\ \dots \\ a_n\lambda_n \end{pmatrix}, \dots, \quad A^{n-1}\zeta = \begin{pmatrix} a_1\lambda_1^{n-1} \\ a_2\lambda_2^{n-1} \\ \dots \\ a_n\lambda_n^{n-1} \end{pmatrix}.$$

In order for them to be linearly dependent, there must exist $c_1, \dots, c_n$ not all zero such that $c_1\zeta + c_2(A\zeta) + \dots + c_n(A^{n-1}\zeta) = 0$. This corresponds to n systems of equations $1 \le i \le n$, $c_1a_i + c_2a_i\lambda_i + c_3a_i\lambda_i^2 + \dots + c_na_i\lambda_i^{n-1} = a_i(c_1 + c_2\lambda_i + c_3\lambda_i^2 + \dots + c_n\lambda_i^{n-1}) = 0$. Since none of the a_i are zero this is equivalent to the condition that $\lambda_1, \dots, \lambda_n$ must be the roots of the polynomial $c_1 + c_2\lambda + \dots c_n\lambda^{n-1}$ of degree $(n-1)$. This is impossible since $\lambda_1, \dots, \lambda_n$ are all distinct (by the fundamental theorem of algebra). Therefore $\zeta, A\zeta, A^2\zeta, \dots, A^{n-1}\zeta$ are linearly independent. □

The example above propagates an idea that if the vector field V 'avoids' tangencies to certain invariant sets (that correspond to the invariant subspaces in the example), all the invariant measures in the system should be absolutely continuous with respect to m. However, a single tangency of a vector field to any of the invariant subspaces is unlikely to produce a singular invariant measure. In order to obtain relatively general conditions on the vector field V that guarantee that existence of absolutely continuous invariant measures, we are going to use the results from §3.

4.2. *Hyperbolic toral automorphisms, C^2 vector field.* In order to formulate the result, we need the following definitions.

Definition. Tangential coincidence. Given a foliation $\mathfrak{F}$ and a vector field V on M, we are going to say that V has a tangential coincidence with the foliation if there exist a curve γ along the vector field flow that fully belongs to a single foliation leaf.

If f is a hyperbolic toral automorphism on $\mathbb{T}^n$, then any power of f, f^k is also a hyperbolic toral automorphism. It makes sense to talk about toral subgroups of $\mathbb{T}^n$ invariant under powers of f. Let $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n$ be the quotient map.

Definition. Foliation $\mathfrak{F}_G$. Given a proper compact subgroup G of $\mathbb{T}^n$ invariant under f^k, let W be the corresponding subspace of $\mathbb{R}^n$, with $\pi(W) = G$. Consider a foliation $\overline{\mathfrak{F}_G}$ of $\mathbb{R}^n$ by the hyper-planes parallel to the subspace $\mathrm{Span}\{W \cup E^s\}$, where E^s is the stable subspace under f. Define $\mathfrak{F}_G = \pi(\overline{\mathfrak{F}_G})$ to be the projection of $\overline{\mathfrak{F}_G}$ to $\mathbb{T}^n$.

THEOREM 3. *Let f be a hyperbolic toral automorphism and $\mathcal{F}$ be its perturbation of size ϵ along the vector field V. Assume V has no tangential coincidences with the stable*

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foliation as well as with foliations $\mathfrak{F}_G$ for all proper compact toral subgroup G of $\mathbb{T}^n$ invariant under powers of f. Assume further that there exists $x \in \mathbb{T}^n$ such that $n(x) < \infty$ (i.e. $S \neq \mathbb{T}^n$). Then all the invariant measures under $\mathcal{F}$ are absolutely continuous with respect to m.

Remark. The condition $(\exists x\colon n(x) < \infty)$ is equivalent to $(\exists x\colon n_0(x) < \infty)$ by definition of $n(x)$.

Idea of proof. Given a rank-one random perturbation of an Anosov diffeomorphism f, let S the set of 'deficient' points discussed in §3. If $S \neq \emptyset$, then Theorem 1 states that S is closed, forward invariant and forward invariant under the perturbed dynamics. Consider a subset $S_{in} = \bigcap_{i=0}^{\infty} f^i S \subset S$. By the forward invariance of S, $S \supset f(S) \supset f^2(S) \supset \cdots$ is a sequence of nested sets and by compactness of S (closed subset of a compact manifold) S_{in} is compact, non-empty, and invariant under f. We are going to show that if the vector field V has no tangential coincidences with the stable foliation as well as the foliations $\mathfrak{F}_G$ for all proper compact toral subgroups G of $\mathbb{T}^n$ invariant under powers of f, then $S_{in} \neq \emptyset$ implies $S = \mathbb{T}^n$. Since Theorem 3 assumes that the later is not the case, S must be empty.

PROPOSITION 4.2. *If $S \neq \emptyset$ there are three possibilities.*

(*) *$S_{in} \neq \mathbb{T}^n$ is invariant and contains a line segment J parallel to E^u.*

(**) *V has tangential coincidences with the stable foliation.*

(***) $S = S_{in} = \mathbb{T}^n$.

In order to prove this proposition we are going to use the following lemma.

LEMMA 4.1. [**2**] *Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a hyperbolic toral automorphism and let $C \in \mathbb{T}^n$ be a C^2 curve that is nowhere tangent to the stable foliation leaves. Then there exist a sequence of positive integers $n_j \to \infty$ as $j \to \infty$ and arcs $C_j \subset C$ such that $J = \lim_{j\to\infty} f^{n_j}(C_j)$ is a straight line segment parallel to E^u.*

The proof of this lemma is contained in the proof of [**2**, Lemma 2].

Proof of Proposition 4.2. Assume V has no tangential coincidences to the stable foliation, i.e. case (**) does not hold. If $S \neq \emptyset$, then it contains a C^2 arc along the flow of the vector field V by Corollary 3.1. Such curve cannot fully belong to any stable leaf because V has no tangential coincidences with the stable foliation. Thus there exists a subcurve $\gamma \subset I_\epsilon(x)$ that is nowhere tangent to the stable foliation. By Lemma 4.1, there exist n_j and $\gamma_j \subset \gamma$ such that $J = \lim_{j\to\infty} f^{n_j}(\gamma_j)$ is a straight line segment parallel to E^u. Therefore, the set $S_{in} = \bigcap_{n\geq 0} f^n(S) \subset S$ contains J and is, in fact, a compact invariant set that contains a line segment.

We showed that if (**) is not the case, then S_{in} contains a line segment. Since in the situation (***), $S_{in} = \mathbb{T}^n$ certainly contains a line segment, we can break the situations (*) and (***) apart. □

We plan to eliminate the possibility (**) by imposing the condition that V cannot have tangential coincidences with the stable foliation and the possibility (***) by assuming that there exists x such that $n(x) < \infty$, i.e. by simply assuming that (***) is not the case. In Lemma 3.4 we ensured that for an open and dense set of vector fields (***) does not occur.

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The conditions of Theorem 3 eliminate the possibilities (**) and (***). In order to eliminate the possibility (*), we are going to try 'reconstructing' S from S_{in} in the following fashion.

Let

$$\begin{gathered} S_0 = S_{in}, \quad S_1 = \bigcup_{x\in S_0} I_\epsilon(fx), \quad S_2 = \bigcup_{x\in S_1} I_\epsilon(fx), \\ \ldots, \quad S_k = \bigcup_{x\in S_{k-1}} I_\epsilon(fx), \quad \ldots. \end{gathered} \tag{1}$$

Let $S'_{in} = \overline{\bigcup_k S_k}$.

LEMMA 4.2. *S'_{in} is compact, forward invariant and forward invariant under the perturbed dynamics. Moreover, $S'_{in} \subset S$ and $S_{in} = \bigcap_{i=0}^{\infty} f^i(S'_{in})$.*

Proof. S'_{in} is closed by definition and compact because it is a subset of a compact manifold $\mathbb{T}^n$. The set $\bigcup_k S_k$ is forward invariant and forward invariant under the perturbed dynamics because for all $x \in S_k$, $I_\epsilon(fx) \subset \bigcup_{x\in S_k} I_\epsilon(fx) = S_{k+1}$. Suppose $x \in (\overline{\bigcup_k S_k}) \setminus (\bigcup_k S_k)$ and $x_j \to x$, $x_j \subset \bigcup_k S_k$. Then for all j, $I_\epsilon(fx_j) \subset S'_{in}$. Since $I_\epsilon(\cdot)$ are defined along the flow of C^2 vector field and S'_{in} is closed, $I_\epsilon(fx) \subset S'_{in}$. Thus S'_{in} is forward invariant and forward invariant under the perturbed dynamics.

By Theorem 1 S is closed, forward invariant and forward invariant under the perturbed dynamics. $S_{in} \subset S$ implies that $S_k \subset S$, for all k and thus $\bigcup_k S_k \subset S$. Since S is closed, $S'_{in} = \overline{\bigcup_k S_k} \subset S$.

To show that $S_{in} = \bigcap_{i=0}^{\infty} f^i(S'_{in})$, we need to show that both inclusions work.

- $S_{in} \subset \bigcap_{i=1}^{\infty} f^i(S'_{in})$ because $S_{in} \subset S'_{in}$ is invariant.
- $\bigcap_{i=0}^{\infty} f^i(S'_{in}) \subset S_{in}$ because $S'_{in} \subset S$ and $S_{in} = \bigcap_k f^k(S)$.

Thus $S_{in} = \bigcap_{i=0}^{\infty} f^i(S'_{in})$. □

We can perform the same construction for any compact invariant set $T \subset \mathbb{T}^n$. That is, define

$$\begin{gathered} T_0 = T, \quad T_1 = \bigcup_{x\in T_0} I_\epsilon(fx), \quad T_2 = \bigcup_{x\in T_1} I_\epsilon(fx), \\ \ldots, \quad T_k = \bigcup_{x\in T_{k-1}} I_\epsilon(fx), \quad \ldots. \end{gathered} \tag{2}$$

Let $T' = \overline{\bigcup_k T_k}$.

LEMMA 4.3. *If T is invariant, then T' is closed, forward invariant and forward invariant under the perturbed dynamics.*

The proof is exactly the same as for Lemma 4.2.

To describe the compact f-invariant sets, we are going to use following theorem by John Franks.

FRANK'S THEOREM. [**2**, Theorem 1] *If $f : \mathbb{T}^n \to \mathbb{T}^n$ is a hyperbolic toral automorphism and K is a compact invariant set which contains a C^2 arc (line segment for our purposes), then K contains a torus of dimension ≥ 2 which is a coset of a subgroup of $\mathbb{T}^n$ which is invariant under some power of f.*

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Using this result we are going to show that if T is a non-empty compact invariant set that contains a C^2 arc and V satisfies the assumptions of Theorem 3, then $T' = \mathbb{T}^n$. If we let $T = S_{in}$, it would imply that $T' = S'_{in} = \mathbb{T}^n$ and therefore $S = \mathbb{T}^n$ since $S'_{in} \subset S$ by Lemma 4.2. Thus the situation (*) cannot occur.

To understand when $\mathbb{T}^n$ contains a proper subgroup invariant under some power of f is, we will use the following.

Definition. For an automorphism g of $\mathbb{T}^n$ induced by a matrix A, we say that g is reducible if the characteristic polynomial of A is reducible over $\mathbb{Z}$; g is irreducible otherwise.

LEMMA 4.4. [**3**] *A hyperbolic toral automorphism $f : \mathbb{T}^n \to \mathbb{T}^n$ induced by a matrix A has a proper invariant toral subgroup if and only if the characteristic polynomial of A is reducible over $\mathbb{Z}$.*

In eliminating the possibility (*), let us first treat the simple case when every power of f is irreducible. In this situation there are no proper subgroups of $\mathbb{T}^n$ invariant under some power of f by Lemma 4.4 and thus no proper f-invariant subsets that contain curves by Frank's theorem.

Now suppose that f^k is reducible for some power k. Then by Lemma 4.4 $\mathbb{T}^n$ has non-trivial subgroups invariant under f^k. For each proper compact subgroup G of $\mathbb{T}^n$, we defined the foliation $\mathfrak{F}_G$ in the following way.

Let $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n$ be the quotient map. If G is a proper compact subgroup of $\mathbb{T}^n$, invariant under f^k, let W be the corresponding subspace of $\mathbb{R}^n$, with $\pi(W) = G$. Consider a foliation $\overline{\mathfrak{F}_G}$ of $\mathbb{R}^n$ by the hyper-planes parallel to the subspace $\mathrm{Span}\{W \cup E^s\}$. Let $\mathfrak{F}_G = \pi(\overline{\mathfrak{F}_G})$ be the projection of $\overline{\mathfrak{F}_G}$ to $\mathbb{T}^n$.

PROPOSITION 4.3. *If V has no tangential coincidences with any of the foliations $\mathfrak{F}_G$ for all proper subgroups of G of $\mathbb{T}^n$ invariant under powers of f, then for any invariant set T that contains a C^2 curve, $T' = \mathbb{T}^n$.*

To prove this proposition, we are going to use the following lemma.

LEMMA 4.5. *Let C be a C^2 curve in $\mathbb{T}^n$ that is never tangent to the foliation $\mathfrak{F}_G$ for some proper toral subgroup G invariant under f^k. Let* $\mathrm{Span}\{W \cup E^s\}$ *be the corresponding lift of the foliation $\mathfrak{F}_G$ to $\mathbb{R}^n$. Then there exist a sequence of positive integers $n_j \to \infty$ as $j \to \infty$ and arcs $C_j \subset C$ such that $J = \lim_{j\to\infty} f^{n_j}(C_j)$ is a straight line segment not parallel to* $\mathrm{Span}\{W \cup E^s\}$.

The proof of this lemma is again contained in [**2**, proof of Lemma 2].

Proof of Proposition 4.3. Let $\prec$ be the partial order by inclusion on the subspaces W with $G = \pi(W)$ being proper toral subgroups invariant under powers of f. Then if V has no tangential coincidences with the $\mathfrak{F}_G$ induced by the hyper-planes parallel to $\mathrm{Span}\{W \cup E^s\}$ for all W with non-dense projections maximal under this partial order, then it does not have tangential coincidences with all such foliations $\mathfrak{F}_G$. If W is a maximal element of the partial order $\prec$ and $v \notin \mathrm{Span}\{W \cup E^s\}$, then $\mathrm{Span}\{v \cup W\}$ cannot project to a proper toral subgroup invariant under some power of f by the maximality of W. Thus $\pi(\mathrm{Span}\{v \cup W\})$ must be dense in $\mathbb{T}^n$.

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Given any compact invariant set T, by Lemma 4.3, $T' = \overline{\bigcup_k T_k}$ is compact forward invariant and forward invariant under the perturbed dynamics. Let G be a proper compact f^k-invariant toral subgroup with the corresponding subspace W, $\pi(W) = G$, maximal under $\prec$. Since $I_\epsilon(fx) \in T'$, for all $x \in T'$ and we assumed that V has no tangential coincidences to $\mathfrak{F}_G$, T' contains a curve γ that is never tangent to $\mathfrak{F}_G$. Thus by Lemma 4.5, T' contains a line segment J not parallel to $\mathrm{Span}\{W \cup E^s\}$. In particular, J is not parallel to W. Therefore if v is a vector tangent to J, $\pi(\mathrm{Span}\{v \cup W\})$ is dense in $\mathbb{T}^n$, implying that $T' = \mathbb{T}^n$. Since this argument can be performed with any proper compact toral subgroup G with corresponding maximal W, the conclusion follows. □

Now we are ready to prove the Theorem 3.

Proof of Theorem 3. Suppose $S \neq \emptyset$. Then $S_{in} = \bigcap_{i=0}^{\infty} f^i(S) \neq \emptyset$ and one of the cases (*), (**), or (***) must apply. If S_{in} is a compact f-invariant set that contains a line segment, then by Proposition 4.3, $S'_{in} = \mathbb{T}^n$. Since $S'_{in} \subset S$ by Lemma 4.2 it follows that $S = \mathbb{T}^n$. So case (*) cannot occur. Case (**) cannot occur because we assumed that V has no tangential coincidences with the stable foliation. And case (***) cannot occur because there exists $x \in \mathbb{T}^n$ such that $n(x) < \infty$. This contradicts the fact that one of the three cases must apply. Thus $S = \emptyset$ and all the invariant measures under $\mathcal{F}$ are absolutely continuous with respect to the Riemannian measure m. □

4.3. *Anosov diffeomorphisms with codimension-one stable manifolds.* In the previous subsection we gave a condition for hyperbolic toral automorphisms that ensures that all invariant measures for the perturbed dynamics are absolutely continuous with respect to m. For that we were required to avoid tangential coincidences with a countable number of foliations, which were obtained using the algebraic structure of hyperbolic toral automorphisms. It is natural to expect that nonlinear Anosov diffeomorphisms similarly have non-trivial compact invariant subsets that contain curves since most of the known examples of Anosov diffeomorphisms are topologically conjugate to hyperbolic toral automorphisms and their algebraic generalizations. However we do not have enough tools to describe such non-trivial compact invariant subsets in great generality. Thus in this subsection we will only study Anosov diffeomorphisms with stable manifolds of codimension one.

THEOREM 4. *Let $f : M \to M$ be C^2 Anosov diffeomorphism with codimension-one stable manifolds. Let $\mathcal{F}$ be a rank-one perturbation of f given a C^2 vector field V and $\epsilon > 0$. Assume V has no tangential coincidences with the stable foliation and there exists $x \in M$ such that $n(x) < \infty$ (i.e. $S \neq M$). Then $S = \emptyset$ and all invariant measures are absolutely continuous with respect to the Riemannian measure m on M.*

LEMMA 4.6. *Given the setting above, consider $f(S) = \{x \in S : f^{-1}x \in S\}$. Then if $V(x) \notin E^s$ for at least one $x \in f(S)$, then there exists a piece of curve $\gamma \subset S$ that is everywhere tangent to the unstable foliation, meaning that $\gamma \subset W^u_{loc}(y)$ for some $y \in \gamma$.*

Remark. Note that $f(S)$ is non-empty if S is non-empty because S is forward invariant by Theorem 1.

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Proof. Suppose there exists $x \in f(S)$ with $V(x) \notin E^s$. Since S is forward invariant under the perturbed dynamics (Theorem 1), $I_\epsilon(x)$ is a C^2 curve contained in S. We assumed that $V(x) \notin E^s$, thus there exists an open subcurve $\gamma \subset I_\epsilon(x)$, such that for all $z \in \gamma$ the angle between $V(z)$ and E^s is bounded away from zero. Since S is forward invariant, all the forward images of γ under f must belong to S as well. Because f is Anosov, the product structure exists everywhere and there is an exponential contraction/expansion along the stable/unstable manifold. So for all $z \in \gamma$, the angles between $V(f^k z)$ and W^u must converge uniformly to 0 as $k \to \infty$. By compactness of M, there exists a subsequence $f^{k_i}x$ of $f^k x$ that converges to some $x_0 \in M$ and $x_0 \in S$ because S is closed. Denote $\{z \in f^{k_i}(\gamma) : d(f^{k_i}x, z) < \epsilon\}$ by γ_i and let γ_0 be the curve formed by all the possible limits of sequences $\{x_i\}$, $x_i \in \gamma_i$. The limits exist because each γ_i is C^2 and the angles between tangent vectors to γ_i and W^u tend uniformly to zero. Thus γ_0 is everywhere tangent to W^u and $\gamma_0 \subset S$. □

Proof of Theorem 4. Codimension-one Anosov diffeomorphisms are one-sided topologically transitive. We would like to use this fact to show that forward images of γ_0 must be dense. Let V be a neighborhood we would like to reach. Then it contains a sub-neighborhood W' such that $\text{dist}(\overline{W'}, W^c) \geq \delta$ for some $\delta > 0$. Let $U = \bigcup_{x \in \gamma_0} W^s_\delta(x)$ be a δ neighborhood of γ_0. Then by one-sided topological transitivity, there exists an $n \geq 0$ such that $f^n(U) \cap W' \neq \emptyset$. Since the curves along the stable manifolds cannot expand under f, by definition of U and W', $f^n(\gamma_0) \cap W \neq \emptyset$. Thus the forward images of this curve $\gamma_0 \subset S$ are dense.

Since S is closed and must include all these images, $S = M$. We assumed this is not the case: there exists $x \in M$ such that $n(x) < \infty$. Therefore S must be empty. By Theorem 2 all invariant measures are absolutely continuous with respect to m. This completes the proof of the Theorem 4. □

5. *Genericity of the conditions in §§4.2 and 4.3*

5.1. *Statement of results.* In the previous sections, we provided answers to the question (Q) for the cases of hyperbolic toral automorphisms and of Anosov diffeomorphisms with stable manifolds of codimension one. The conclusion was that in order to ensure that all invariant measures are absolutely continuous with respect the Riemannian measure m, we have to 'avoid' tangential coincidences with a countable number of certain foliations.

In this section we will establish that for a residual set of C^2 vector fields the conditions required in Theorems 3 and 4 hold and thus all the invariant measures are absolutely continuous with respect to m.

THEOREM 5. *Let f be a hyperbolic toral automorphism. Given V and $\epsilon > 0$, define $\mathcal{F}_V$ to be the corresponding random perturbation of f. Then for a residual subset of vector fields $\mathcal{V}$, all the invariant measures under $\mathcal{F}_V$, $V \in \mathcal{V}$, are absolutely continuous with respect to m.*

THEOREM 6. *Let f be an Anosov diffeomorphism with stable manifolds of codimension one. Given V and $\epsilon > 0$, define $\mathcal{F}_V$ to be the corresponding random perturbation of f.*

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Then for a residual subset of vector fields $\mathcal{V}$, all the invariant measures under $\mathcal{F}_V$, $V \in \mathcal{V}$, are absolutely continuous with respect to m.

In order to prove Theorems 5 and 6, we are going to establish that the set of C^2 vector fields that do not have tangential coincidences with a given continuous foliation by C^2 leaves is residual without making any assumptions on foliation smoothness (i.e. on how the leaves are packed).

PROPOSITION 5.1. *Let $\mathfrak{F}$ be a continuous foliation of a C^2 manifold M by C^2 leaves. Then a residual set of C^2 vector fields have no tangential coincidences with $\mathfrak{F}$.*

Proof of Theorem 5. In Theorem 3 we established that if f is a hyperbolic toral automorphism and $\mathcal{F}$ its random perturbation along V with size ϵ, then if V has no tangential coincidences to a countable number of foliations and $S \neq M$, then all invariant measures under $\mathcal{F}$ are absolutely continuous. Applying Proposition 5.1 and Lemma 3.4, the conclusion follows. □

Proof of Theorem 6. This is a direct consequence of Theorem 4, Proposition 5.1, and Lemma 3.4. □

The rest of the section will be devoted to proving Proposition 5.1.

5.2. *Proof of Proposition 5.1, Part 1.* Let M be an n-dimensional C^2 manifold and let $\mathfrak{F}$ be a continuous foliation by C^2 leaves. In particular, $\mathfrak{F}$ could be the stable or the unstable foliation of a C^2 Anosov diffeomorphism. Given a C^2 vector field V with at least one tangential coincidence with the foliation $\mathfrak{F}$, we are going to demonstrate a simple argument that all the tangential coincidences can be removed by a randomly small C^2 perturbation of V. That will ensure, in particular, that the set of vector fields that do not have tangential coincidences is dense in the set of all C^2 vector fields.

Note that in this argument we use the word 'perturbation' of the vector field V to mean that there exists another vector field V' such that $\|V - V'\|_{C^2} < \epsilon$, for some $\epsilon > 0$ small. We do not deal with any random perturbations of dynamical systems here.

LEMMA 5.1. *Let M be an n-dimensional C^2 manifold, $\mathfrak{F}$ be a continuous foliation of M by C^2-leaves, and V be a C^2 vector field on M. Then if V has a tangential coincidence with $\mathfrak{F}$, there exists a perturbation of V any small such that the perturbed vector field has no open set coincidences with $\mathfrak{F}$.*

For better visualization we a going to assume that the foliation $\mathfrak{F}$ has codimension one in M. Later we will show that this assumption is not essential.

The following argument was kindly suggested by Charles Pugh.

Since V is a C^2 vector field, there exist an open neighborhood $\tilde{U}$ around any point $x \in M$ and a C^2 diffeomorphism $\varphi : \tilde{U} \to \tilde{W} \subset \mathbb{R}^n$ such that $\varphi(V) = X$ is a straight line vector field in $\tilde{W}$. We can always pick $\tilde{U}$ and φ such that $\|\varphi\| > 1$ and $\tilde{W} = ([0, 1] \times \tilde{D})$, with X parallel to the axis containing $[0, 1]$, call it x-axis, and $\tilde{D}$ some $(n-1)$-dimensional disk perpendicular to the x-axis.

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Let $\tilde{U}$ be such a neighborhood in M that contains a tangential coincidence of the vector field V with the foliation $\mathfrak{F}$ leaf. We assumed that $\varphi(\mathfrak{F})$ has $(n-1)$-dimensional leaves and that one of them contains a tangential coincidence with X along the x-axis. Let $W \subset \tilde{W}$ containing a tangential coincidence be small enough so that we can pick another axis, call it y-axis, in $\mathbb{R}^n$ such that $\varphi(\mathfrak{F})$ is everywhere transversal to the y-direction in W and has the angles between the y-direction and the tangent spaces of the foliation leaves bounded away from zero. Define $U = \varphi^{-1}W \subset \tilde{U}$.

LEMMA 5.2. *Given a perturbation size δ, there exists a vector field X' δ-near X in $W = ([0,1] \times D)$ such that X' has no tangential coincidences with the foliation $\varphi(\mathfrak{F})$ in $[1/3, 2/3] \times D'$, where $D' \subset D$ is such that* $\operatorname{dist}(D', \partial D) < 2\delta$. *In addition to that, $X = X'$ on ∂W.*

Proof. Consider 'slicing' the foliation leaves by the xy-planes. Each plane parallel to the xy-plane intersects each $\varphi(\mathfrak{F})$-foliation leaf on some C^2 curve. If R is such a plane parallel to the xy-plane, then $R \cap \varphi(\mathfrak{F})$ is a foliation of R by C^2 curves such that each curve can be considered as a function from x to y (because the angle between the foliation curves and y-axis is bounded away from zero).

Consider the disk $((1/2) \times D)$. Through each point $z \in ((1/2) \times D)$ passes the unique plane R parallel to the xy-plane and the unique $\varphi(\mathfrak{F})$-foliation leaf $L(x)$. Their intersection gives the unique C^2 curve $\gamma(z)$ described by some function $y = g(x)$ in R. Let $h_z : \mathbb{R} \to \mathbb{R}$ be a function defined by $h_z(x) = g(x) - g(1/2)$.

Let T be a space of C^2 functions from $[1/3, 2/3]$ to I with $f(1/2) = 0$, where I is some compact interval tht contains the projection of $[1/3, 2/3] \times D$ onto the y-axis. Define $H : ((1/2) \times D) \to T$ by $H(z) = (h_z)|_{[1/3,2/3]}$. The disc $((1/2) \times D)$ is compact while the space T of all planar C^2 curves through $[1/2, 0]$ is not locally compact in the C^2-topology. Therefore there exist a curve, γ, of norm less than δ, with $\gamma(1/3) = \gamma(2/3) = 0$, $\gamma'(1/3) = \gamma'(2/3) = 0$, and $\gamma''(1/3) = \gamma''(2/3) = 0$, that does not coincide with any of the curves in the image of H on an open set.

Change the vector field X to X' such that X' is always tangent to the translations of this curve on $([1/3, 2/3] \times D')$, where $D' \subset D$ and $\operatorname{dist}(D', \partial D) < 2\delta$. Also make X' C^2-vary on $[1/3, 2/3] \times (D \setminus D')$ such that $X' = X$ on $[1/3, 2/3] \times \partial D$, i.e. parallel to the x-axis. Let $X' = X$ on $([0, 1/3] \times D) \cup ([2/3, 1] \times D)$. That makes X' a C^2 vector field on $([0,1] \times D)$ such that there are no tangential coincidences of X' with $\varphi(\mathfrak{F})$ on $[1/3, 2/3] \times D'$. This completes the proof of Lemma 5.2. □

The vector field V' defined to be $V' = \varphi^{-1}(X')$ in U and $V' = V$ in U^c is a C^2 vector field on M such that $\|V - V'\|_{C^2} < \delta'$, where $\delta' = \delta/\|\varphi\|$. We chose φ such that $\|\varphi\| > 1$, and thus $\delta' < \delta$.

The assumption that $\mathcal{F}$ has codimension one creates a better visual picture, while not essential for the fact. Indeed, if the leaves of $\mathcal{F}$ have dimension $k < n$ and if we pick the 'flow box' $W = ([0,1] \times D)$ small enough, then we can pick $(n-k)$ perpendicular axes $y_1, \ldots, y_{n-k}$ (instead of a single y-axis) that have angles with the $\varphi(\mathcal{F})$-leaves bounded away from zero. Thus we can 'slice' the foliation by the hyper-planes parallel to $x \times y_1 \times \cdots \times y_{n-k}$, getting the C^2 curves in the intersection (which can also be

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represented as C^2 functions from x-axis to $y_1 \times \cdots \times y_{n-k}$). The space of such curves is similarly not locally compact in the C^2 topology, thus we can change the vector field in the neighborhood $[0, 1] \times D$ in a similar fashion. This way we can obtain a δ-small perturbation of X that breaks the tangential coincidences with $\varphi(\mathfrak{F})$ in $([1/3, 2/3] \times D')$.

Proof of Lemma 5.1. Fix $\epsilon > 0$ to be the maximum perturbation size allowed. Each open set coincidence can be surrounded by neighborhoods U_α as above and the tangential coincidences can be eliminated on $U'_\alpha = \varphi^{-1}([1/3, 2/3] \times D')$ sets. For each point $x \in M$, there exist U_x and U'_x of this kind such that $x \in U'_x \subset U_x$ (we just need to ensure, for instance, that $\varphi(x)$ is a point with the coordinate $1/2$ on the x-axis). Cover M with such U'_α neighborhoods. Then by compactness of M, there exists a finite subcovering $U'_1, \ldots, U'_n$. Let $\delta = \epsilon/n$. Following the procedure of the Lemma 5.2 we can eliminate the tangential coincidences with $\mathfrak{F}$ in all U_i, one by one. Thus we would obtain a vector field V', $\|V - V'\| < \epsilon$, such that V' has no tangential coincidences with $\mathfrak{F}$. □

5.3. *Proof of Proposition 5.1, Part 2.* Lemma 5.1 implies that the set of C^2 vector fields with no tangential coincidences to a given foliation by C^2 leaves is dense in the set of all C^2 vector fields with C^2 topology. To prove Theorem 5.1 we need to extend this result to show that a residual set of vector fields has no tangential coincidences to a given foliation. We will proceed using upper semi-continuous functions (referred to by Amie Wilkinson). The following argument exactly parallels the discussion in [**6**, §10.1] with lower semi-continuous functions.

Let $\mathcal{C}_M$ be a collection of all compact subsets of M. Let $\mathcal{X}$ be the space of all C^2 vector fields on M.

Definition. For $A_n \subset \mathcal{C}_M$, $n \geq 1$, define

$$\limsup_{n\to\infty} A_n = \left\{ y \in M : \text{there exists } y_n \in A_n,\ n \geq 1,\ \text{such that } y = \lim_{n\to\infty} y_n \right\}.$$

Definition. A set-valued function $\Gamma : \mathcal{X} \to \mathcal{C}_M$ is called upper semi-continuous at x if $\Gamma(X) \supset \limsup_{n\to\infty} \Gamma(X_n)$ for every $X_n \to X$.

LEMMA 5.3. [**1**] *Suppose $\Gamma : \mathcal{X} \to \mathcal{C}_M$ is an upper semi-continuous set-valued function. Let $\mathcal{R} \subset \mathcal{X}$ be the points of continuity of Γ. Then $\mathcal{R}$ is residual.*

For any C^2 vector field V, define $\Gamma(V) = \{x \in M : V \text{ has a tangency with } \mathfrak{F} \text{ at } x\} = \Gamma^0 X$ as defined in the previous subsection. Then Γ is upper semi-continuous. Indeed, if $V_n \to V$ in C^2 topology and V_n have tangencies to $\mathfrak{F}$ at $x_n \to x$, then V must have a tangency to $\mathfrak{F}$ at x (because C^2 convergence requires the first and second derivatives converge too).

Assume V has a tangential coincidence with $\mathfrak{F}$. Given $\epsilon_n \to 0$, $n \geq 1$, by Lemma 5.1 there exist a sequence of vector field V_n with no tangential coincidences with $\mathfrak{F}$ such that $\|V - V_n\|_{C^2} < \epsilon_n$. Thus the vector fields with tangential coincidences belong to the discontinuity set of Γ.

By Lemma 5.3, the set of C^2 vector fields with no tangential coincidences to $\mathfrak{F}$ is residual, which concludes the proof of Theorem 5.1. □

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6. *Rank-k random perturbations and further generalizations*

In this section we are going to generalize the types of perturbations our results apply to.

6.1. *Rank-k random perturbations.* Let M be a compact n-dimensional Riemannian manifold, let f be a C^2 Anosov diffeomorphism, and let $V_1, \dots, V_k$ be C^2 unit vector fields on M never tangent to each other, i.e. for any x, $\mathrm{Span}\{V_1(x), \dots, V_k(x)\}$ is k-dimensional. Let $\epsilon_1, \dots, \epsilon_k > 0$ be fixed.

In order to define the rank-k random perturbation given $V_1, \dots, V_k$ and $\epsilon_1, \dots, \epsilon_k$, consider k different rank-one perturbed dynamics $\mathcal{F}_1, \dots, \mathcal{F}_k$ with identity maps and ϵ_i perturbations along the vector fields V_i, $1 \le i \le k$ defined as in §3. Then if we start with measure δ_x and push it forward by all of these perturbations in order, we will obtain a k-dimensional distribution supported on some C^2 submanifold of M. Let Q_x be such a distribution, i.e. $Q_x = (\mathcal{F}_k)_*(\cdots(\mathcal{F}_2)_*((\mathcal{F}_1)_*\delta_x))$ and let $I_\epsilon(x)$ denote the corresponding C^2 submanifold supporting Q_x, where ϵ represents a k-tuple $(\epsilon_1, \dots, \epsilon_k)$. Denote the random perturbation of f with transition probabilities $P(\cdot|x) = Q_{fx}$ by $\mathcal{F}$.

With the setting as above, we can define $n_0(x)$ as in §3 if we consider the corresponding span for all k vector fields $V_1, \dots, V_k$ simultaneously.

Definition. $n_0(x)$ for rank k perturbations. For every $x \in M$, $\dim M = n$, let $n_0(x)$ be the minimum l such that

$$\mathrm{Span}\{Df_{fx}^{l-1}(V_1(fx)), \dots, Df_{fx}^{l-1}(V_k(fx)), Df_{f^2x}^{l-2}(V_1(f^2x)), \dots, Df_{f^2x}^{l-2}(V_k(f^2x)),$$

$$\dots, Df_{f^{l-1}x}(V_1(f^{l-1}x)), \dots, Df_{f^{l-1}x}(V_k(f^{l-1}x)), V_k(f^lx), \dots, V_k(f^lx)\} = \mathbb{R}^n.$$

If $\mathrm{Span}\{Df_{fx}^{l-1}(V_1(fx)), \dots, Df_{fx}^{l-1}(V_k(fx)), Df_{f^2x}^{l-2}(V_1(f^2x)), \dots, Df_{f^2x}^{l-2}(V_k(f^2x)),$ $\dots, Df_{f^{l-1}x}(V_1(f^{l-1}x)), \dots, Df_{f^{l-1}x}(V_k(f^{l-1}x)), V_k(f^lx), \dots, V_k(f^lx)\}$ is a proper subspace of $\mathbb{R}^n$ for all l, define $n_0(x) = \infty$.

With the definition of $n_0(x)$ as above, define $n(x)$ and S as in §3.

THEOREM 7. *S is closed, forward invariant and forward invariant under the perturbed dynamics.*

THEOREM 8. *If $S = \emptyset$, then any invariant measure is absolutely continuous with respect to m.*

The proofs of Theorems 7 and 8 exactly parallel the proofs of Theorems 1 and 2. The only facts required in the proof that does not come directly from the definitions of $n(x)$ and S are as follows.

- If $n(x) < \infty$, there is an open set U_x around x such that for all $y \in U_x$, $n(y) \le n(x)$.
- If μ is an invariant measure under $\mathcal{F}$ and $\mu = \mu_\perp + \mu_{\ll}$, then both $\mu_\perp$ and $\mu_{\ll}$ are invariant under $\mathcal{F}$.

The proofs of these fact are very similar to the proofs of Lemmas 3.3 and 2.3 for rank-one perturbations.

We can also formulate the following corollary similar to the Corollary 3.1.

COROLLARY 6.1. *If $S \neq \emptyset$, S contains a k-dimensional C^2 disk.*

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Remark. Since S also has to be closed and forward invariant, S is likely to be empty for large k because there might be very few or no closed and forward invariant subsets containing C^2 disks. When $k = n$, the perturbations are not degenerate and $S = \emptyset$.

In order to obtain the results for Anosov diffeomorphisms, the definition of tangential coincidence should be modified to suit the rank-k situation.

Definition. Tangential coincidence of $I_\epsilon(x)$. Given a foliation $\mathfrak{F}$ and a family of C^2 disks $\{I_\epsilon(x)\}$, we say that $\{I_\epsilon(x)\}$ has a tangential coincidence with the $\mathfrak{F}$ if at least one member $I_\epsilon(x)$ coincides with a foliation leaf on a set open in $I_\epsilon(x)$.

When $\{I_\epsilon(x)\}$ is a family of C^2 disks supporting $\{Q_x\}$ for rank-k random perturbation and has a tangential coincidence with a continuous foliation $\mathfrak{F}$, then the vector fields $V_1, \ldots, V_k$ all have tangential coincidences with $\mathfrak{F}$. Using Proposition 5.1, this immediately implies the following.

PROPOSITION 6.1. *Let $\mathfrak{F}$ be a continuous foliation of C^2 manifold M by C^2 leaves. Then a family of C^2 disks $\{I_\epsilon(x)\}$ generated by a generic (in a residual sense) k-tuple of C^2 vector fields has no tangential coincidences with $\mathfrak{F}$.*

The key ingredient in proving Theorems 3 and 4 was ensuring that if $S \neq \emptyset$ there exists a C^2 curve in S that is never tangent to any of the 'forbidden' foliations. This in turn implied that S must be equal to the whole manifold M, which is assumed not to be the case in the above theorems. Suppose $\{I_\epsilon(x)\}$ is a family of C^2 disks supporting $\{Q_x\}$ for rank-k random perturbation and has no tangential coincidence with a foliation $\mathfrak{F}$. If $S \neq \emptyset$, say $x \in S$, then $I_\epsilon(fx) \in S$. Since $I_\epsilon(fx)$ cannot coincide with any foliation leaf on an open set, there exists a C^2 curve in S that is never tangent to $\mathfrak{F}$. The rest of the arguments of Theorems 3 and 4 follow through. Thus we obtain the following results.

THEOREM 9. *Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a hyperbolic toral automorphism and $\mathcal{F}$ be its rank-k perturbation. Assume $\{I_\epsilon(x)\}$ has no tangential coincidences with the stable foliation as well as with foliations $\mathfrak{F}_G$ for all proper compact toral subgroup G of $\mathbb{T}^n$ invariant under powers of f. Assume further that there exists $x \in \mathbb{T}^n$ such that $n(x) < \infty$ (i.e. $S \neq \mathbb{T}^n$). Then all the invariant measures under $\mathcal{F}$ are absolutely continuous with respect to m.*

THEOREM 10. *Let $f : M \to M$ be C^2 Anosov diffeomorphism with codimension-one stable manifolds. Let $\mathcal{F}$ be a rank-k perturbation of f. Assume $\{I_\epsilon(x)\}$ has no tangential coincidences with the stable foliation and there exists $x \in M$ such that $n(x) < \infty$ (i.e. $S \neq M$). Then $S = \emptyset$ and all invariant measures are absolutely continuous with respect to Riemannian measure m on M.*

THEOREM 11. *Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a hyperbolic toral automorphism. Given a k-tuple of C^2 vector fields that are never tangent to each other $V = (V_1, \ldots, V_k)$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_k)$, $\epsilon_1, \ldots, \epsilon_k > 0$, define $\mathcal{F}_V$ to be the corresponding rank-k random perturbation of f. Then for a residual subset of such k-tuples of C^2 vector fields, $\mathcal{V}$, all the invariant measures under $\mathcal{F}_V$, $V \in \mathcal{V}$ are absolutely continuous with respect to m.*

THEOREM 12. *Let f be an Anosov diffeomorphism with stable manifolds of codimension one. Given a k-tuple of C^2 vector fields that are never tangent to each other*

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$V = (V_1, \ldots, V_k)$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_k)$, $\epsilon_1, \ldots, \epsilon_k > 0$, define $\mathcal{F}_V$ to be the corresponding rank-k random perturbation of f. Then for a residual subset of such k-tuples of C^2 vector fields, $\mathcal{V}$, all the invariant measures under $\mathcal{F}_V$, $V \in \mathcal{V}$ are absolutely continuous with respect to m.

6.2. *Further generalizations.* There are several ways to generalize the definition of rank-k perturbation keeping all or most of the Theorems 7–12. First, note that we can relax the uniformity assumption on $\{Q_x\}$ provided that each Q_x vary continuously with x, is absolutely continuous with respect to the Riemannian measure on $I_\epsilon(x)$, and $I_\epsilon(x) = \text{supp}(Q_x)$. All the Theorems 7–12 still apply in this situation.

Another simple generalization is relaxing the perturbation along the vector field to the perturbations along the C^2 disks that are centered at and vary C^2 with x. For such perturbation, there is no problem defining $n_0(x)$, $n(x)$, and S is a similar manner and since we assume that disks vary C^2 with x, the openness condition (if $n(x) < \infty$, there is an open set U_x around x such that for all $y \in U_x$, $n(y) \leq n(x)$) similarly holds. Therefore Theorems 7 and 8 and Corollary 6.1 hold for such generalized perturbations. As discussed in the previous subsection, with the above results established Theorems 9 and 10 follow. However, we cannot establish genericity results for such generalized perturbations since our proof of Proposition 5.1 relies heavily on the rigidity of the vector fields in our proofs. In fact, they might fail under such mild assumptions on the perturbation.

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